EXTREMAL PERIMETERS OF QUADRANGLES IN THE PONCELET PORISM

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Abstract

Extremal problems for quadrangles circuminscribed in a circular annulus with the Poncelet porism property are considered. Quadrangles with the maximal and the minimal perimeters are determined. Two conjectures end the paper.

1. Introduction

In this paper we will consider circular annuli, that is, annuli bounded by two circles. A polygon is said to be described in an annulus if it is circumscribed on the inner circle and inscribed in the outer circle. We recall the Poncelet closure theorem in a version used in this paper. Let us consider two circles C_1 and C_2 and let the circle C_2 lie inside the circle C_1. From any point M_1 on C_1, draw a tangent to C_2 and extend it to C_1 in the opposite direction. From this point we draw another tangent, and so on. Thus, we construct a Poncelet transverse $(M_1, t_1, M_2, t_2, M_2, t_3, \ldots)$ such that $M_i \in C_1$ and t_i is tangent to the circle C_2 with $M_{i+1} = t_i \cap t_{i+1}$. We say that the Poncelet transverse closes after n steps if $M_{n+1} = M_1$. With this notation, we have the theorem as follows.

PONCELET CLOSURE THEOREM (See, for example, [2, 6]). *If a transverse starting at a point M_1 of the circle C_1 closes after n steps, then any Poncelet transverse starting at any point of the circle C_1 closes after n steps.*

Sometimes this theorem is called Poncelet's porism. According to WolframMath World [1], the word porism is an archaic term with two meanings: 'a *corollary* and "a *proposition*" affirming the possibility of finding such conditions as will render a certain problem indeterminate, or capable of innumerable solutions' (Playfair, 1792). Unfortunately, this definition is slightly inaccurate, because the proposition actually states the conditions, rather than affirming the possibility of finding them'. In our
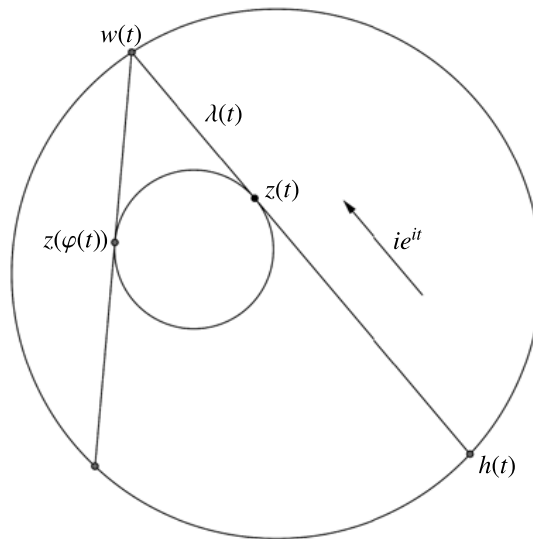
FIGURE 1. Definition of $\lambda(t)$, $h(t)$, $w(t)$, $\varphi(t)$.

investigations we will say that *porism* means that *if a certain assertion holds for one point then it holds for any point of a considered set.*

Radić [7] solved the extremal problem connected with areas of triangles determined by Poncelet's closure theorem in a circular annulus. He expressed the area of the triangle in terms of the lengths of parts of Poncelet transversals and then he determined the extremal triangle. In this paper we solve the extremal problem connected with perimeters of quadrangles determined by the Poncelet closure theorem, using the differential equation given in [3]. The essential difficulty in the approach used by Radić and our approach in this paper is the necessity to use *n*-porism formulas in explicit form (see [2, 8]). Two conjectures based on Radić's and our results are given at the end of this paper.

2. Geometric interpretation of the function *b*

We consider a circular annulus bounded by two circles, namely $C_r : x^2 + y^2 = r^2$ and $C_{a,R} : (x - a)^2 + y^2 = R^2$. We parametrise C_r and $C_{a,R}$ by means of $z(t) = re^{it}$ and $w(t) = z(t) + \lambda(t)ie^{it}$, respectively, where

$$\lambda(t) = \sqrt{R^2 - (r - a\cos t)^2} - a\sin t; \tag{2.1}$$

$\lambda(t)$ is indicated in Figure 1.

