ON THE STRUCTURE AND EXISTENCE OF SOME AMICABLE ORTHOGONAL DESIGNS

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Abstract

The structure is determined for the existence of some amicable weighing matrices. This is then used to prove the existence and non-existence of some amicable orthogonal designs in powers of two.

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1. Introduction

An orthogonal design of order n and type $(u_1, u_2, ..., u_s)$ $(u_i > 0)$ *on the commuting variables* $x_1, x_2, ..., x_s$ *is an* $n \times n$ *matrix* A *with entries from* $\{0, \pm x_1, ..., \pm x_s\}$ *such that*

$$AA^{\mathrm{T}} = \sum_{i=1}^{s} (u_i x_i^2) I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely u_i entries of type $\pm x_i$.

In Geramita et al. (1975–76), where this was first defined and many examples and properties of such designs were investigated, it is mentioned that

$$A^{\mathrm{T}} A = \sum_{i=1}^{s} (u_i x_i^2) I_n,$$

and so the alternative description of A applies as well to the columns of A. It is also shown in the same paper that $s \leqslant \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d,$$

when $n = 2^a . b$, b odd, $a = 4c + d$, $0 \leqslant d < 4$. In Wolfe (1975) it is shown that if $n \equiv 4 \pmod 8$ and if there exists an orthogonal design of order n and type (a_1, a_2, a_3, a_4) then $\prod_{i<j} (a_i, a_j)_p = 1$ at all primes p where $(a_i, a_j)_p$ is the Hilbert norm residue symbol. D. Shapiro (private communication, 1975–76) has found conditions on possible $\rho(n)$-tuples which can be the types of orthogonal designs in all orders n. In Geramita and Verner (1976) it is observed that if there exists an

118

orthogonal design of type $(u_1, u_2, ..., u_t)$ in order $n \equiv 0$ (mod 4) with $\sum_{i=1}^{t} u_i = n-1$ there exists a design of type $(1, u_1, u_2, ..., u_t)$ in order n. Robinson (1976b) has shown that orthogonal designs of type $(1, 1, 1, 1, 1, n-5)$ cannot exist in orders $n > 40$.

Wolfe (1976) defines two orthogonal designs X, Y, of the same order and types $(x_1, ..., x_s)$ and $(y_1, ..., y_t)$ respectively satisfying

$$XY^{\mathrm{T}} = YX^{\mathrm{T}},$$

to be *amicable orthogonal designs* of types $((x_1, ..., x_s); (y_1, ..., y_t))$. He gives conditions on the number of variables in such designs. Such amicable orthogonal designs were first used (in another guise) to construct Hadamard matrices. Some are given in Geramita et al. (1975–76) and some infinite families constructed in Wallis (1975).

A *weighing matrix of weight k and order n* is a square $\{0, 1, -1\}$ matrix, $W = W(n, k)$, or order n satisfying

$$WW^{\mathrm{T}} = kI_n.$$

In Geramita et al. (1975–76) it is shown that the existence of an orthogonal design of order n and type $(u_1, ..., u_s)$ is equivalent to the existence of weighing matrices $A_1, ..., A_s$, of order n, where A_i has weight u_i and the matrices, $\{A_i\}_{i=1}^{s}$, satisfying the matrix equation

$$XY^{\mathrm{T}} + YX^{\mathrm{T}} = 0,$$

in pairs.

2. Preliminaries

Let

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We shall write $+$ for $+1$ and $-$ for -1. Also we use I for the identity matrix and $\oplus_k Z$ for the direct sum of k copies of Z.

