Canad. J. Math. Vol. 67 (5), 2015 pp. 1109–1143
http://dx.doi.org/10.4153/CJM-2015-004-2

Goldman Systems and Bending Systems

Yuichi Nohara and Kazushi Ueda

Abstract. We show that the moduli space of parabolic bundles on the projective line and the polygon space are isomorphic, both as complex manifolds and as symplectic manifolds equipped with structures of completely integrable systems, if the stability parameters are small.

1 Introduction

Let $\mathcal{N}_{\boldsymbol{\alpha}}$ be the moduli space of semi-stable parabolic bundles of rank 2 on the projective line X with n marked points $z_1, \dots, z_n$, where $\boldsymbol{\alpha} \in (0, 1/2)^n$ is the parameter for the parabolic weight. The moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ is a smooth projective manifold for a generic choice of $\boldsymbol{\alpha}$. Mehta and Seshadri [MS80] gave a construction of $\mathcal{N}_{\boldsymbol{\alpha}}$ using geometric invariant theory and showed that it is diffeomorphic to the moduli space of unitary representations of the fundamental group of the punctured Riemann surface $X \setminus \{z_1, \dots, z_n\}$.

With any pair-of-pants decomposition of the punctured Riemann surface $X \setminus \{z_1, \dots, z_n\}$ one can associate a completely integrable system on $\mathcal{N}_{\boldsymbol{\alpha}}$ called the *Goldman system* [Gol86]. The Goldman system resembles the moment map of a toric variety [Wei92, JW92, JW94, JW97], although the natural complex structure on $\mathcal{N}_{\boldsymbol{\alpha}}$ is not preserved by the action of the Goldman's Hamiltonians. Even worse, the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ as a complex manifold usually does not admit a structure of a toric variety at all.

A pair-of-pants decomposition of the punctured Riemann surface $X \setminus \{z_1, \dots, z_n\}$ is described by a trivalent graph Γ with n leaves in such a way that nodes correspond to pairs of pants and edges show how they are glued together, as shown in Figure 1. In this paper, we consider the case when the genus of X is zero, so that Γ is a tree. The corresponding Goldman system will be denoted by $\Theta_\Gamma \colon \mathcal{N}_{\boldsymbol{\alpha}} \to \mathbb{R}^{n-3}$.

The moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ is closely related to the moduli space $\mathcal{M}_{\boldsymbol{w}}$ of ordered n points on the projective line, which is constructed as the geometric invariant theory quotient

$$\mathcal{M}_{\boldsymbol{w}} = \operatorname{Proj}\Big(\bigoplus_{k=0}^{\infty} \Gamma\big((\mathbb{P}^1)^n, \mathcal{O}(kw_1, \dots, kw_n) \big)^{PGL_2} \Big).$$

Here, $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{Q}^n$ is the parameter for the PGL_2-linearization, which determines the stability condition and the ample line bundle on the quotient.

The moduli space $\mathcal{M}_{\boldsymbol{w}}$ has a natural symplectic structure as a polarized projective variety. As such, it admits an interpretation as the moduli space of polygons in $\mathbb{R}^3$ with

Received by the editors May 28, 2014; revised January 21, 2015.
Published electronically April 30, 2015.
This research is supported by Grant-in-Aid for Young Scientists (No.23740055 and No.24740043).
AMS subject classification: **53D30**, 14H60.
Keywords: toric degeneration.

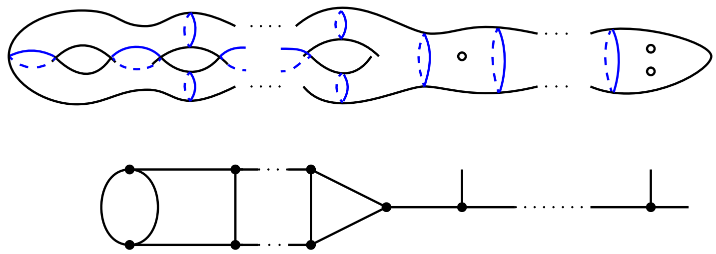

Figure 1: A pair-of-pants decomposition and its dual graph

side lengths $(w_1, \dots, w_n)$. Fix a convex planar n-gon P called the *reference polygon*. We identify the set of triangulations of the reference polygon with the set of trivalent trees with n leaves by assigning the dual graph to a triangulation. For any triangulation Γ of the reference polygon, Klyachko [Kly94] and Kapovich and Millson [KM96] introduced a completely integrable system $\Phi_\Gamma : \mathcal{M}_{\boldsymbol{w}} \to \mathbb{R}^{n-3}$ called the *bending system*.

To relate a completely integrable system with a toric variety, the notion of a *toric degeneration of an integrable system* was introduced in [NNU10, Definition 1.1]. For each triangulation Γ of the reference polygon P, we have given a toric degeneration of the corresponding bending system in [NU14, Corollary 1.3]. The toric degeneration of $\mathcal{M}_{\boldsymbol{w}}$ underlying this toric degeneration of the bending system is the one given in [HK97, KY02, SS04, FH05, HMM11].

The main result in this paper is the following theorem.

Theorem 1.1 *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/2)^n$ be a parabolic weight satisfying $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n < 1$. Then for any triangulation Γ of the reference polygon P, there is a symplectomorphism $\psi : \mathcal{N}_{\boldsymbol{\alpha}} \to \mathcal{M}_{\boldsymbol{w}}$ such that $\psi^* \Phi_\Gamma = \Theta_\Gamma$.*

Combining with [NU14, Corollary 1.3], we have the following corollary.

Corollary 1.2 *Suppose that $\boldsymbol{\alpha} \in (0, 1/2)^n$ satisfies $|\boldsymbol{\alpha}| < 1$. Then there exists a continuous family $\pi : \mathfrak{Y} \to [0, 1]$ of symplectic varieties equipped with completely integrable systems $F_t : Y_t = \pi^{-1}(t) \to \mathbb{R}^{n-3}$ such that $(Y_1, F_1) = (\mathcal{N}_{\boldsymbol{\alpha}}, \Theta_\Gamma)$, and (Y_0, F_0) is a pair of a toric variety and a toric moment map whose moment polytope is $\Theta_\Gamma(\mathcal{N}_{\boldsymbol{\alpha}})$. Moreover, there is a continuous family of maps $\psi_t : Y_1 \to Y_{1-t}$ that are symplectomorphisms on an open dense subset and satisfy $\psi_t^* F_{1-t} = F_1 = \Theta_\Gamma$.*

As a corollary, we obtain a new proof of the $|\boldsymbol{\alpha}| < 1$ case of the result of Jeffrey and Weitsman [JW92] stating that the numbers of lattice points on the moment polytope of the Goldman system is equal to the number of sections of the natural ample line bundle on $\mathcal{N}_{\boldsymbol{\alpha}}$ provided by GIT construction.

