

ON A PROBLEM BY R. A. HIRSCHFELD:
COUNTEREXAMPLE

M. Edelstein

The following question was asked by R. A. Hirschfeld in [1]: "E and F are Banach spaces, F reflexive, D is a subset of E and $T: D \rightarrow F$ a nonlinear contraction, i. e.,

$$\|Tx_1 - Tx_2\|_F \leq \|x_1 - x_2\|_E \text{ whenever } x_1, x_2 \in D.$$

Can T be extended to a contraction $\tilde{T}: E \rightarrow F$ (for $E=F=$ Hilbert space the answer is yes)."

Counterexample.

Let E be the space of all ordered pairs $x = (x_1, x_2)$ of real numbers with $\|x\| = \max(|x_1|, |x_2|)$ and let $D = \{(-1, 1), (1, 1), (1, -1)\}$. Let T map D onto the vertices of an equilateral triangle Δ of side-length = 2 in the Euclidean plane F.

It is clear that no extension \tilde{T} of the desired type can exist. For $\tilde{T}((0, 0))$ must lie in each disc of radius one centered at the vertices of Δ and the intersection of these discs is empty.

REFERENCE

1. R. A. Hirschfeld: Extension of nonlinear contractions. Research Problem 5, Bull. Amer. Math. Soc. 71, (1965), p. 495.

Dalhousie University