Through a point $z(t) \in C_r$ we draw the tangent line. This line intersects the circle $C_{a,R}$ at two points; one of them is $w(t)$, the second point will be denoted by $h(t)$. Next, from $w(t)$ we draw the tangent line to C_r and it is tangent to C_r at a point denoted by $z(\varphi(t))$. It was shown in [3] that the function φ satisfies the following differential
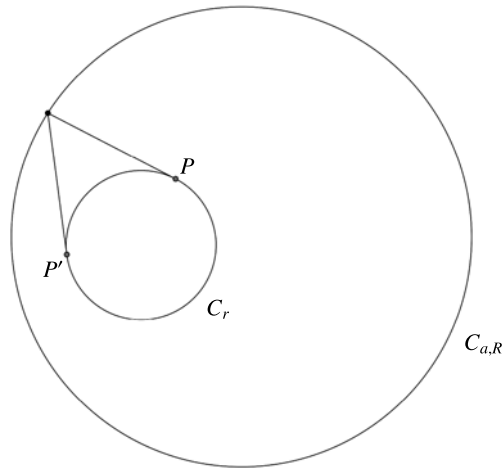
equation

$$\varphi' = \frac{b}{b \circ \varphi}, \tag{2.2}$$

where

$$b(t) = \left(1 - \left(\frac{r - a\cos t}{R}\right)^2\right)^{-1/2}. \tag{2.3}$$

It is easy to see that the length of a segment joining the points $w(t)$ and $h(t)$ is given by the formula

$$|w(t)h(t)| = 2R\sqrt{1 - \sigma^2(t)},$$

where

$$\sigma(t) = \frac{1}{R}(r - a\cos t). \tag{2.4}$$

Hence,

$$b(t) = \frac{2R}{|w(t)h(t)|}. \tag{2.5}$$

Note that, from (2.3) and (2.4),

$$b' = \frac{a}{R}b^3\sigma\sin t. \tag{2.6}$$

Moreover, it follows from the formula (2.2) that the function b determines an invariant measure on the circle C_r with respect to a mapping $f : C_r \to C_r$ defined by the formula $f(P) = P'$, given in Figure 2.

An invariant measure connected with the Poncelet porism was considered by Kołodziej [5] and King [4].
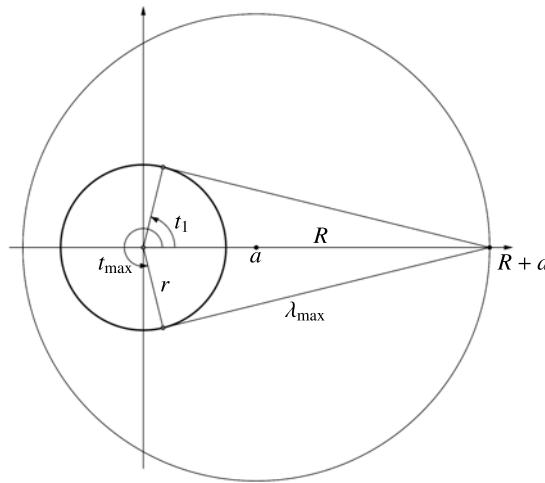
FIGURE 3. Illustration of $t_{\max}$ and $\lambda_{\max}$.

3. Properties of the function λ

We find the extrema of the function λ given by (2.1). The equation $\lambda'(t) = 0$ can be written in the form

$$\left(\frac{r - a\cos t}{R} - \cos t\right)\left(\frac{r - a\cos t}{R} + \cos t\right) = 0.$$

Hence, the two extreme points are given by

$$\cos t_1 = \frac{r}{R + a} \tag{3.1}$$

and

$$\cos t_2 = \frac{-r}{R - a}.$$

Let

$$t_{\max} = 2\pi - t_1.$$

These relations will turn out to be very important in the further parts of the paper.

Radić [7] claimed that the longest segment $\lambda_{\max}$ joining $z(t)$ and $w(t)$ is given by the formula

$$\lambda_{\max} = \sqrt{(R + a)^2 - r^2}.$$

Indeed, we have $\lambda_{\max} = \lambda(t_{\max}) = \lambda(2\pi - t_1) = \sqrt{(R + a)^2 - r^2}$ (see Figure 3).

LEMMA 3.1. *With the above notation,*

$$\varphi'(t_{\max}) = 1. \tag{3.2}$$

PROOF. It is easy to see that

$$\sigma(t_{\max}) = \frac{1}{R}(r - a\cos t_{\max}) = \frac{1}{R}(r - a\cos t_1) = \frac{r}{R+a} = \cos t_1 = \cos t_{\max},$$

that is,

$$\sigma(t_{\max}) = \cos t_{\max} \tag{3.3}$$

and

$$\sigma(t_1) = \cos t_1. \tag{3.4}$$

The formulas (2.3), (2.4) and (3.4) imply that

$$b(t_{\max}) = (1 - \sigma^2(t_{\max}))^{-1/2} = \frac{1}{\sin t_1}. \tag{3.5}$$

Since we have $\varphi(t_{\max}) = 2\pi + t_1$, (3.5) implies that

$$b(\varphi(t_{\max})) = (1 - \sigma^2(\varphi(t_{\max})))^{-1/2} = (1 - \sigma^2(t_1))^{1/2} = \frac{1}{\sin t_1},$$

that is,

$$b(\varphi(t_{\max})) = b(t_{\max}).$$

Now, from the differential equation (2.2) we get (3.2).   □

Note also that

$$w \circ \varphi = w + (\lambda + \lambda \circ \varphi)ie^{i\varphi},$$

illustrated in Figure 4.