This paper is organized as follows. In Section 2, we recall the description of coherent sheaves on smooth rational orbifold curves due to Geigle and Lenzing [GL87], who call such curves *weighted projective lines*. In Section 3, we recall the relation between quasi-parabolic bundles and orbifold bundles. In Section 4, we recall the definition of parabolic weights and stability conditions. In Section 5, we recall the relation between flat $SU(2)$-bundles and parabolic bundles of rank two and parabolic degree zero. In Section 6, we show that the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ is the projective space $\mathbb{P}^{n-3}$ for a suitable choice of a stability parameter. In Section 7, we recall wall-crossing phenomena in moduli spaces of parabolic bundles following Bauer [Bau91]; the space of stability parameters is divided into finitely many chambers by walls, and the change in moduli spaces under wall-crossing can be described explicitly as a blow-down followed by a blow-up. More general results on variation of geometric invariant theory quotients are obtained by Thaddeus [Tha96] and Dolgachev and Hu [DH98]. In Section 8, we use the wall-crossing phenomena to give an explicit description of $\mathcal{N}_{\boldsymbol{\alpha}}$ for general $\boldsymbol{\alpha}$. The strategy is to start with the stability parameter in Section 6 and successively cross walls in the space of stability parameters to arrive at any stability parameter. This strategy was used in Bauer [Bau91], and the main difference between his work and ours is that we make extensive use of the language of weighted projective lines developed by Geigle and Lenzing [GL87], and the chamber that we start with is different from that of Bauer. In Section 9, we give a description of the moduli space $\mathcal{M}_{\boldsymbol{w}}$ parallel to that of $\mathcal{N}_{\boldsymbol{\alpha}}$. This immediately shows that $\mathcal{M}_{\boldsymbol{w}}$ and $\mathcal{N}_{\boldsymbol{\alpha}}$ are isomorphic if $\boldsymbol{w} = \boldsymbol{\alpha}$ and $|\boldsymbol{\alpha}| < 1$. In Section 10, we recall the construction of the bending system on $\mathcal{M}_{\boldsymbol{w}}$. In Section 11, we recall the description of the symplectic structure given by Guruprasad, Huebschmann, Jeffrey, and Weinstein [GHJW97], and the Goldman system. In Section 12 we recall extended moduli spaces defined by Jeffrey [Jef94] and Hurtubise and Jeffrey [HJ00] to construct $\mathcal{N}_{\boldsymbol{\alpha}}$ as a finite dimensional symplectic reduction, and as a quasi-Hamiltonian reduction [AMM98]. In Section 13, we see the walls in Section 7 from the view point of quasi-Hamiltonian reduction. In Section 14, we study the Goldman system via gluing of Riemann surface, following the idea of [HJ00] and [AMM98]. In Section 15, we construct a symplectomorphism between $\mathcal{N}_{\boldsymbol{\alpha}}$ and $\mathcal{N}_{t\boldsymbol{\alpha}}$ $(0 < t \leq 1)$ that identifies the Goldman systems in the case where $|\boldsymbol{\alpha}| < 1$. In Section 16, we show that $\mathcal{M}_{\boldsymbol{\alpha}}$ and $\mathcal{N}_{\boldsymbol{\alpha}}$ are symplectomorphic in such a way that the Goldman system on $\mathcal{N}_{\boldsymbol{\alpha}}$ and the bending system on $\mathcal{M}_{\boldsymbol{w}}$ are identified for sufficiently small $\boldsymbol{\alpha}$. Combining with the result in Section 15, Theorem 1.1 is proved.

2 Orbifold Projective Lines

Let $\mathbb{X}$ be a smooth Deligne–Mumford stack of dimension one without generic stabilizer. We assume that $\mathbb{X}$ is rational, so that the coarse moduli space X of $\mathbb{X}$ is isomorphic to $\mathbb{P}^1$. Such a stack was studied in detail by Geigle and Lenzing [GL87] under the name *weighted projective lines*, and we summarize some of their results in this subsection. One can also see [Len11] and references therein for more on this subject. Orbifold points of $\mathbb{X}$ will be denoted by $w_1, \dots, w_n$, and their images in X will be denoted by $z_1, \dots, z_n$. The absence of generic stabilizer implies that the stabilizer group Γ_{p_i} at w_i for any $i = 1, \dots, n$ is a cyclic group, whose order will be denoted by p_i.

Locally around the orbifold point w_i, we can take an orbifold chart $[\mathbb{A}/\Gamma_{w_i}] \hookrightarrow \mathbb{X}$ where $\mathbb{A} = \operatorname{Spec} \mathbb{C}[u]$ is an affine space and Γ_{w_i} acts linearly by a primitive p_i-th root of unity. Following [GL87], we let $\mathcal{O}_{\mathbb{X}}(\vec{x}_i)$ for $i = 1, \dots, n$, denote the dual of $\mathcal{O}(-\vec{x}_i)$, defined as the kernel of the natural morphism $\mathcal{O}_{\mathbb{X}} \to \mathcal{O}_{w_i}$ to the skyscraper sheaf $\mathcal{O}_{w_i} = [(\operatorname{Spec}\mathbb{C}[u]/(u))/\Gamma_{w_i}]$:

$$0 \to \mathcal{O}_{\mathbb{X}}(-\vec{x}_i) \to \mathcal{O}_{\mathbb{X}} \to \mathcal{O}_{w_i} \to 0.$$

We also define $\mathcal{O}_{\mathbb{X}}(\vec{c})$ as the line bundle $\mathcal{O}_{\mathbb{X}}(x)$, which does not depend on the choice of a point $x \in \mathbb{X} \setminus \{w_1, \dots, w_n\}$. One has relations

$$\mathcal{O}_{\mathbb{X}}(p_i \vec{x}_i) = \mathcal{O}_{\mathbb{X}}(\vec{c}), \quad i = 1, \dots, n,$$

and the Picard group of $\mathbb{X}$ is given by

$$L = \operatorname{Pic}\mathbb{X} = \mathbb{Z}\vec{x}_1 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n \oplus \mathbb{Z}\vec{c}/(p_1\vec{x}_1 - \vec{c}, \dots, p_n\vec{x}_n - \vec{c}).$$

Choose a global coordinate on $X \cong \mathbb{P}^1$ so that the points $z_1, \dots, z_n$ on X are given in this coordinate by $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3, \dots, \lambda_n \in \mathbb{A}^1 \setminus \{0\}$. The total coordinate ring of $\mathbb{X}$ is given by

$$S = \bigoplus_{\vec{k} \in L} H^0(\mathcal{O}_{\mathbb{X}}(\vec{k})) = k[X_0, X_1, \dots, X_n] \Big/ \left(X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}\right)_{i=2}^n,$$

which is graded by the abelian group L as $\deg X_i = \vec{x}_i$ for $i = 1, \dots, n$. The stack $\mathbb{X}$ is recovered as the quotient stack

$$\mathbb{X} = \left[\left(\operatorname{Spec} S \setminus \{\mathbf{0}\}\right) \big/ G\right]$$

by the affine algebraic group $G = \operatorname{Spec}\mathbb{C}[L]$. The graded ring S is Gorenstein with parameter $\vec{\omega} = (n-2) - \sum_{i=1}^n \vec{x}_i$, and Serre duality on $\mathbb{X}$ is given by

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{F}) \cong H^0\big(\mathcal{F}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{X}}(\vec{\omega})\big)^{\vee}$$

for any coherent sheaves $\mathcal{E}$ and $\mathcal{F}$.

3 Quasi-Parabolic Bundles as Orbifold Bundles

In this section, we recall the relation between quasi-parabolic bundles on punctured curves and orbifold bundles on orbi-curves. Although this is well known to experts and essentially goes back to [MS80], we provide a sketch of proof here for the reader's convenience.

Let $\widetilde{U} = \operatorname{Spec}\mathbb{C}[u]$ be an affine line and let $\mathbb{U} = [\widetilde{U}/\Gamma]$ be the quotient stack of $\widetilde{U}$ with respect to the $\Gamma = \mathbb{Z}/p_j\mathbb{Z}$-action, which acts on points of $\widetilde{U}$ as

$$u \mapsto \zeta^{-1}u, \quad \zeta = \exp(2\pi\sqrt{-1}/p_j).$$

A complex analytic neighborhood of the origin in $\mathbb{U}$ is identified with a complex analytic neighborhood of w_j in $\mathbb{X}$. The coarse moduli space of $\mathbb{U}$ is given by $U = \operatorname{Spec}\mathbb{C}[v]$, where $\mathbb{C}[v] = \mathbb{C}[u]^{\Gamma}$ for $v = u^{p_j}$ is the invariant ring.

The action of Γ on $\widetilde{U}$ induces an action on the coordinate ring $\mathbb{C}[u]$ in such a way that an element $\gamma \in \Gamma$ sends a function f to its pull-back $(\gamma^{-1})^* f$ by $\gamma^{-1} \colon \widetilde{U} \to \widetilde{U}$. It follows from the definition of sheaves on quotient stacks that a locally-free sheaf $\mathcal{E}$ on $\mathbb{U}$ corresponds to a Γ-equivariant locally-free sheaf on $\widetilde{U}$. Since $\widetilde{U}$ is affine, a

Γ-equivariant locally-free sheaf on $\widetilde{U} = \operatorname{Spec}\mathbb{C}[u]$ is the same as a free $\mathbb{C}[u]$-module M, equipped with an action of Γ satisfying

$$\gamma \cdot (fm) = (\gamma \cdot f)(\gamma \cdot m), \tag{1}$$

for any $\gamma \in \Gamma$, $f \in \mathbb{C}[u]$ and $m \in M$. Here, $\cdot$ is the Γ-action on $\mathbb{C}[u]$ and M. The crossed product algebra $\mathbb{C}[u] \rtimes \Gamma$ consists of elements of the form $f \otimes \gamma$ for $f \in \mathbb{C}[u]$ and $\gamma \in \Gamma$, with relations

$$(f \otimes \gamma) \circ (g \otimes \delta) = f(\gamma \cdot g) \otimes (\gamma\delta). \tag{2}$$

It follows from (1) and (2) that a Γ-equivariant $\mathbb{C}[u]$-module can be identified with a $\mathbb{C}[u] \rtimes \Gamma$-module.