Matrices A, B which satisfy $AB^{\mathrm{T}} = BA^{\mathrm{T}}$ will be said to be *amicable*. A *best pair* is a pair of amicable weighing matrices (A, B) of weights i, j respectively satisfying

$$A^{\mathrm{T}} = -A, \quad B^{\mathrm{T}} = B, \quad AB^{\mathrm{T}} = BA^{\mathrm{T}}.$$

A *best pair family of order n* is a set of best pairs (A_i, B_j) of order n and weights i and j where

$$i = 1, 2, 3, ..., n-1, \quad j = 1, 2, 3, ..., n$$

and

$$A_i^{\mathrm{T}} = -A_i, \quad B_j^{\mathrm{T}} = B_j, \quad A_i B_j^{\mathrm{T}} = B_j A_i^{\mathrm{T}}.$$

Let $R, P_1, P_2, ..., H$ be orthogonal designs of order n and types $(r_1, r_2, ...)$, $(p_{i1}, p_{i2}, ...)$ for $i = 1, 2, ..., (h_1, h_2, ...)$ respectively. Then $(R; (P_1; P_2; ...); H)$ are *repeat designs* of order n and types $(r_1, r_2, ...; (p_{11}, p_{12}, ...; p_{21}, p_{22}, ...; ...); h_1, h_2, ...)$

(i) $R * P_i = 0$ for $i = 1, 2, ...$ (* the Hadamard product);

(ii) $R + P_i$ for $i = 1, 2, ...$ are orthogonal designs;

(iii) $R + P_i$ and H for $i = 1, 2, ...$ are amicable orthogonal designs;

(iv) $P_i P_j^T = P_j P_i^T$ for $i \neq j$.

Repeat designs are defined and used extensively in Robinson and Seberry (to appear) to study orthogonal designs. For convenience we will call a repeat design $(P; (R; S); H)$ of type $(1; (r; s); h)$ in order n a *triplet* when $P = I$. Alternatively, a triplet is three weighing matrices (R, S, H) of order n and weights (r, s, h) respectively which are pairwise amicable; R, S are skew symmetric and H is symmetric. We note that

LEMMA 1. *There are triplets in orders $n = 2$ and 4 for weights (i, j, k) where $i, j = 1, 2, ..., n-1$ and $k = 1, 2, ..., n$. Hence, there is a best pair family of orders 2 and 4.*

PROOF. For order 2 consider the pairs (K, M) and (K, H). The required matrices for order 4 are (the weights are given in brackets):

1. $(1, 2, 1)$ $\{K \times I, K \times H; L \times I\}$,
2. $(1, 2, 2)$ $\{K \times I, K \times H; L \times H\}$,
3. $(1, 2, 3)$ $\{K \times I, K \times H; L \times H + M \times I\}$,
4. $(1, 2, 4)$ $\{K \times I, K \times H; H \times H\}$,
5. $(1, 3, 1)$ $\{K \times L, K \times I + H \times K; M \oplus -L\}$,
6. $(1, 3, 2)$ $\{K \times L, L \times I + H \times K; H \times L\}$,
7. $(1, 3, 3)$ $\{K \times L, K \times I + H \times K; M \oplus K + I \times M + K \times K\}$,
8. $(1, 3, 4)$ $\{K \times L, L \times I + H \times K; L \times I + M + L + I \times M + K \times K\}$,
9. $(2, 3, 1)$ $\{I \times L + L \times M, L \times I + H \times L; L \oplus -M\}$,
10. $(2, 3, 2)$ $\{K + H, L \times K + M \times K + K \times I; M \times I - L \times I\}$,
11. $(2, 3, 3)$ $\{I \times L + L \times M, L \times I + H \times L; M \oplus L + I \times M + L \times L\}$,
12. $(2, 3, 4)$ $\{I \times K + K \times M, K \times I + H \times K; L \times I + M \times L + I \times M + K \times K\}$.

Because of the extremely powerful constructions that arise from repeat designs we wished to extend this lemma to higher powers of two. This effort led to the results that follow. First, we give two useful lemmas and note a useful lemma of one of us (Robinson).