We have

$$\begin{aligned}
0 = |w \circ \varphi - a|^2 - R^2 &= \langle w - a + (\lambda + \lambda \circ \varphi)ie^{i\varphi}, w - a + (\lambda + \lambda \circ \varphi)ie^{i\varphi}\rangle - R^2 \\
&= 2(\lambda + \lambda \circ \varphi)\langle w - a, ie^{i\varphi}\rangle + (\lambda + \lambda \circ \varphi)^2,
\end{aligned}$$

that is,

$$\lambda + \lambda \circ \varphi + 2\langle w - a, ie^{i\varphi}\rangle = 0, \tag{3.6}$$

where $\langle\,,\rangle$ denotes the Euclidean scalar product. Since $w = re^{it} + \lambda ie^{it}$, we have immediately $\langle w - a, ie^{i\varphi}\rangle = a\sin\varphi + \lambda\cos(\varphi - t) - r\sin(\varphi - t)$. Now, from (3.6),

$$\lambda \circ \varphi - \lambda + 2[(1 + \cos(\varphi - t))\lambda - r\sin(\varphi - t)] + 2a\sin\varphi = 0.$$

It is easy to verify from Figure 4 that the term in the square brackets is equal to zero, whence

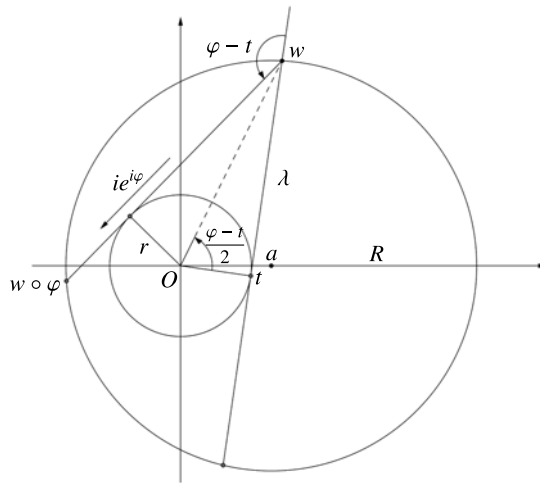$$\lambda \circ \varphi = \lambda - 2a\sin\varphi. \tag{3.7}$$

FIGURE 4. Geometric meaning of $w \circ \varphi$.

4. Critical points of the perimeter L

In this section we consider a circular annulus $C_r C_{a,R}$ having the Poncelet porism for quadrangles. We take a quadrangle $ABCD$ with an edge containing the segment $\lambda_{\max}$ and let

$$\begin{cases} \varphi^{[0]}(t) = t, \\ \varphi^{[1]} = \varphi, \\ \varphi^{[n+1]} = \varphi^{[n]} \circ \varphi \end{cases}$$

for $n = 1, 2, 3, \ldots$ and $\lambda_k = \lambda(\varphi^{[k]}(t_1))$ for $k = 1, 2, 3$, as in Figure 5. With the above notation, note that

$$\begin{cases} \varphi(t_1) = t_1 + \dfrac{\pi}{2}, \\ \varphi(t_{\max}) = 2\pi + t_1, \\ \varphi^{[2]}(t_{\max}) = 2\pi + \dfrac{\pi}{2} + t_1, \\ \varphi^{[3]}(t_{\max}) = 2\pi + \dfrac{3\pi}{2} - t_1. \end{cases} \tag{4.1}$$

We denote by $L(t)$ the perimeter of a circuminscribed quadrangle in $C_r C_{a,R}$ passing through a point $z(t)$. The perimeter L is given by the formula

$$\tfrac{1}{2}L = \lambda + \lambda \circ \varphi + \lambda \circ \varphi^{[2]} + \lambda \circ \varphi^{[3]}.$$

Applying (3.7),

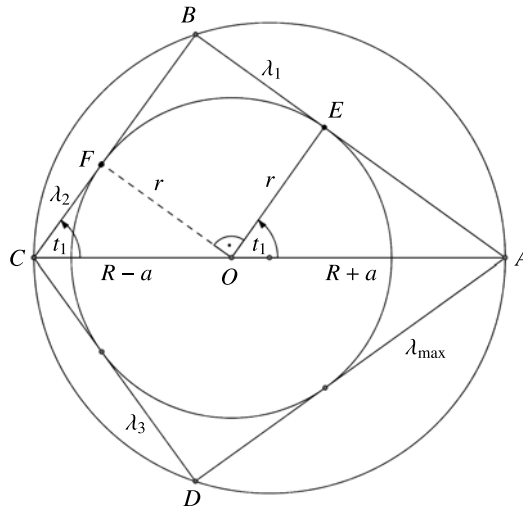$$L = 8\lambda - 4a(3\sin\varphi + 2\sin\varphi^{[2]} + \sin\varphi^{[3]}). \tag{4.2}$$

FIGURE 5. Illustration of the segments $\lambda_1, \lambda_2, \lambda_3$.