Let P be a finitely-generated $\mathbb{C}[u] \rtimes \Gamma$-module. As a Γ-module, it has a direct sum decomposition $P = \bigoplus_{i=1}^{p_j} P_i$ into isotypical components, where the generator $[1] \in \Gamma$ acts on P_i by multiplication by $\exp(2\pi\sqrt{-1}(i-1)/p_j)$. The $\mathbb{C}[u]$-module structure is determined by the action of u, which is just a collection of $\mathbb{C}$-linear maps

$$u\colon P_i \longrightarrow P_{i-1}, \qquad i \in \mathbb{Z}/p_j\mathbb{Z}.$$

Each P_i is a $\mathbb{C}[v]$-module, and multiplication by u is a homomorphism of $\mathbb{C}[v]$-modules, which must satisfy $u^m = v\colon P_i \to P_{i-m}$. In terms of sheaves $\mathcal{P}_i$ of $\mathcal{O}_U$-modules associated with $\mathbb{C}[v]$-modules P_i, this gives a *quasi-parabolic sheaf*, defined as an infinite sequence

$$\cdots \xrightarrow{u} \mathcal{P}_i \xrightarrow{u} \mathcal{P}_{i+1} \xrightarrow{u} \cdots \tag{3}$$

such that $\mathcal{P}_{i+p_j} = \mathcal{P}_i(-z_j)$ and the composition

$$\mathcal{P}_{i+p_j} \xrightarrow{u^{p_j}} \mathcal{P}_i$$

is equal to the multiplication

$$\mathcal{P}_i(-z_j) \xrightarrow{v} \mathcal{P}_i$$

by v for any $i \in \mathbb{Z}$. A *morphism* of quasi-parabolic sheaves is a collection of morphisms $f_i\colon \mathcal{P}_i \to \mathcal{Q}_i$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "u"] & \mathcal{P}_i \arrow[r, "u"] \arrow[d, "f_i"'] & \mathcal{P}_{i+1} \arrow[r, "u"] \arrow[d, "f_{i+1}"'] & \cdots \\
\cdots \arrow[r, "u"] & \mathcal{Q}_i \arrow[r, "u"] & \mathcal{Q}_{i+1} \arrow[r, "u"] & \cdots
\end{tikzcd}$$

commutative. Under the correspondence between $\mathbb{C}[v]$-modules with quasi-parabolic structures and $\mathbb{C}[u] \rtimes \Gamma$-modules, a morphism of quasi-parabolic sheaves can be identified with a morphism of $\mathbb{C}[u] \rtimes \Gamma$-modules. By using this correspondence around each orbifold points, one obtains the following proposition.

Proposition 3.1 *The category of quasi-parabolic sheaves on X is equivalent to the category of coherent sheaves on $\mathbb{X}$.*

If P is locally-free, then multiplication by v is an injection, so that (3) gives a filtration

$$\mathcal{P}_1(-z_j) \cong \mathcal{P}_{p_j+1} \hookrightarrow \mathcal{P}_{p_j} \hookrightarrow \cdots \hookrightarrow \mathcal{P}_2 \hookrightarrow \mathcal{P}_1$$

of sheaves, which in turn gives a filtration

$$0 = F_{p_j+1}(\mathcal{P}_{z_j}) \subset F_{p_j}(\mathcal{P}_{z_j}) \subset \cdots \subset F_1(\mathcal{P}_{z_j}) = \mathcal{P}_{z_j}$$

of the fiber $\mathcal{P}_{z_j} = \mathcal{P}_1/\nu \cdot \mathcal{P}_1$ of $\mathcal{P}_1$ at z_j. A pair consisting of a locally-free sheaf and a filtration at each z_j is called a *quasi-parabolic bundle*. A morphism of quasi-parabolic bundles $\mathcal{P}$ and $\mathcal{Q}$ can be described, in terms of a filtration at each z_j, as a morphism ϕ of the underlying vector bundle such that $\phi(F_i(\mathcal{P}_{z_j})) \subset F_i(\mathcal{Q}_{z_j})$. The equivalence in Proposition 3.1 restricts to an equivalence between the category of vector bundles on $\mathbb{X}$ and the category of quasi-parabolic bundles on X.

4 Parabolic Weights and Stability Conditions

Assume that the stabilizer groups at all orbifold points are cyclic groups of order two: $\Gamma_{w_j} = \mathbb{Z}/2\mathbb{Z}$ for $j = 1, \dots, n$. A vector bundle on $\mathbb{X}$ corresponds to a quasi-parabolic bundle consisting vector bundle $\mathcal{P}$ on X and 2-step flags

$$0 = F_3(\mathcal{P}_{z_j}) \subset F_2(\mathcal{P}_{z_j}) \subset F_1(\mathcal{P}_{z_j}) = \mathcal{P}_{z_j}$$

for each $j = 1, \dots, n$. The Picard group of $\mathbb{X}$ is given by

$$L = \operatorname{Pic} \mathbb{X} = \mathbb{Z}\vec{x}_1 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n \oplus \mathbb{Z}\vec{c}/(2\vec{x}_1 - \vec{c}, \dots, 2\vec{x}_n - \vec{c}).$$

The structure sheaf $\mathcal{O}_{\mathbb{X}}$ corresponds to the trivial bundle $\mathcal{P} = \mathcal{O}_X$ equipped with the filtration $F_2(\mathcal{P}_{z_j}) = 0$ for any z_j. The line bundle $\mathcal{O}_{\mathbb{X}}(\vec{x}_i)$ corresponds to the trivial bundle $\mathcal{P} = \mathcal{O}_X$ equipped with the filtration

$$F_2(\mathcal{P}_{z_j}) = \begin{cases} \mathcal{P}_{z_j} & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

A parabolic bundle is a quasi-parabolic bundle together with a choice of parabolic weights

$$(a_{j,1}, a_{j,2}) \in \mathbb{Q}^2, \qquad 0 \le a_{j,1} < a_{j,2} < 1$$

for each $j = 1, \dots, n$. In this paper, we always assume that a parabolic weight satisfies $a_{j,1} + a_{j,2} = 1$ for $j = 1, \dots, n$. Any subbundle $\mathcal{E}$ of a parabolic bundle $\mathcal{P}$ has a natural parabolic structure whose quasi-parabolic structure is given by

$$F_i(\mathcal{E}_{z_j}) = F_i(\mathcal{P}_{z_j}) \cap \mathcal{E}_{z_j}$$

with the same parabolic weight as $\mathcal{P}$. The *parabolic degree* of $\mathcal{P}$ is defined by

$$\operatorname{par deg} \mathcal{P} = \deg \mathcal{P} + \sum_{j=1}^{n} \big[a_{j,1}(\dim F_1(\mathcal{P}_{z_j}) - \dim F_2(\mathcal{P}_{z_j})) + a_{j,2} \dim F_2(\mathcal{P}_{z_j}) \big].$$

For example, if $\operatorname{rank} \mathcal{P} = 2$ and

$$\dim F_1(\mathcal{P}_{z_j}) - \dim F_2(\mathcal{P}_{z_j}) = \dim F_2(\mathcal{P}_{z_j}) = 1, \qquad j = 1, \dots, n,$$

then the parabolic degree of $\mathcal{P}$ is given by

$$\operatorname{par deg} \mathcal{P} = \deg \mathcal{P} + \sum_{j=1}^{n} (a_{j,1} + a_{j,2}) = \deg \mathcal{P} + n.$$

A parabolic bundle is *semi-stable* if one has

$$\frac{\operatorname{par}\deg \mathcal{E}}{\operatorname{rank}\mathcal{E}} \le \frac{\operatorname{par}\deg \mathcal{P}}{\operatorname{rank}\mathcal{P}} \tag{4}$$

for any subbundle $\mathcal{E} \subset \mathcal{P}$. It is *stable* if the strict inequality holds in (4) for any non-trivial subbundle $0 \neq \mathcal{E} \subsetneq \mathcal{P}$.