LEMMA 2. *Suppose that X and Y are amicable orthogonal designs in order n, where X is of type $(1, u_1, ..., u_s)$ on the variables $x_0, ..., x_s$ and Y is of type $(v_1, ..., v_t)$ on the variables $y_1, ..., y_t$ then there exist monomial matrices P and Q (that is, with elements $0, \pm 1$ and only one non-zero element per row and column) so that*

$$PXQ = x_0 I + \sum x_i M_i, \quad PYQ = \sum y_i N_i,$$

where the M_i's and N_i's satisfy

(0) $M_i * M_j = 0, i \neq j, \ N_l * N_k = 0, l \neq k$;

(i) $M_i^T = -M_i \ \forall i, \ N_j^T = N_j \ \forall j$;

(ii) $M_i M_j^T = u_i I_n, \ N_j N_j^T = v_j I_n$;

(iii) $M_i N_j^T = N_j M_i^T$;

(iv) $M_i M_j^T + M_j M_i^T = 0, i \neq j, \ N_l N_k^T + N_k N_l^T = 0, l \neq k$.

PROOF. Choose P and Q so that the variable x_0 occurs on the diagonal of PXQ and the rest follows immediately.

CONSTRUCTION 3. *If A, B, C is a triplet of weights $(a, b, 1)$ in order n, then (A, B, AC) and $(A, B, AC+C)$ are triplets of weights (a, b, a) and $(a, b, a+1)$.*

Since by a lemma of P. J. Robinson (1976a) there is no best pair of weights $(7, 5)$ in order 8 we have, regarding a best pair (A, B) as a triplet (X, A, B):

COROLLARY 4. *There are no triplets of weight $(x, 7, 5)$, $(4, 7, 1)$ or $(5, 7, 1)$ in order 8.*

In fact in order 8 we can say

LEMMA 5. *In order 8 all triplets (R, S, H) of weights (r, s, h), $0 \leqslant r, s \leqslant 7, 0 \leqslant k \leqslant 8$ exist except*

(i) $(3, 7, 1)$, $(4, 7, 1)$, $(5, 7, 1)$, $(6, 7, 1)$ *and* $(r, 7, 5)$ $1 \leqslant r \leqslant 6$ *which do not exist; and possibly*

(ii) $(5, 7, 2)$, $(6, 7, 2)$, $(6, 7, 4)$, $(3, 7, 3)$, $(4, 7, 3)$, $(5, 6, 3)$, $(5, 7, 3)$, $(6, 7, 3)$, $(1, 5, 7)$, $(3, 5, 7)$, $(3, 7, 7)$, $(4, 7, 7)$, $(5, 7, 7)$, $(6, 7, 7)$ *which are undecided.*

PROOF. Part (i) follows from the previous corollary and Theorem 15 which shows that $(3, 7, 1)$ and $(6, 7, 1)$ do not exist. Lemma 12(i), (iii), (iv), (v) of Robinson and Seberry (to appear) and the above construction give all those that exist except $(3, 5, 1)$, $(3, 5, 5)$, $(1, 5, 1)$, $(1, 7, 1)$, $(2, 7, 1)$ and $(1, 7, 3)$. Now we note that a repeat design of type $(r; (p_1, \ldots; q); w_1, \ldots)$ in order n gives a repeat design of type $(r; (q; p_1, \ldots, q); w_1, \ldots)$ in order $2n$ by Lemma 12 of Robinson and Seberry (to appear). Hence the repeat design of type $(1; (2; 3); 1)$ in order 4 gives the $(1; (3; 5); 1)$ in order 8 and hence the triplet $(3, 5, 1)$ and by Construction 3 the $(3, 5, 5)$ in 8. We now give specific constructions for $(1, 5, 1)$, $(1, 7, 1)$, $(2, 7, 1)$ and $(1, 7, 3)$.

Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad R^2 = I,$$

and K, L, M be as in Section 2. Write $E = -S + S^2 + S^3$ and $G = S + S^2 + S^3$. Then

$$(L \times S^2, I_2 \times (S + S^3) + K \times ER, L \times S^2)$$

and

$$(L \times S^2, L \times G + K \times (E+I)^T R, (S^3 + S - I)^T R \oplus (-S^2 - S + I)^T R)$$

have weights $(1, 5, 1)$ and $(1, 7, 3)$ respectively. Amnon Neeman found the following $(1, 7, 1)$:

$$
\left[
\begin{array}{cccc|cccc}
0 & 1 &   &   &   &   &   &   \\
- & 0 &   &   &   &   &   &   \\
  &   & 0 & 1 &   &   &   &   \\
  &   & - & 0 &   &   &   &   \\ \hline
  &   &   &   & 0 & 1 &   &   \\
  &   &   &   & - & 0 &   &   \\
  &   &   &   &   &   & 0 & 1 \\
  &   &   &   &   &   & - & 0 \\
\end{array}
\right],
\left[
\begin{array}{cccc|cccc}
0 & 1 & 1 & - & - & 1 & 1 & 1 \\
- & 0 & 1 & 1 & - & - & - & 1 \\
- & - & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & - & - & 0 & - & 1 & - & 1 \\ \hline
1 & 1 & - & 1 & 0 & - & 1 & 1 \\
- & 1 & - & - & 1 & 0 & - & 1 \\
- & 1 & - & 1 & - & 1 & 0 & - \\
- & - & - & - & - & - & 1 & 0 \\
\end{array}
\right],
\left[
\begin{array}{cccc|cccc}
0 & 0 & 0 & 1 &   &   &   &   \\
0 & 0 & 1 & 0 &   &   &   &   \\
0 & 1 & 0 & 0 &   &   &   &   \\
1 & 0 & 0 & 0 &   &   &   &   \\ \hline
  &   &   &   & 0 & 1 & 0 & 0 \\
  &   &   &   & 1 & 0 & 0 & 0 \\
  &   &   &   & 0 & 0 & 1 & 0 \\
  &   &   &   & 0 & 0 & 0 & - \\
\end{array}
\right].
$$

The following three matrices give a $(2, 7, 1)$:

$$
\left[
\begin{array}{cccc|cccc}
0 & 1 & 1 & 0 &   &   &   &   \\
- & 0 & 0 & - &   &   &   &   \\
- & 0 & 0 & 1 &   &   &   &   \\
0 & 1 & - & 0 &   &   &   &   \\ \hline
  &   &   &   & 0 & 0 & 1 & 1 \\
  &   &   &   & 0 & 0 & 1 & - \\
  &   &   &   & - & - & 0 & 0 \\
  &   &   &   & - & 1 & 0 & 0 \\
\end{array}
\right],
\left[
\begin{array}{cccc|cccc}
0 & 1 & 1 & - & 1 & - & - & 1 \\
- & 0 & 1 & - & - & 1 & 1 & 1 \\
- & - & 0 & 1 & 1 & 1 & - & 1 \\
1 & 1 & - & 0 & 1 & 1 & 1 & 1 \\ \hline
- & 1 & - & - & 0 & 1 & - & - \\
1 & - & - & - & - & 0 & - & 1 \\
1 & - & 1 & - & 1 & 1 & 0 & - \\
- & - & - & - & 1 & - & 1 & 0 \\
\end{array}
\right],
\left[
\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 &   &   &   &   \\
1 & 0 & 0 & 0 &   &   &   &   \\
0 & 0 & 0 & 1 &   &   &   &   \\
0 & 0 & 1 & 0 &   &   &   &   \\ \hline
  &   &   &   & 0 & - & 0 & 0 \\
  &   &   &   & - & 0 & 0 & 0 \\
  &   &   &   & 0 & 0 & 1 & 0 \\
  &   &   &   & 0 & 0 & 0 & - \\
\end{array}
\right].
$$

This gives the results of the enunciation after using Construction 3 to obtain $(1, 5, 5)$, $(1, 7, 7)$ and $(2, 7, 7)$.

REMARK. This lemma indicates that the existence problem for triplets (R, S, H) which are repeat designs $(I; (R; S); H)$ is very difficult and far from resolved.
But this lemma does allow us to say:

COROLLARY 6. *There are best pairs for all* (a, b), $0 \leqslant a \leqslant 7$, $0 \leqslant b \leqslant 8$ *in order* 8 *except* $(7, 5)$. *There are amicable weighing matrices for all* (a, b), $0 \leqslant a$, $b \leqslant 8$ *in order* 8.