Differentiating (4.2),

$$L' = 8\lambda' - 4aK\varphi',$$

where

$$K = 3\cos\varphi + 2\varphi' \circ \varphi \cdot \cos\varphi^{[2]} + \varphi' \circ \varphi^{[2]} \cdot \varphi' \circ \varphi\cos\varphi^{[3]}.$$

The aim of this section is to prove that $t_{\max}$ and 0 are the critical points of L'.

Case 1. $t_{\max}$ *is a critical point of* L'.

It is sufficient to prove that $K(t_{\max}) = 0$. From (4.1), (2.2) and (2.5),

$$
\begin{aligned}
K(t_{\max}) &= 3\cos t_1 + 2\frac{b(t_1)}{b((\pi/2) + t_1)}\cos\left(\frac{\pi}{2} + t_1\right) + \frac{b(t_1)}{b((3\pi/2) - t_1)}\cos\left(\frac{3\pi}{2} - t_1\right) \\
&= 3\cos t_1 - 2\frac{2R}{|AB|}\frac{|BC|}{2R}\sin t_1 - \frac{2R}{|AB|}\frac{|CD|}{2R}\sin t_1 \\
&= 3\cos t_1 - 3\frac{|BC|}{|AB|}\sin t_1 = 3\cos t_1 - 3\cot t_1\sin t_1 = 0.
\end{aligned}
$$

Case 2. 0 *is a critical point of* L'.

We consider a quadrangle determined by $t = 0$; see Figure 6.

We have $\lambda_0 = \lambda_3$ and $\lambda_1 = \lambda_2$, where $\lambda_m = \lambda(\varphi^{[m]}(0))$ for $m = 0, 1, 2, 3$. Consider the trapezium $ABCD$ shown in Figure 7.

With the above notation,

$$\lambda_0 = \sqrt{R^2 - (r-a)^2}, \quad \lambda_1 = \sqrt{R^2 - (r+a)^2} \tag{4.3}$$

FIGURE 6. Illustration of the angle τ.

FIGURE 7. Notation used in the proof of Case 2.

and $(\lambda_0 + \lambda_1)^2 = (\lambda_0 - \lambda_1)^2 + 4r^2$. Hence,

$$\lambda_0 \lambda_1 = r^2. \tag{4.4}$$

Using (4.4),

$$\lambda_1^2 |BC|^2 - (r^2 + \lambda_1^2)^2 = \lambda_1^2(\lambda_0 + \lambda_1)^2 - (r^2 + \lambda_1^2)^2$$
$$= (\lambda_0 \lambda_1)^2 + 2\lambda_0 \lambda_1^3 + \lambda_1^4 - r^4 - 2r^2 \lambda_1^2 - \lambda_1^4 = 2\lambda_1^2(\lambda_0 \lambda_1 - r^2) = 0,$$

that is,

$$\lambda_1 |BC| = r^2 + \lambda_1^2. \tag{4.5}$$

Now, we prove that

$$K(0)\varphi'(0) = -2.$$

Since $\varphi(0) = \pi - 2\omega, \varphi^{[2]}(0) = \pi, \varphi^{[3]}(0) = \pi + 2\omega$ and $\tan \omega = \lambda_1/r$, using (2.5), (4.4)

and (4.5),

$$K(0)\varphi'(0) = \left(-3\cos 2\omega - 2\frac{b(\pi - 2\omega)}{b(\pi)} - \cos 2\omega\right)\frac{b(0)}{b(\varphi(0))}$$

$$= -4\left(\frac{r^2 - \lambda_1^2}{r^2 + \lambda_1^2} + \frac{\lambda_1}{|BC|}\right)\frac{|BC|}{2\lambda_0} = -2\frac{r^2}{\lambda_0\lambda_1} = -2.$$

Finally,

$$L'(0) = 8\lambda'(0) - 4aK(0)\varphi'(0) = 0.$$

5. Extrema of the perimeter L

In this section we prove that the perimeter L reaches its local maximum at $t_{\max}$ and its local minimum at 0. We recall that for quadrangles the 4-porism formula has the following form:

$$(R^2 - a^2)^2 = 2r^2(R^2 + a^2) \tag{5.1}$$

(see [2, 8]). This formula can be rewritten in the equivalent form

$$R^2 = a^2 + r^2 + r\sqrt{r^2 + 4a^2}. \tag{5.2}$$

We find the sign of L''. From (2.5),

$$L = \frac{2R}{b} + \frac{2R}{b \circ \varphi} + \frac{2R}{b \circ \varphi^{[2]}} + \frac{2R}{b \circ \varphi^{[3]}}.$$