The Picard group L of $\mathbb{X}$ acts on $\mathbb{Q}^n$ by

$$\vec{x}_i(\boldsymbol{\alpha}) = \boldsymbol{\alpha}', \quad \alpha'_j = \begin{cases} \alpha_j & i \neq j, \\ -\alpha_j & i = j. \end{cases}$$

Note that this action factors through $L/(2\vec{c}) \cong (\mathbb{Z}/2\mathbb{Z})^n$. Any element of L can be written uniquely as $\vec{k} = k_1\vec{x}_1 + \cdots + k_n\vec{x}_n + k_0\vec{c}$, where $k_i \in \{0,1\}$ for $i = 1,\dots,n$ and $k_0 \in \mathbb{Z}$, and the parabolic degree of the line bundle $\mathcal{O}(\vec{k})$ is given by

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{O}(\vec{k}) = |\vec{k}| + |\vec{k}(\boldsymbol{\alpha})| := k_0 + k_1 + \cdots + k_n + (-1)^{k_1}\alpha_1 + \cdots + (-1)^{k_n}\alpha_n.$$

5 Moduli Spaces of Parabolic Bundles

Mehta and Seshadri [MS80] constructed the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ of semistable parabolic bundles, which is a normal projective variety parametrizing S-equivalence classes of semistable parabolic bundles. They have also shown that the open subvariety $\mathcal{N}^s_{\boldsymbol{\alpha}} \subset \mathcal{N}_{\boldsymbol{\alpha}}$ parametrizing stable parabolic bundles of parabolic degree zero is diffeomorphic to the moduli space of irreducible unitary representations of the fundamental group of $X^\circ := X \setminus \{z_1,\dots,z_n\}$:

$$\mathcal{N}^s_{\boldsymbol{\alpha}} \cong \left\{\rho \in \operatorname{Hom}(\pi_1(X^\circ), SU(2))_{\mathrm{irred}} \;\middle|\; \rho(\gamma_j) \in \mathcal{C}_{\alpha_j}\right\}/\!\sim. \tag{5}$$

Here $\gamma_j \in \pi_1(X^\circ)$ is a loop around z_j, and $\mathcal{C}_{\alpha_j} \subset SU(2)$ is the conjugacy class containing $\exp\left[2\pi\sqrt{-1}\operatorname{diag}(a_{j,1}, a_{j,2})\right]$. The equivalence relation $\sim$ is defined by conjugation; two representations ρ and ρ' are equivalent if there is some $g \in SU(2)$ such that $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for any $\gamma \in \pi_1(X^\circ)$. A parabolic weight is *generic* if semistability implies stability. If the parabolic weight $\boldsymbol{\alpha}$ is generic, then the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ is smooth.

The diffeomorphism (5) is given as follows. For any irreducible unitary representation ρ of $\pi_1(X^\circ)$, one has the flat $\mathbb{C}^2$-bundle E_ρ on X° associated with ρ. By tensoring E_ρ with the structure sheaf $\mathcal{O}_{X^\circ}$ over the constant sheaf $\mathbb{C}_{X^\circ}$, one obtains a coherent sheaf $\mathcal{E}^\circ := E_\rho \otimes_{\mathbb{C}_{X^\circ}} \mathcal{O}_{X^\circ}$ on X°. Around each puncture $z_j \in X$, we take a coordinate v centered at z_j, and consider following the universal cover of a small disk centered at z_j:

$$\left\{x + \sqrt{-1}y \in \mathbb{C} \;\middle|\; y \gg 1\right\} \to X^\circ,\; x + \sqrt{-1}y \mapsto v = \exp\left[2\pi\sqrt{-1}\left(x + \sqrt{-1}y\right)\right].$$

Let $g = \rho(\gamma_j) \in SU(2)$ be the holonomy of the flat bundle E_ρ around z_j. Then a holomorphic section of $\mathcal{E}^\circ$ near z_j is a holomorphic function $f\colon \{x + \sqrt{-1}y \in \mathbb{C} \mid y \gg 1\} \to \mathbb{C}^2$ satisfying $f((x+1) + \sqrt{-1}y) = g \cdot f(x + \sqrt{-1}y)$, and one defines the locally-free extension $\mathcal{E}$ of $\mathcal{E}^\circ$ by saying that f gives a holomorphic section of $\mathcal{E}$ near z_j if f is bounded. By a suitable choice of a coordinate of $\mathbb{C}^2$, one can assume that g is diagonal; $g = \exp\left[2\pi\sqrt{-1}\operatorname{diag}(a_{j,1}, a_{j,2})\right]$. Then the space of holomorphic

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

sections of $\mathcal{E}$ is spanned by $v \mapsto (v^{\alpha_j+k}, v^{(1-\alpha_j)+l})$ for non-negative integers k and l. The quasi-parabolic structure of $\mathcal{E}$ at z_j is defined as the one-dimensional subspace $\mathbb{C}\cdot(1,0)$ in

$$\bigoplus_{k,l=0}^{\infty} \mathbb{C}\cdot\big(v^{\alpha_j+k}, v^{(1-\alpha_j)+l}\big) \big/ v\cdot \bigoplus_{k,l=0}^{\infty} \mathbb{C}\cdot\big(v^{\alpha_j+k}, v^{(1-\alpha_j)+l}\big) \cong \mathbb{C}^2.$$

6 The Moduli Space for a Distinguished Stability Parameter

Let $\mathcal{N}_{\boldsymbol{\alpha}}$ be the moduli space of semistable parabolic bundles of rank two and parabolic degree zero on $X = \mathbb{P}^1$ with n marked points $(z_1, \dots, z_n)$. Here, the stability parameter $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/2)^n$ is related to the parabolic weight $a = ((a_{i,1}, a_{i,2}), \dots, (a_{n,1}, a_{n,2})) \in ((0,1)\times(0,1))^n$ by $(a_{i,1}, a_{i,2}) = (\alpha_i, 1-\alpha_i)$. The vector bundle $\mathcal{P}$ on $\mathbb{X}$ corresponding to a parabolic bundle in $\mathcal{N}_{\boldsymbol{\alpha}}$ has the same class as $\mathcal{O}\oplus\mathcal{O}(-\vec{s})$ in the Grothendieck group $K(\mathbb{X})$ where $\vec{s} = \vec{x}_1 + \dots + \vec{x}_n$. Consider the line bundle $\mathcal{L} = \mathcal{O}(-\vec{s} + \vec{x}_n)$. Since

$$\begin{aligned}
H^0(\mathcal{O}(-\vec{x}_n)) &= 0,\\
H^1(\mathcal{O}(-\vec{x}_n)) &\cong H^0(\mathcal{O}(\vec{\omega} + \vec{x}_n))^\vee = H^0(\mathcal{O}((n-2)\vec{c} - \vec{s} + \vec{x}_n)^\vee = 0,\\
H^0(\mathcal{O}(\vec{s} - \vec{x}_n)) &\cong \mathbb{C},\\
H^1(\mathcal{O}(\vec{s} - \vec{x}_n)) &\cong H^0(\mathcal{O}(\vec{\omega} - \vec{s} + \vec{x}_n))^\vee = H^0(\mathcal{O}((n-2)\vec{c} - \vec{s} - \vec{s} + \vec{x}_n))^\vee\\
&= H^0(\mathcal{O}((n-2)\vec{c} - n\vec{c} + \vec{x}_n))^\vee = H^0(\mathcal{O}(\vec{x}_n - 2\vec{c}))^\vee = 0,
\end{aligned}$$

where $\mathcal{O}(\vec{\omega}) = \mathcal{O}((n-2)\vec{c} - \vec{s})$ is the dualizing sheaf, one has

$$\chi(\mathcal{L}, \mathcal{P}) = \chi\big(\mathcal{L}, \mathcal{O}\oplus\mathcal{O}(\vec{s} - n)\big) = \chi\big(\mathcal{O}(\vec{s} - \vec{x}_n)\big) + \chi\big(\mathcal{O}(-\vec{x}_n)\big) = 1,$$

so that $\operatorname{Hom}(\mathcal{L}, \mathcal{P}) \neq 0$. By taking the saturation of the image of a non-zero morphism $\phi \in \operatorname{Hom}(\mathcal{L}, \mathcal{P})$, one obtains a subbundle of $\mathcal{P}$ of the form $\mathcal{L}(\vec{k})$, where $\vec{k} \in \mathbb{N}\vec{x}_1 + \dots + \mathbb{N}\vec{x}_n$. Note that