The results on repeat designs in Robinson and Seberry (to appear) allow us to say:

THEOREM 7. *There are amicable weighing matrices for all* (a, b), $0 \leqslant a$, $b \leqslant 2^t$.

3. The structure of amicable weighing matrices

A combinatorial argument lets us obtain the following result.

THEOREM 8. *Let R be a monomial matrix of order $n \equiv 0 \pmod{4}$. Let A be a symmetric weighing matrix of order n. Suppose that $RA^T = -AR^T$. Then $A * R = 0$. Further, if A has weight $n-1$ then R is symmetric if A has any diagonal element zero and R is skew-symmetric otherwise.*

THEOREM 9. *The existence of amicable orthogonal designs of order $n \equiv 0 \pmod{4}$ and types $((1, n-1); (1, n-1))$ is equivalent to the existence of a symmetric weighing matrix of order n and weight $n-1$ with at least one zero on the diagonal.*

PROOF. Let A be the symmetric weighing matrix. Use the theorem of Delsarte et al. (1971) to see that we can find monomial matrices P and Q so that $B = PAQ$ is skew-symmetric. Let $R = P^{-1}Q^{-1}$. Then R is a monomial matrix and $B^T = (PAQ)^T = Q^T AP^T = Q^{-1}AP^{-1} = -PAQ$ so $RA^T = -AR^T$. Hence by the previous theorem $A * R = 0$ and R is symmetric. So $uR + vA$ and $xI + yAR$ are the required amicable orthogonal designs.

Now if $xU + yV$ and $uN + vM$ are amicable orthogonal designs of order n and types $((1, n-1); (1, n-1))$. We pre and postmultiply both matrices by monomial matrices P and Q where $I = PUQ$. Then the amicable matrices can be written in the form $xI + yPVQ = xI + yS$ and $uPNQ + vPMQ = uR + vA$. Now the amicability and orthogonality gives us R and A are symmetric and $AR^T = -RA^T$. We now assume A has weight $n-1$ and no zero on the diagonal. Then considering the orthogonality conditions on the rows of $A + R$ leads to a contradiction and we have the result.

REMARK. We note that this proof also shows that the existence of a symmetric weighing matrix C of order n and weight $n-1$ with a zero on the diagonal is equivalent to the existence of a best pair of weights $(n-1, 1)$ in order n.

We recall (from Wallis, 1975) that amicable orthogonal designs of types $((1, p); (1, p))$ do exist in orders $p+1$ when $p \equiv 3 \pmod{4}$ is a prime power. Hence we have

COROLLARY 10. *There is a symmetric weighing matrix with a zero on the diagonal of order $p+1$ and weight p where $p \equiv 3 \pmod{4}$ is a prime power.*

We now observe that symmetric weighing matrices of order n and weight $n-1$ with a zero on the diagonal do not always exist. We first observe:

LEMMA 11. *If (A, B) is a $(n-1, 1)$ best pair in order $n \equiv 0 \pmod{4}$ then we may assume*

$$B = X \oplus \begin{bmatrix} \oplus & Y \\ \frac{1}{2}n-1 & \end{bmatrix}$$

where

$$X = \begin{bmatrix} - & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

PROOF. Firstly, we note that if (A, B) is a best pair and P is any permutation matrix then (PAP^T, PBP^T) is a best pair of the same type.

It is obvious that we can find a P such that

$$PBP^T = \bigoplus_{1 \leqslant i \leqslant \frac{1}{2}n} X_i$$

where $X_i = X$ for $1 \leqslant i \leqslant k$ and $X_i = Y$ for $k+1 \leqslant i \leqslant \frac{1}{2}n$.

Since the top left-hand corner of A is $\pm \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}$ AB could not be symmetric if $X_1 = \pm I_2$. So X_i must be X or Y for all i.

Assume now that $X_i = Y$ for $1 \leqslant i \leqslant \frac{1}{2}n$ and

$$A = \begin{bmatrix} 0 & 1 & & & \\ - & 0 & A_1 & A_2 & \cdots \\ & \tilde{A}_1^T & & & \\ & \tilde{A}_2^T & & Z & \\ & \vdots & & & \end{bmatrix}$$

where A_i are (2×2) matrices with entries ± 1.