Differentiating the function L and using (2.6) and (2.2),

$$L' = 2R\left(\frac{-a}{R}b\sigma\sin t - \frac{a}{R}b \circ \varphi \cdot \sigma \circ \varphi \cdot \frac{b}{b \circ \varphi}\sin\varphi\right.$$

$$- \frac{a}{R}b \circ \varphi^{[2]} \cdot \sigma \circ \varphi^{[2]} \cdot \frac{b \circ \varphi}{b \circ \varphi^{[2]}} \cdot \frac{b}{b \circ \varphi}\sin\varphi^{[2]}$$

$$\left.- \frac{a}{R}b \circ \varphi^{[3]} \cdot \sigma \circ \varphi^{[3]} \cdot \frac{b \circ \varphi^{[2]}}{b \circ \varphi^{[3]}} \cdot \frac{b \circ \varphi}{b \circ \varphi^{[2]}} \cdot \frac{b}{b \circ \varphi}\sin\varphi^{[3]}\right)$$

$$= -2ab(\sigma\sin t + \sigma \circ \varphi \cdot \sin\varphi + \sigma \circ \varphi^{[2]} \cdot \sin\varphi^{[2]} + \sigma \circ \varphi^{[3]} \cdot \sin\varphi^{[3]}),$$

that is,

$$L' = -2abM,$$

where

$$M = \sigma\sin t + \sigma \circ \varphi \cdot \sin\varphi + \sigma \circ \varphi^{[2]} \cdot \sin\circ\varphi^{[2]} + \sigma \circ \varphi^{[3]} \cdot \sin\circ\varphi^{[3]}. \tag{5.3}$$

The derivative M' of M is a sum $M_1 + M_2$, where

$$M_1 = \frac{a}{R}\left(\sin^2 t + \frac{b}{b \circ \varphi}\sin^2\varphi + \frac{b}{b \circ \varphi^{[2]}}\sin^2\varphi^{[2]} + \frac{b}{b \circ \varphi^{[3]}}\sin^2\varphi^{[3]}\right) \tag{5.4}$$

and

$$M_2 = \sigma \cos t + \sigma \circ \varphi \frac{b}{b \circ \varphi} \cos \varphi + \sigma \circ \varphi^{[2]} \frac{b}{b \circ \varphi^{[2]}} \cos \varphi^{[2]} + \sigma \circ \varphi^{[3]} \frac{b}{b \circ \varphi^{[3]}} \cos \varphi^{[3]}.$$
(5.5)

Thus, L'' has the following form:

$$L'' = -2a(b'M + (M_1 + M_2)b).$$

Now, we divide our considerations into two steps.

Step 1. *The perimeter L reaches its local minimum at 0.*

First, we prove that

$$|BC|^2 = 2(R^2 - a^2)$$
(5.6)

(see Figure 6). From (4.3) and (4.4),

$$|BC|^2 = (\lambda_0 + \lambda_1)^2 = (\lambda_0 - \lambda_1)^2 + 4r^2 = \lambda_0^2 - 2r^2 + \lambda_1^2 + 4r^2 = 2R^2 - 2a^2.$$

Let $\tau = \varphi(0)$, as in Figure 6. We prove that

$$\cos \tau = \frac{2r^2 + a^2 - R^2}{2ar}.$$
(5.7)

It follows immediately from (5.1) that the numerator in (5.7) is negative. From (2.5) and (5.6),

$$b(\tau) = \frac{2R}{|BC|} = \frac{2R}{\sqrt{2R^2 - 2a^2}}.$$
(5.8)

On the other hand, by (2.3),

$$\frac{1}{b(\tau)} = \sqrt{1 - \left(\frac{r - a \cos \tau}{R}\right)^2}.$$

Comparing these two formulas and using the formula (5.1) for a 4-porism gives (5.7).

Now, we prove that

$$M'(0) = M_1(0) + M_2(0) < 0.$$

Since $\varphi(0) = \tau$, $\varphi^{[2]}(0) = \pi$ and $\varphi^{[3]}(0) = 2\pi - \tau$, it follows from (5.4) that

$$M_1(0) = \frac{2a}{R} \frac{b(0)}{b(\tau)} \sin^2 \tau.$$
(5.9)

Next, from (5.5),

$$M_2(0) = \sigma(0) + 2\sigma(\tau)\frac{b(0)}{b(\tau)} \cos \tau - \sigma(\pi)\frac{b(0)}{b(\pi)}.$$
(5.10)

From (5.9) and (5.10),

$$M'(0) = \frac{b(0)}{b(\tau)}\left(\frac{2a}{R}(1 - \cos^2 \tau) + \frac{b(\tau)}{b(0)}\sigma(0) + 2\frac{r - a \cos \tau}{R} \cos \tau - \sigma(\pi)\frac{b(\tau)}{b(\pi)}\right)$$

$$= \frac{2}{R}\varphi'(0)\left(-2a \cos^2 \tau + r \cos \tau + a + \frac{1}{2}Rb(\tau)\left(\frac{\sigma(0)}{b(0)} - \frac{\sigma(\pi)}{b(\pi)}\right)\right).$$