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{L}(\vec{k}) > \operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{L} = -\alpha_1 - \dots - \alpha_{n-1} + \alpha_n,$$

so that $\mathcal{L}(\vec{k})$ destabilizes $\mathcal{P}$ if $\alpha_1 + \dots + \alpha_{n-1} < \alpha_n$. This defines a chamber in the space of stability parameters, where every bundle is unstable and $\mathcal{N}_{\boldsymbol{\alpha}} = \varnothing$. The quotient bundle is

$$\mathcal{Q} = \mathcal{P}/\mathcal{L}(\vec{k}) \cong \mathcal{O}(-\vec{x}_n - \vec{k}),$$

and the destabilizing sequence is

$$0 \to \mathcal{O}(-\vec{s} + \vec{x}_n + \vec{k}) \to \mathcal{P} \to \mathcal{O}(-\vec{x}_n - \vec{k}) \to 0.$$

Consider vector bundles obtained as extensions

$$0 \to \mathcal{O}(-\vec{s} + \vec{x}_n) \to \mathcal{P} \to \mathcal{O}(-\vec{x}_n) \to 0,$$

which are classified by

$$\begin{aligned}
e_{\mathcal{P}} &\in \operatorname{Ext}^1\big(\mathcal{O}(-\vec{x}_n), \mathcal{O}(-\vec{s} + \vec{x}_n)\big)\\
&= H^1\big(\mathcal{O}(-\vec{s} + \vec{x}_n + \vec{x}_n)\big) = H^1\big(\mathcal{O}(\vec{c} - \vec{s})\big)\\
&= H^0\big(\mathcal{O}((n-2)\vec{c} - \vec{s} - \vec{c} + \vec{s})\big)^\vee = H^0\big(\mathcal{O}((n-3)\vec{c})\big)^\vee.
\end{aligned}$$

Given a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-\vec{s}+\vec{x}_n) & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{O}(-\vec{x}_n) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}(-\vec{s}+\vec{x}_n) & \longrightarrow & \mathcal{P}' & \longrightarrow & \mathcal{O}(-\vec{x}_n) & \longrightarrow & 0 \end{array}$$

between two such bundles $\mathcal{P}$ and $\mathcal{P}'$, one obtains a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-\vec{s}+\vec{x}_n) & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{O}(-\vec{x}_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-\vec{s}+\vec{x}_n) & \longrightarrow & \mathcal{P}' & \longrightarrow & \mathcal{O}(-\vec{x}_n) & \longrightarrow & 0 \end{array}$$

since

$$\operatorname{Hom}\big(\mathcal{O}(-\vec{s}+\vec{x}_n),\mathcal{O}(-\vec{x}_n)\big) = H^0\big(\mathcal{O}(\vec{s}-2\vec{x}_n)\big) = 0.$$

It follows that the isomorphism classes of such $\mathcal{P}$ are classified by

$$\mathbb{P}\operatorname{Ext}^1\big(\mathcal{O}(-\vec{x}_n),\mathcal{O}(-\vec{s}+\vec{x}_n)\big) \cong \mathbb{P}H^0\big(\mathcal{O}((n-3)\vec{c})\big)^\vee \cong \operatorname{Sym}^{n-3}\mathbb{P}^1 \cong \mathbb{P}^{n-3}.$$

Proposition 6.1 *One has $\mathcal{N}_{\boldsymbol{\alpha}} \cong \mathbb{P}^{n-3}$ if $2\alpha_n < |\boldsymbol{\alpha}| < 1$ and $|\boldsymbol{\alpha}| - 2\alpha_i - 2\alpha_n < 0$ for any $i = 1,\dots,n-1$.*

Proof Let $\mathcal{P}$ be a rank 2 bundle on $\mathbb{X}$ obtained as an extension

$$0 \to \mathcal{O}(-\vec{s}+\vec{x}_n) \to \mathcal{P} \to \mathcal{O}(-\vec{x}_n) \to 0. \tag{6}$$

Note that

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{O}(-\vec{s}+\vec{x}_n) = -\alpha_1 - \cdots - \alpha_{n-1} + \alpha_n = -|\boldsymbol{\alpha}| + 2\alpha_n,$$

so that $\mathcal{O}(-\vec{s}+\vec{x}_n)$ does not destabilize $\mathcal{P}$ if $2\alpha_n < |\boldsymbol{\alpha}|$. If a line bundle $\mathcal{L}$ other than $\mathcal{O}(-\vec{s}+\vec{x}_n)$ has a non-trivial morphism to $\mathcal{P}$, then $\mathcal{L}$ has a non-trivial morphism to $\mathcal{O}(-\vec{x}_n)$, so that it can be written as $\mathcal{O}(-\vec{x}_n - \vec{k})$ for some $\vec{k} = k_1\vec{x}_1 + \cdots + k_n\vec{x}_n + k_0\vec{c}$ where $k_i \in \{0,1\}$ for $i = 1,\dots,n$ and $k_0 \in \mathbb{N}$. Its parabolic degree is given by

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{O}(-\vec{x}_n - \vec{k}) = \begin{cases} -k_0 + |\boldsymbol{\alpha}| - 2\sum_{i\in I}\alpha_i - 2\alpha_n & k_n = 0, \\ -k_0 - 1 + |\boldsymbol{\alpha}| - 2\sum_{i\in I}\alpha_i & k_n = 1, \end{cases}$$

where $I = \{i \in \{1,\dots,n-1\} \mid k_i = 1\}$. Note that if the extension (6) does not split, then one has $\vec{k} \neq 0$. For $\vec{k} \neq 0$, the conditions $|\boldsymbol{\alpha}| - 2\alpha_i - 2\alpha_n < 0$ for any $i \in \{1,\dots,n-1\}$ and $|\boldsymbol{\alpha}| < 1$ imply that

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} \mathcal{O}(-\vec{x}_n - \vec{k}) < 0,$$

so that the line bundle $\mathcal{O}(-\vec{x}_n - \vec{k})$ does not destabilize $\mathcal{P}$. The same condition also implies that the line bundle $\mathcal{O}(-\vec{s}+\vec{x}_n+\vec{k})$ destabilizes any vector bundle $\mathcal{P}$ obtained as an extension

$$0 \to \mathcal{O}(-\vec{s}+\vec{x}_n+\vec{k}) \to \mathcal{P} \to \mathcal{O}(-\vec{x}_n-\vec{k}) \to 0$$

for any non-zero $\vec{k} \in \mathbb{N}\vec{x}_1 + \cdots + \mathbb{N}\vec{x}_n$, and Proposition 6.1 is proved. ∎

7 Wall-crossings in Moduli Spaces of Parabolic Bundles

The space $A = [0, 1/2)^n$ of stability parameters is divided into chambers by walls

$$H_{I,k} = \left\{ \boldsymbol{\alpha} \in A \;\middle|\; \sum_{j \in J} \alpha_j - \sum_{i \in I} \alpha_i = k \right\},$$

where $I \subset \{1, \dots, n\}$, $J = \{1, \dots, n\} \setminus I$ and k is a non-negative integer. Let C_+ and C_- be two chambers separated by the wall $W_{I,k}$ and take stability parameters $\boldsymbol{\alpha}_+ \in C_+$, $\boldsymbol{\alpha}_- \in C_-$ and $\boldsymbol{\alpha}_0 \in W_{I,k}$. There is a diagram

(7)
$$\begin{tikzcd} \mathcal{N}_{\boldsymbol{\alpha}_+} \arrow[dr, "\phi_+"'] & & \mathcal{N}_{\boldsymbol{\alpha}_-} \arrow[dl, "\phi_-"] \\ & \mathcal{N}_{\boldsymbol{\alpha}_0} & \end{tikzcd}$$

where $\phi_\pm \colon \mathcal{N}_{\boldsymbol{\alpha}_\pm} \to \mathcal{N}_{\boldsymbol{\alpha}_0}$ are natural projective morphisms sending a $\boldsymbol{\alpha}_\pm$-stable bundle to the S-equivalence class of the same bundle considered as an $\boldsymbol{\alpha}_0$-semistable bundle. Let $\Sigma_{\boldsymbol{\alpha}_\pm} \subset \mathcal{N}_{\boldsymbol{\alpha}_\pm}$ be the subscheme parametrizing $\boldsymbol{\alpha}_\mp$-unstable bundles.

Proposition 7.1 (Bauer [Bau91, Proposition 2.7]) *The following hold.*

(i) *If we set* $\Sigma_{\boldsymbol{\alpha}_0} := \phi_+(\Sigma_{\boldsymbol{\alpha}_+})$, *then one has* $\Sigma_{\boldsymbol{\alpha}_0} = \phi_-(\Sigma_{\boldsymbol{\alpha}_-})$.