Let

$$A_1 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Therefore

$$A_1 Y = \begin{bmatrix} x_2 & x_1 \\ x_4 & x_3 \end{bmatrix} \quad \text{and} \quad \tilde{A}_1^T Y = \begin{bmatrix} \bar{x}_3 & \bar{x}_1 \\ \bar{x}_4 & \bar{x}_2 \end{bmatrix}.$$

Because AB is symmetric, we must have $x_2 = -x_3$ and $x_1 = -x_4$, that is,

$$A_1 = \pm \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} \quad \text{or} \quad \pm \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix}.$$

This reasoning is also true for the other A_i's.

Now $\sum_{i=1}^{\frac{1}{2}n-1} A_i A_i^T = (n-2) I_n$ by the orthogonality of A, and

$$A_i A_i^T = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

But since there is an odd number of A_i's

$$\sum A_i A_i^T \neq (n-2).$$

Therefore at least one of the X_i's is X.

We now assume $X_1 = \pm X_2 = X$. Then we see the product AB is not symmetric. Therefore

$$X_1 = X \quad \text{and} \quad X_i = Y \quad \text{for } 2 \leqslant i \leqslant \tfrac{1}{2}n.$$

Hence we may assume B is of the form given in (1).

We use this lemma to show that symmetric weighing matrices of order n and weight $n-1$ with a zero on the diagonal do not always exist since

THEOREM 12. *There is no best pair of weights* $(15, 1)$ *in order* 16. *Equivalently there is no symmetric weighing matrix of order* 16 *and weight* 15 *with a zero on the diagonal.*

PROOF. The proof is long and combinatorial. It is given in detail in Robinson (1977b).

4. Some non-existence results

Lemma 11 can be used to produce a result similar to a theorem of Wolfe (1976a):

THEOREM 13. *There are no amicable orthogonal designs of order* $n \equiv 0$ (mod 8) *and types* $((1); (1, a, n-a-1))$, $a = 2, 3, 4$ *or* 5.

PROOF. We assume that the matrix of weight 1 is of the form given by the lemma. A long, careful combinatorial argument gives the result. A fuller proof is given in Robinson (1977b).

COROLLARY 14. *There are no amicable orthogonal designs of order* $n \equiv 0$ (mod 4) *and types* $((1); (1, a, b, c))$, $a+b+c = n-1$, $a, b, c \neq 0$, *and* abc *odd*.

PROOF. In the proof of the theorem we can see that each variable appears an even number of times off the diagonal 2×2 block and therefore one of a, b and c is odd.

We are grateful to Amnon Neeman for the proof of another result using Lemma 11:

THEOREM 15. *Let $n \equiv 0$ (mod 8). Then it is not possible to have triplets of weights $(a, n-1, 1)$ in order n where $a = n-4$, $n-3$, $n-2$, nor of weights $(3, 7, 1)$ in order 8.*

PROOF. Lemma 11 allows us to consider the triplet (X, Y, Z) where

$$Z = \left[\begin{array}{c} \oplus \\ \tfrac{1}{2}n-1 \end{array} \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right] \oplus \left[\begin{array}{cc} - & 0 \\ 0 & 1 \end{array} \right].$$

It is now possible, since $XZ^T = ZX^T$ and $X^T = -X$, to decide the structure of the 2×2 submatrices of X. A careful analysis of the necessary structure for X in each case of the enunciation shows it is not possible.

5. Some existence results

We take this opportunity to obtain some amicable orthogonal designs and a few repeat designs.