Next, using (5.7) and (5.1),

$$
\begin{aligned}
-2a\cos^2\tau + r\cos\tau + a &= \frac{-(2r^2 + a^2 - R^2)^2}{2ar^2} + \frac{2r^2 + a^2 - R^2}{2a} + a \\
&= \frac{1}{2ar^2}(-2r^4 - 3r^2(a^2 - R^2) - 2r^2(R^2 + a^2)) + a \\
&= \frac{R^2 - 2r^2 - 3a^2}{2a}.
\end{aligned}
$$

Now, using (5.8) and (5.2),

$$
\begin{aligned}
M'(0) &= \frac{2}{R}\varphi'(0)\left(\frac{R^2 - 2r^2 - 3a^2}{2a}\right. \\
&\quad \left. + \frac{1}{2}R\frac{2R}{\sqrt{2R^2 - 2a^2}}\left(\frac{r-a}{R}\sqrt{1 - \left(\frac{r-a}{R}\right)^2} - \frac{r+a}{R}\sqrt{1 - \left(\frac{r+a}{R}\right)^2}\right)\right) \\
&= \frac{2}{R}\varphi'(0)\left(\frac{R^2 - 2r^2 - 3a^2}{2a} + \frac{(r-a)\sqrt{R^2 - (r-a)^2} - (r+a)\sqrt{R^2 - (r+a)^2}}{\sqrt{2R^2 - 2a^2}}\right) \\
&= \frac{2}{R}a\varphi'(0)\left(\frac{1}{2}\cdot f\left(\frac{r}{a}\right)\right),
\end{aligned}
$$

where

$$
f(x) = x\sqrt{4 + x^2} - x^2 - 2 + \frac{(x-1)\sqrt{2 + \sqrt{4 + x^2}} - (x+1)\sqrt{\sqrt{4 + x^2} - 2}}{\sqrt{2}\sqrt{x + \sqrt{4 + x^2}}}.
$$

It is easy to see that $x\sqrt{4 + x^2} - x^2 - 2 < 0$ for $x > 0$. Moreover, we can simply prove in two steps (for $x \in (0, 1]$ and $x \in (1, +\infty)$) that the last term of f is negative. It follows from (2.6) that $b'(0) = 0$. Hence,

$$
L''(0) = -2a(b'(0)M(0) + b(0)M'(0)) = -2ab(0)M'(0) > 0,
$$

so the perimeter L reaches its local minimum at $t = 0$.

Step 2.   The perimeter L reaches its local maximum at $t_{\max}$.

First, we prove that

$$
M_1(t_{\max}) = 4ar^2\frac{R^2 + 3a^2}{(R - a)^2(R + a)^3}. \tag{5.11}
$$

For this purpose, note that $\sin t_1 = r/(R - a)$ (see Figure 5). From (3.3), (4.1), (2.3)

and (2.5),

$$
\begin{aligned}
M_1(t_{\max}) &= \frac{a}{R}\Big(\sin^2 t_{\max} + \sin^2 \varphi(t_{\max}) + \frac{b(t_{\max})}{b(\varphi^{[2]}(t_{\max}))}\sin^2 \varphi^{[2]}(t_{\max}) \\
&\qquad + \frac{b(t_{\max})}{b(\varphi^{[3]}(t_{\max}))}\sin^2 \varphi^{[3]}(t_{\max})\Big) \\
&= \frac{a}{R}b(2\pi - t_1)\Big(\frac{\sin^2 t_1}{b(2\pi - t_1)} + \frac{\sin^2 t_1}{b(t_1)} + \frac{\cos^2 t_1}{b((\pi/2)+t_1)} + \frac{\cos^2 t_1}{b((3\pi/2)-t_1)}\Big) \\
&= \frac{a}{R}\frac{2R}{|AD|}\Big(\frac{|AD|}{2R}\sin^2 t_1 + \frac{|AB|}{2R}\sin^2 t_1 + \frac{|BC|}{2R}\cos^2 t_1 + \frac{|CD|}{2R}\cos^2 t_1\Big) \\
&= \frac{2a}{R}(\sin^2 t_1 + \cot t_1 \cos^2 t_1) = 4ar^2 \frac{R^2 + 3a^2}{(R-a)^2(R+a)^3}.
\end{aligned}
$$