(ii) *Any point in* $\Sigma_{\boldsymbol{\alpha}_0}$ *can be written as* $[\mathcal{S} \oplus \mathcal{Q}]$, *where* $\operatorname{par} \deg_{\boldsymbol{\alpha}_+}(\mathcal{S}) = -\operatorname{par} \deg_{\boldsymbol{\alpha}_+}(\mathcal{Q}) < 0$ *and* $\operatorname{par} \deg_{\boldsymbol{\alpha}_-}(\mathcal{S}) = -\operatorname{par} \deg_{\boldsymbol{\alpha}_-}(\mathcal{Q}) > 0$.

(iii) $\phi_+^{-1}([\mathcal{S} \oplus \mathcal{Q}]) \cong \mathbb{P} \operatorname{Ext}^1(\mathcal{Q}, \mathcal{S})^\vee$.

(iv) $\phi_-^{-1}([\mathcal{S} \oplus \mathcal{Q}]) \cong \mathbb{P} \operatorname{Ext}^1(\mathcal{S}, \mathcal{Q})^\vee$.

Proof For any bundle $\mathcal{P}$ in $\Sigma_{\boldsymbol{\alpha}_+}$, let

(8)
$$0 \to \mathcal{S} \to \mathcal{P} \to \mathcal{Q} \to 0$$

be the $\boldsymbol{\alpha}_-$-destabilizing sequence. Since $\mathcal{P}$ is of rank two, both the destabilizing subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$ are line bundles. Any point in the fiber of ϕ_+ above the point $[\mathcal{S} \oplus \mathcal{Q}] \in \mathcal{N}_{\boldsymbol{\alpha}_0}$ is given by the extension of the form (8), and any such extension is $\boldsymbol{\alpha}_+$-stable, so that one has $\phi_+^{-1}([\mathcal{S} \oplus \mathcal{Q}]) \cong \mathbb{P} \operatorname{Ext}^1(\mathcal{Q}, \mathcal{S})^\vee$. The fiber of ϕ_- is obtained by exchanging the roles of $\mathcal{S}$ and $\mathcal{Q}$, and Proposition 7.1 is proved. ∎

If $\boldsymbol{\alpha}_0$ does not lie on any other wall, then $\Sigma_{\boldsymbol{\alpha}_0}$ consists of one point, and the diagram (7) is a blow-down followed by a blow-up. It may also happen that ϕ_+ or ϕ_- is an isomorphism.

8 Detailed Description of the Wall-crossing

Recall that X is the coarse moduli space of $\mathbb{X}$, and one has a natural isomorphism $H^0(\mathcal{O}_{\mathbb{X}}((n-3)\vec{c})) \cong H^0(\mathcal{O}_X(n-3))$. Since X is a projective line, one has

$$\mathbb{P} H^0\big(\mathcal{O}_X(n-3)\big) \cong \operatorname{Sym}^{n-3} X \cong \mathbb{P}^{n-3}.$$

The Veronese embedding is the diagonal map $X \hookrightarrow \operatorname{Sym}^{n-3} X$ sending a point $x \in X$ to $[x, \dots, x] \in \operatorname{Sym}^{n-3} X$. For $\vec{k} = \sum_{i \in I} \vec{x}_i + k_0 \vec{c} \in L$, where $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

and $k_0 \in \mathbb{Z}$, the $\vec{k}$-th secant variety $V(\vec{k}) \subset \operatorname{Sym}^{n-3} X$ is defined by

$$V(\vec{k}) = \begin{cases} z_{i_1} * \cdots * z_{i_r} * \operatorname{Sec}_{k_0}(X) & k_0 \geq 0, \\ \varnothing & k_0 < 0, \end{cases}$$

where X and the marked points $z_k \in X$ are considered as subvarieties of $\operatorname{Sym}^{n-3} X$ by the Veronese embedding. Here, the join $A * B$ of two subvarieties of a projective space is the union $\bigcup_{a \in A,\ b \in B} \ell_{a,b}$ of lines $\ell_{a,b}$ passing through points $a \in A$ and $b \in B$, and the k_0-th secant variety $\operatorname{Sec}_{k_0}(X) = X * \cdots * X$ is the join of k_0 copies of X.

Let $I = \{i_1, \ldots, i_r\}$ be a subset of $\{1, \ldots, n\}$ and let

$$J = \{j_1, \ldots, j_{n-r}\} = \{1, \ldots, n\} \setminus I$$

be its complement. Assume that one has

$$-\sum_{i \in I} \alpha_{+,i} + \sum_{j \in J} \alpha_{+,j} - k < 0 \quad \text{and} \quad -\sum_{i \in I} \alpha_{-,i} + \sum_{j \in J} \alpha_{-,j} - k > 0.$$

If a vector bundle $\mathcal{P}$ admits a non-trivial homomorphism from the line bundle

$$\mathcal{L} = \mathcal{O}\Big(-\vec{s} + \sum_{j \in J} \vec{x}_j - k\vec{c}\Big), \quad \operatorname{par}\deg_{\alpha} \mathcal{L} = -\sum_{i \in I} \alpha_i + \sum_{j \in J} \alpha_j - k,$$

then its saturation destabilizes the bundle $\mathcal{P}$ with respect to the stability parameter $\boldsymbol{\alpha}_-$. Assume that $\mathcal{P}$ is given as an extension

$$0 \to \mathcal{O}(-\vec{s} + \vec{x}_n) \to \mathcal{P} \to \mathcal{O}(-\vec{x}_n) \to 0$$

classified by an element

$$e_{\mathcal{P}} \in \operatorname{Ext}^1(\mathcal{O}(-\vec{x}_n), \mathcal{O}(-\vec{s} + \vec{x}_n)) \cong H^0(\mathcal{O}((n-3)\vec{c}))^\vee,$$

and $\mathcal{O}(-\vec{s}+\vec{x}_n)$ does not destabilize $\mathcal{P}$ with respect to the stability parameter $\boldsymbol{\alpha}_-$. Then one has $\operatorname{Hom}(\mathcal{L}, \mathcal{O}(-\vec{s} + \vec{x}_n)) = 0$ and the $\boldsymbol{\alpha}_-$-destabilizing morphism $\mathcal{L} \to \mathcal{P}$ must come from a non-trivial morphism $\mathcal{L} \to \mathcal{O}(-\vec{x}_n)$. Conversely, a non-trivial morphism $\phi \in \operatorname{Hom}(\mathcal{L}, \mathcal{O}(-\vec{x}_n))$ lifts to a non-trivial morphism $\phi \in \operatorname{Hom}(\mathcal{L}, \mathcal{P})$ if and only if $e_{\mathcal{P}} \circ \phi \in \operatorname{Ext}^1(\mathcal{L}, \mathcal{O}(-\vec{s} + \vec{x}_n))$ vanishes. Under the isomorphisms

$$\operatorname{Hom}(\mathcal{L}, \mathcal{O}(-\vec{x}_n)) = H^0\Big(\mathcal{O}\Big(\sum_{i \in I} \vec{x}_i - \vec{x}_n + k\vec{c}\Big)\Big),$$

$$\operatorname{Ext}^1\big(\mathcal{O}(-\vec{x}_n), \mathcal{O}(-\vec{s} + \vec{x}_n)\big) \cong H^0\big(\mathcal{O}((n-3)\vec{c})\big)^\vee,$$

$$\operatorname{Ext}^1\big(\mathcal{L}, \mathcal{O}(-\vec{s} + \vec{x}_n)\big) \cong H^0\Big(\mathcal{O}\Big((n-3)\vec{c} - \Big(\sum_{i \in I} \vec{x}_i - \vec{x}_n + k\vec{c}\Big)\Big)\Big)^\vee,$$

the Yoneda product

$$\operatorname{Hom}\big(\mathcal{L}, \mathcal{O}(-\vec{x}_n)\big) \otimes \operatorname{Ext}^1\big(\mathcal{O}(-\vec{x}_n), \mathcal{O}(-\vec{s} + \vec{x}_n)\big) \to \operatorname{Ext}^1\big(\mathcal{L}, \mathcal{O}(-\vec{s} + \vec{x}_n)\big)$$

corresponds to the composition

$$H^0\Big(\mathcal{O}\Big((n-3)\vec{c} - \Big(\sum_{i \in I} \vec{x}_i - \vec{x}_n + k\vec{c}\Big)\Big)\Big) \otimes H^0\Big(\mathcal{O}\Big(\sum_{i \in I} \vec{x}_i - \vec{x}_n + k\vec{c}\Big)\Big) \to H^0\big(\mathcal{O}((n-3)\vec{c})\big),$$

so that there is a non-trivial morphism $\mathcal{L} \to \mathcal{P}$ if and only if

$$[e_{\mathcal{P}}] \in \mathbb{P}H^0\big(\mathcal{O}_{\mathbb{X}}(n-3)\big) \cong \operatorname{Sym}^{n-3} X \cong \mathbb{P}^{n-3}$$

belongs to the secant variety $V\left(\sum_{i\in I}\vec{x}_i - \vec{x}_n + k\vec{c}\right)$.