Consider:

$$\begin{aligned}
\mathbf{X}_1 &= \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_1 & -x_2 \\ x_3 & -x_3 & x_2 & -x_1 \end{array} \right], & \mathbf{Y}_1 &= \left[\begin{array}{cccc} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_2 & y_1 \\ -y_3 & y_3 & y_1 & -y_2 \end{array} \right], \\
&= x_1 X_1 + x_2 X_2 + x_3 X_3 & &= y_1 Y_1 + y_2 Y_2 + y_3 Y_3 \\
\mathbf{X}_2 &= \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_3 \\ -x_2 & x_1 & x_3 & -x_3 \\ -x_3 & -x_3 & x_1 & -x_2 \\ -x_3 & x_3 & -x_2 & x_1 \end{array} \right], & \mathbf{Y}_2 &= \left[\begin{array}{cccc} y_1 & y_2 & y_3 & y_3 \\ y_2 & -y_1 & y_3 & -y_3 \\ y_3 & y_3 & -y_2 & -y_1 \\ y_3 & -y_3 & -y_1 & y_2 \end{array} \right], \\
&= x_1 U_1 + x_2 U_2 + x_3 U_3 & &= y_1 V_1 + y_2 V_2 + y_3 V_3
\end{aligned} \right\} \quad (2)$$

where $X_1, \ldots, V_3$ are the obvious $(0, 1)$-matrices.

Then we have:

LEMMA 16. *Suppose that there is a set of pairwise amicable weighing matrices (or orthogonal designs) $\{M_1, \ldots, M_s; N_1, \ldots, N_t\}$ of order m and weights $(m_1, \ldots, m_s; n_1, \ldots, n_t)$ where*

$$M_i^T = -M_i \ \forall i \quad \text{and} \quad N_j^T = N_j \ \forall j.$$

Then there are amicable orthogonal designs of order $4m$ and types
  (i) $(1, m_a, m_b, 2m_c)$ *and* $(n_d, m_e, 2m_f)$;
  (ii) $(1, m_a, n_b, 2n_c)$ *and* $(n_d, n_e, 2n_f)$;
  (iii) $(1, m_a, n_b, 2m_c)$ *and* $(n_d, n_e, 2m_f)$;
  (iv) $(1, m_a, m_b, 2n_c)$ *and* $(n_d, m_e, 2n_f)$;
where $m_i \in \{m_1, \ldots, m_s\}$, $i \in \{1, \ldots, s\}$ and $n_j \in \{n_1, \ldots, n_t\}$, $j \in \{1, \ldots, t\}$.

PROOF. Use the matrices defined by (2) above. The required designs are

(i)
$$u_1 I \times I + u_2 V_1 \times M_a + u_3 V_2 \times M_b + u_4 V_3 \times M_c$$
and
$$v_1 U_1 \times N_d + v_2 U_2 \times M_e + v_3 U_3 \times M_f;$$

(ii)
$$u_1 I \times I + u_2 U_1 \times M_a + u_3 U_2 \times N_b + u_4 U_3 \times N_c$$
and
$$v_1 V_1 \times N_d + v_2 V_2 \times N_e + v_3 V_3 \times N_f;$$

(iii)
$$u_1 I \times I + u_2 X_1 \times M_a + u_3 X_2 \times N_b + u_4 X_3 \times M_c$$
and
$$v_1 Y_1 \times N_d + v_2 Y_2 \times N_e + v_3 Y_3 \times M_f;$$

(iv)
$$u_1 I \times I + u_2 Y_1 \times M_a + u_3 Y_2 \times M_b + u_4 Y_3 \times N_c$$
and
$$v_1 X_1 \times N_d + v_2 X_2 \times M_e + v_3 X_3 \times N_f;$$

where $u_1, u_2, u_3, u_4, v_1, v_2, v_3$ are commuting variables.

These give with the results quoted above and Lemma 12 of Robinson and Seberry (to appear):

LEMMA 17. *In order* 16 *there exist best pairs* (a, b) $(= repeat \ designs$ $(1; (a); b))$ *for all* $a = 1, 2, ..., 15$ *and* $b = 1, 2, ..., 16$ *except possibly the pairs* (a, b): $(13, 1)$, $(13, 5)$, $(13, 9)$, $(15, 7)$, $(15, 9)$, $(15, 15)$ *which are undecided and* $(15, 1)$ *which does not exist*.

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