Now, we prove that

$$
M_2(t_{\max}) = \frac{-4ar^2}{(R-a)(R+a)^2}. \tag{5.12}
$$

Similarly to Step 1,

$$
\begin{aligned}
M_2(t_{\max}) &= \sigma(t_{\max})\cos t_{\max} + \sigma(\varphi(t_{\max}))\cos \varphi(t_{\max}) \\
&\qquad + \sigma(\varphi^{[2]}(t_{\max}))\frac{b(t_{\max})}{b(\varphi^{[2]}(t_{\max}))}\cos \varphi^{[2]}(t_{\max}) \\
&\qquad + \sigma(\varphi^{[3]}(t_{\max}))\frac{b(t_{\max})}{b(\varphi^{[3]}(t_{\max}))}\cos \varphi^{[3]}(t_{\max}) \\
&= b(2\pi - t_1)\Big(\sigma(t_1)\frac{\cos t_1}{b(2\pi - t_1)} + \sigma(t_1)\frac{\cos t_1}{b(t_1)} - \sigma\Big(\frac{\pi}{2}+t_1\Big)\frac{\sin t_1}{b(\frac{\pi}{2}+t_1)} \\
&\qquad - \sigma\Big(\frac{3\pi}{2}+t_1\Big)\frac{\sin t_1}{b(\frac{3\pi}{2}-t_1)}\Big).
\end{aligned}
$$

Now, using (2.5) and (3.1),

$$
\begin{aligned}
M_2(t_{\max}) &= 2b(t_1)\Big(\sigma(t_1)\frac{\cos t_1}{b(t_1)} - \sigma\Big(\frac{\pi}{2}+t_1\Big)\frac{\sin t_1}{b(\frac{\pi}{2}+t_1)}\Big) \\
&= 2\frac{2R}{|AB|}\Big(\frac{r-a}{R}\frac{|AB|}{2R}\cos t_1 - \frac{r+a\sin t_1}{R}\frac{|BC|}{2R}\sin t_1\Big) \\
&= \frac{-2a}{R}\frac{r}{R+a}\Big(\frac{r}{R+a} + \frac{r}{R-a}\Big) = \frac{-4ar^2}{(R-a)(R+a)^2}.
\end{aligned}
$$

Next, we prove that

$$
b'(t_{\max}) = \frac{-ar^5}{R(R-a)^4(R+a)}.
$$

Using (2.6) and (3.5),

$$b'(t_{\max}) = \frac{a}{R}\left(\sqrt{1 - \sigma^2(t_{\max})}\right)^3 \sigma(t_{\max}) \sin t_{\max}$$

$$= \frac{a}{R}(\sin t_{\max})^3 \cos t_{\max} \sin t_{\max} = \frac{-ar^5}{r(R - a)^4(R + a)}.$$

In this substep, we prove that

$$M(t_{\max}) = \frac{-2r^2}{(R - a)(R + a)}.$$

Indeed,

$$M(t_{\max}) = \sigma(t_{\max}) \sin t_{\max} + \sigma(\varphi(t_{\max})) \sin \varphi(t_{\max})$$

$$+ \sigma(\varphi^{[2]}(t_{\max})) \sin \varphi^{[2]}(t_{\max}) + \sigma(\varphi^{[3]}(t_{\max})) \sin \varphi^{[3]}(t_{\max})$$

$$= -\sigma(t_1) \sin t_1 + \sigma(t_1) \sin t_1 + \sigma\left(\frac{\pi}{2} + t_1\right) \sin\left(\frac{\pi}{2} + t_1\right)$$

$$+ \sigma\left(\frac{3\pi}{2} - t_1\right) \sin\left(\frac{3\pi}{2} - t_1\right)$$

$$= -2\sigma\left(\frac{\pi}{2} + t_1\right) \cos t_1 = -2\frac{r + a \sin t_1}{R} \cos t_1 = \frac{-2r^2}{(R - a)(R + a)}.$$

Finally, we prove that $L''(t_{\max}) < 0$. To see this, note that

$$M'(t_{\max}) = 4ar^2 \frac{R^2 + 3a^2}{(R - a)^2(R + a)^3} - \frac{4ar^2}{(R - a)(R + a)^2} = \frac{16a^3 r^2}{(R - a)^2(R + a)^3} > 0$$

and $b'(t_{\max})M(t_{\max}) > 0$, so $L''(t_{\max}) < 0$.

6. The number of extrema of L

In the previous section we proved that $L' = -2abM$, where M is given by the formula (5.3). Substituting (2.4) into (5.3),

$$M = \sigma(t) \sin t + \sigma \circ \varphi(t) \sin \varphi(t) + \sigma \circ \varphi^{[2]}(t) \sin \varphi^{[2]}(t) + \sigma \circ \varphi^{[3]}(t) \sin \varphi^{[3]}(t)$$

$$= \frac{r}{R}(\sin t + \sin \varphi(t) + \sin \varphi^{[2]}(t) + \sin \varphi^{[3]}(t))$$

$$- \frac{a}{2R}(\sin 2t + \sin 2\varphi(t) + \sin 2\varphi^{[2]}(t) + \sin 2\varphi^{[3]}(t)).$$