Remark 8.1 Bauer uses a different parametrization of the space of stability parameters, and the stability parameter that he has chosen as the starting point is written as

$$\boldsymbol{\alpha} = \begin{cases} \left(\frac{1}{2n-2}, \frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}\right) & \text{if } n \text{ is even,} \\ \left(\frac{n-2}{2n-2}, \dots, \frac{n-2}{2n-2}\right) & \text{if } n \text{ is odd} \end{cases}$$

in the notation here, which does not satisfy $|\boldsymbol{\alpha}| < 1$. The advantage of this stability parameter is that the underlying bundle of a stable parabolic bundle is always given by

$$\mathcal{E} \cong \begin{cases} \mathcal{O}(-n/2) \oplus \mathcal{O}(-n/2) & \text{if } n \text{ is even,} \\ \mathcal{O}(-(n+1)/2) \oplus \mathcal{O}(-(n-1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

For example, if n is even and the underlying bundle is $\mathcal{O}(-n/2-k) \oplus \mathcal{O}(-n/2+k)$ for some $k > 0$, then the parabolic degree of the subbundle $\mathcal{O}(-n/2+k)$ satisfies

$$\begin{aligned} \operatorname{par}\deg \mathcal{O}(-n/2+k) &\geq \deg \mathcal{O}(-n/2+k) + \sum_{j=1}^{n} \alpha_j \\ &= -n/2 + k + \frac{1}{2n-2} + (n-1)\frac{n-2}{2n-2} \\ &= k - 1 + \frac{1}{2n-2} > 0. \end{aligned}$$

The discussion so far can be summarized as Theorem 8.2, which is a variation of [Bau91, Theorem 2.9]. For the sake of simplicity of the exposition, we restrict ourselves to the case $|\boldsymbol{\alpha}| < 1$, which is the case of interest for the purpose of this paper; this allows us to deal only with walls $H_{I,k}$ with $k = 0$.

Theorem 8.2 *The moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$ for any parameter $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ satisfying $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n < 1$ is described as follows.*

(i) *Assume $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ by reordering the points if necessary. Set $\boldsymbol{\beta}_0 = (r\alpha_1, \dots, r\alpha_{n-1}, \alpha_n)$ for a sufficiently small positive number r, so that $\boldsymbol{\beta}_0$ belongs to the chamber described in Proposition 6.1 and one has $\mathcal{N}_{\boldsymbol{\beta}_0} \cong \operatorname{Sym}^{n-3} X \cong \mathbb{P}^{n-3}$.*

(ii) *We first cross walls of the form $H_{\{i,n\},0}$ for $1 \leq i \leq n-1$ satisfying*

$$|\boldsymbol{\alpha}| - 2\alpha_i - 2\alpha_n > 0. \tag{9}$$

When we cross the wall $H_{\{i,n\},0}$, the moduli space is blown-up at the point $z_i \in X \subset \operatorname{Sym}^{n-3} X \cong \mathbb{P}^{n-3}$. After crossing all these walls, we arrive at the stability parameter $\boldsymbol{\beta}_1$ such that $\mathcal{N}_{\boldsymbol{\beta}_1}$ is obtained from $\mathcal{N}_{\boldsymbol{\beta}_0}$ by blowing up the points z_i for $1 \leq i \leq n-1$ satisfying (9).

(iii) *We then cross walls of the form $H_{\{i_1,i_2,n\},0}$ for $1 \leq i_1 < i_2 \leq n-1$ satisfying*

$$|\boldsymbol{\alpha}| - 2\alpha_{i_1} - 2\alpha_{i_1} - 2\alpha_n > 0.$$

When we cross the wall $H_{\{i_1,i_2,n\},0}$, the moduli space is blown-down along the strict transform of the line passing through z_{i_1} and z_{i_2}, and then blown-up in the other direction so that the exceptional divisor is isomorphic to $\mathbb{P}^{n-5}$. In other words, we blow-up *the moduli space along the strict transform of the line passing through z_{i_1} and z_{i_2}, and contract it down in the other direction.*

(iv) *In the r-th step, we cross the walls $H_{\{i_1,\dots,i_r,n\},0}$ for $1 \le i_1 < \cdots < i_r \le n-1$ satisfying*

$$|\boldsymbol{\alpha}| - 2\alpha_{i_1} - \cdots - 2\alpha_{i_r} - 2\alpha_n > 0.$$

Note that this condition can be written as

$$\alpha_{i_1} + \cdots + \alpha_{i_r} + \alpha_n < \alpha_{j_1} + \cdots + \alpha_{j_{n-r-1}}$$

where $\{j_1,\dots,j_{n-r-1}\}$ is the complement of $\{i_1,\dots,i_r,n\}$ in $\{1,\dots,n\}$. When we cross the wall $H_{\{i_1,\dots,i_r,n\},0}$, the moduli space is blown-up along the strict transform of the $(r-1)$-dimensional linear subspace spanned by $z_{i_1},\dots,z_{i_r}$, and then contracted in the other direction. This is a birational transformation that replaces $\mathbb{P}^{r-1}$ with $\mathbb{P}^{n-r-4}$.

(v) *By successively crossing the walls as above, we arrive at the chamber containing $\boldsymbol{\alpha}$.*

9 Wall Crossing in $\mathcal{M}_{\boldsymbol{w}}$

Let $\boldsymbol{w} = (w_1,\dots,w_n) \in \mathbb{Q}^n$ be a stability parameter for the moduli space of ordered n-points on $\mathbb{P}^1$, which can be taken from

$$W = \Big\{ \boldsymbol{w} = (w_1,\dots,w_n) \in \mathbb{Q}^n \;\Big|\; |\boldsymbol{w}| = w_1 + \cdots + w_n = 2 \Big\}$$

by rescaling $\boldsymbol{w}$ if necessary; unlike the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}$, the overall rescaling of $\boldsymbol{w}$ only changes the ample $\mathbb{Q}$-line bundle on $\mathcal{M}_{\boldsymbol{w}}$ and does not affect the moduli space $\mathcal{M}_{\boldsymbol{w}}$. A configuration $(x_1,\dots,x_n)$ of ordered n points on $\mathbb{P}^1$ is $\boldsymbol{w}$-semistable if for any point $x \in \mathbb{P}^1$, one has

$$\sum_{i=1}^{n} \delta_{x,x_i} w_i \le 1.$$

The moduli space $\mathcal{M}_{\boldsymbol{w}}$ contains the configuration space

$$X(2,n) = \big((\mathbb{P}^1)^n \setminus \Delta \big) / PGL_2$$

of n points on $\mathbb{P}^1$ as an open subscheme if and only if $\boldsymbol{w} \in (0,1)^n$, where

$$\Delta = \big\{ (x_1,\dots,x_n) \in (\mathbb{P}^1)^n \mid x_i = x_j \text{ for some } i \ne j \big\}$$

is the big diagonal. Normalizing the last three points as $(x_{n-2}, x_{n-1}, x_n) = (0,1,\infty)$ by the PGL_2-action, one can realize $X(2,n)$ as an open subscheme

$$X(2,n) \cong \big\{ [x_1 : \cdots : x_{n-3} : 1] \in \mathbb{P}^{n-3} \mid x_i \ne 0, 1, x_j \text{ for } i \ne j \big\},$$

which is the complement of a hyperplane arrangement in $\mathbb{P}^{n-3}$.