In order to simplify this formula, we prove the identity

$$\sin t + \sin \varphi(t) + \sin \varphi^{[2]}(t) + \sin \varphi^{[3]}(t) \equiv 0. \qquad (6.1)$$

The formula (3.7) implies that

$$\lambda \circ \varphi^{[4]}(t) = \lambda(t) - 2a \sin \varphi(t) - 2a \sin \varphi^{[2]}(t) - 2a \sin \varphi^{[3]}(t) - 2a \sin \varphi^{[4]}(t)$$

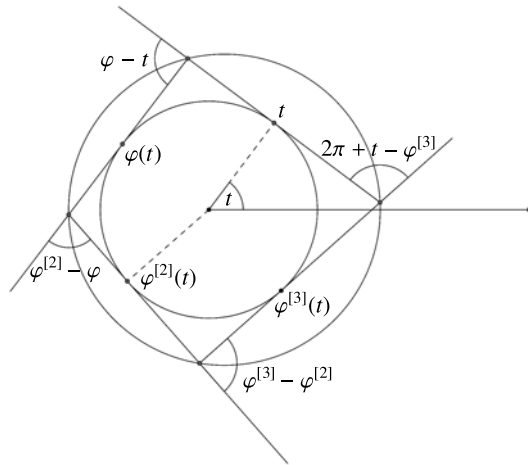and (6.1) follows because $\varphi^{[4]}(t) \equiv t + 2\pi$.

FIGURE 8. Outer angles of the described quadrangle.

Next, we prove that

$$\sin 2t + \sin 2\varphi(t) + \sin 2\varphi^{[2]}(t) + \sin 2\varphi^{[3]}(t)$$
$$= 4\cos(\varphi(t) - t)\cos(\varphi^{[2]}(t) - \varphi(t))\sin(\varphi^{[2]}(t) + t). \tag{6.2}$$

We make use of the well-known trigonometric formula

$$\sin\alpha + \sin\beta + \sin(\alpha + \beta) = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\alpha + \beta}{2}. \tag{6.3}$$

The condition for a quadrangle to be inscribed in a circle (see Figure 8) implies that

$$\varphi^{[3]}(t) - \varphi^{[2]}(t) + \varphi(t) - t \equiv \pi. \tag{6.4}$$

From (6.3) and (6.4), we deduce (6.2):

$$\sin 2t + \sin 2\varphi + \sin 2\varphi^{[2]} + \sin 2\varphi^{[3]}$$
$$= 2\sin(t + \varphi)\cos(t - \varphi) + 2\sin(\varphi^{[2]} + \varphi^{[3]})\cos(\varphi^{[3]} - \varphi^{[2]})$$
$$= 2\sin(\varphi + t)\cos(\varphi - t) + 2\sin(\varphi^{[2]} + \varphi^{[3]})\cos(\pi - \varphi + t)$$
$$= 4\cos(\varphi - t)\sin\tfrac{1}{2}(\varphi + t - \varphi^{[2]} - \varphi^{[3]})\cos\tfrac{1}{2}(\varphi + t + \varphi^{[2]} + \varphi^{[3]})$$
$$= 4\cos(\varphi - t)\sin\tfrac{1}{2}(\varphi - \varphi^{[2]} - \varphi^{[2]} + \varphi - \pi)\cos\tfrac{1}{2}(t + \varphi^{[2]} + \pi + \varphi^{[2]} + t)$$
$$= 4\cos(\varphi - t)\cos(\varphi^{[2]} - \varphi)\sin(\varphi^{[2]} + t).$$

Summarising,

$$L' = 8\frac{a^2}{R}b\cos(\varphi - t)\cos(\varphi^{[2]} - \varphi)\sin(\varphi^{[2]} + t). \tag{6.5}$$

Moreover, $L \circ \varphi = L$, so

$$\varphi'L' \circ \varphi = L'.$$

Now, it is easy to see that if $L' = 0$, then the first and the second trigonometric factors of (6.5) determine the same quadrangle (with the maximal perimeter) and the third factor determines a quadrangle with the minimal perimeter.

Now, we are in position to formulate the main theorem of this paper.

THEOREM 6.1. *Suppose that the annulus $C_r C_{a,R}$ has the Poncelet porism property. A quadrangle circuminscribed in $C_r C_{a,R}$ has the maximal perimeter if it passes through the point $(R + a, 0)$ and it has the minimal perimeter if passes through the point $(r, 0)$.*

7. Conjectures

Results of Radić and our considerations allow us to formulate the following conjectures.

(I)   An n-gon circuminscribed in $C_r C_{a,R}$ has the maximal perimeter (area) if it passes through a point $(R + a, 0)$. One of its edges contains the maximal segment $\lambda_{\max}$.

(II)   An n-gon circuminscribed in $C_r C_{a,R}$ has the minimal perimeter (area) if it passes through a point $(r, 0)$.

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