Walls in the space W of stability parameters are given by

$$H_I = \Big\{ \boldsymbol{w} \in W \;\Big|\; \sum_{i \in I} w_i = 1 \Big\}$$

for a proper subset $I = \{i_1,\dots,i_r\}$ of $\{1,\dots,n\}$. Note that $\sum_{i\in I} w_i = 1$ implies $\sum_{j \in J} w_j = 1$ for $J = \{j_1,\dots,j_{n-r}\} = \{1,\dots,n\} \setminus I$. Let C_+ and C_- be two chambers separated by the wall W_I, and take stability conditions $\boldsymbol{w}_+ \in C_+$, $\boldsymbol{w}_- \in C_-$, and

$\boldsymbol{w}_0 \in W_I$. Assume that $\sum_{i \in I} w_{i,+} > 1$, $\sum_{i \in I} w_{i,-} < 1$, and $\boldsymbol{w}_0$ is not on any other walls. Then one has a diagram

$$\begin{tikzcd} \mathcal{M}_{\boldsymbol{w}_+} \arrow[dr, "\phi_+"'] & & \mathcal{M}_{\boldsymbol{w}_-} \arrow[dl, "\phi_-"] \\ & \mathcal{M}_{\boldsymbol{w}_0} & \end{tikzcd} \tag{10}$$

where ϕ_+ blows-down the subvariety

$$S_{\boldsymbol{w}_+} = \{x_{j_1} = \cdots = x_{j_{n-r}}\} \cong \mathbb{P}^{r-2}$$

of $\mathcal{M}_{\boldsymbol{w}_+}$ to the subvariety

$$S_{\boldsymbol{w}_0} = \left\{x_{i_1} = \cdots = x_{i_r},\ x_{j_1} = \cdots = x_{j_{n-r}}\right\}$$

of $\mathcal{M}_{\boldsymbol{w}_0}$ consisting of just one point, and ϕ_- blows-down the subvariety

$$S_{\boldsymbol{w}_-} = \{x_{i_1} = \cdots = x_{i_r}\} \cong \mathbb{P}^{n-r-2}$$

of $\mathcal{M}_{\boldsymbol{w}_-}$ to the same point in $\mathcal{M}_{\boldsymbol{w}_0}$.

The diagram (10) for the special case $I = \{n\}$ gives a wall-crossing from the empty space $\mathcal{M}_{\boldsymbol{w}_+} = S_{\boldsymbol{w}_+} \cong \mathbb{P}^{-1} = \varnothing$ to the projective space $\mathcal{M}_{\boldsymbol{w}_-} = S_{\boldsymbol{w}_-} \cong \mathbb{P}^{n-3}$ through one point $\mathcal{M}_{\boldsymbol{w}_0} = S_{\boldsymbol{w}_0}$. The chamber C_- containing $\boldsymbol{w}_-$ in this case is defined by

$$C_- = \{\boldsymbol{w} \in W \mid w_n < 1 \text{ and } w_i + w_n > 1 \text{ for any } 1 \le i \le n-1\}. \tag{11}$$

The moduli space $\mathcal{M}_{\boldsymbol{w}}$ for $\boldsymbol{w} \in C_-$ is described explicitly as follows. One can set $x_n = \infty \in \mathbb{P}^1$ by the PGL_2-action. Since one has $x_i \neq x_n$ for any $1 \le i \le n-1$ by the stability condition, one must have $(x_1, \dots, x_{n-1}) \in \mathbb{A}^{n-1}$. One can set $x_{n-1} = 0$ by the residual PGL_2-action, and then one is left with the $\mathbb{G}_m$-action on $\mathbb{A}^{n-2}$. The stability condition prohibits $x_1 = \cdots = x_{n-1}$, so that one cannot have $x_1 = \cdots = x_{n-2} = 0$. This shows that one has

$$\mathcal{M}_{\boldsymbol{w}} = \left(\mathbb{A}^{n-2} \setminus \{0\}\right)/\mathbb{G}_m,$$

which is nothing but the projective space $\mathbb{P}^{n-3}$.

Theorem 9.1 *The moduli space $\mathcal{M}_{\boldsymbol{w}}$ for any stability parameter $\boldsymbol{w} = (w_1, \dots, w_n)$ can be obtained from $\mathbb{P}^{n-3}$ by the following birational transformations: Assume $w_1 \le w_2 \le \cdots \le w_n$ by reordering the points if necessary. We start from the chamber* (11) *and gradually increase $w_1, \dots, w_{n-1}$ and decrease w_n. Set $p_i = [\delta_{i0} : \cdots : \delta_{i,n-2}] \in \mathbb{P}^{n-3}$ for $1 \le i \le n-2$ and $p_{n-1} = [1 : \cdots : 1] \in \mathbb{P}^{n-3}$.*

(i) *We first cross the walls $H_{\{i,n\}}$ for $1 \le i \le n-1$ satisfying $w_i + w_n < 1$. When we cross the wall $H_{\{i,n\}}$, the moduli space is blown-up at the point p_i.*

(ii) *We then cross the walls $H_{\{i_1,i_2,n\}}$ for $1 \le i_1 < i_2 \le n-1$ satisfying $w_{i_1} + w_{i_1} + w_n < 1$. When we cross the wall $H_{\{i_1,i_2,n\}}$, the moduli space is blown-down along the strict transform of the line passing through p_{i_1} and p_{i_2}, and then blown-up in the other direction, so that the exceptional divisor is isomorphic to $\mathbb{P}^{n-5}$. In other words, we blow-up the moduli space along the strict transform of the line passing through p_{i_1} and p_{i_2}, and contract it down in the other direction.*

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

(iii) *In the r-th step, we cross the walls* $H_{\{i_1,\dots,i_r,n\}}$ *for* $1 \le i_1 < \dots < i_r \le n-1$ *satisfying* $w_{i_1} + \dots + w_{i_r} + w_n < 1$*. Note that this condition is equivalent to* $w_{i_1} + \dots + w_{i_r} + w_n < w_{j_1} + \dots + w_{j_{n-r-1}}$*, where* $\{j_1, \dots, j_{n-r-1}\}$ *is the complement of* $\{i_1, \dots, i_r, n\}$ *in* $\{1, \dots, n\}$*. When we cross the wall* $H_{\{i_1,\dots,i_r,n\}}$*, the moduli space is blown-up along the strict transform of the* $(r-1)$*-dimensional linear subspace spanned by* $p_{i_1}, \dots, p_{i_r}$*, and then contracted down in the other direction. This is a birational transformation which replaces* $\mathbb{P}^{r-1}$ *with* $\mathbb{P}^{n-r-4}$*.*

(iv) *By successively crossing the walls as above, we arrive at the chamber containing* $\boldsymbol{w}$*.*

Example 9.2 Set $n = 5$ and

$$\boldsymbol{w}_1 = \Big(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}\Big).$$

We consider the straight line segment $\boldsymbol{w}_t = (1-t)\boldsymbol{w}_0 + t\boldsymbol{w}_1$ starting from the stability parameter

$$\boldsymbol{w}_0 = \Big(\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{3}{4}\Big)$$

in the chamber (11) satisfying $\mathcal{M}_{\boldsymbol{w}_0} \cong \mathbb{P}^2$. The wall-crossing takes place at $t = \frac{5}{21}$ and

$$\boldsymbol{w}_t = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\Big),$$

where the points $x_i = x_j = x_k$ for $1 \le i < j < k \le 4$ are stable for $t \le \frac{5}{21}$ and unstable for $t > \frac{5}{21}$. These points are blown-up by the wall-crossing, so that the point $x_i = x_j = x_k$ is replaced by the exceptional divisor $x_\ell = x_5$ where $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. With respect to the normalization

$$(x_1, x_2, x_3, x_4, x_5) = (x, y, 1, 0, \infty),$$

four points at the center of the blow-up are given by

$$\begin{aligned} x_1 = x_2 = x_3&: \quad [x:y:1] = [1:1:1] \in \mathbb{P}^2, \\ x_1 = x_2 = x_4&: \quad [x:y:1] = [0:0:1] \in \mathbb{P}^2, \\ x_1 = x_3 = x_4&: \quad [x:y:1] = [0:1:0] \in \mathbb{P}^2, \\ x_2 = x_3 = x_4&: \quad [x:y:1] = [1:0:0] \in \mathbb{P}^2. \end{aligned}$$

These points are in general position, so that $\mathcal{M}_{\boldsymbol{w}_1}$ is $\mathbb{P}^2$ blown-up at four points in general position.

We are now ready to prove the following theorem.

Theorem 9.3 *Let* $\boldsymbol{\alpha}$ *be a stability parameter for the moduli space of parabolic bundles satisfying* $|\boldsymbol{\alpha}| < 1$*, and let* $\boldsymbol{w} = 2\boldsymbol{\alpha}/|\boldsymbol{\alpha}|$ *be the corresponding normalized stability parameter for the moduli space ordered n points on* $\mathbb{P}^1$*. Then one has an isomorphism* $\mathcal{N}_{\boldsymbol{\alpha}} \cong \mathcal{M}_{\boldsymbol{w}}$ *of algebraic varieties.*

Proof Since the wall-crossing in $\mathcal{N}_{\boldsymbol{\alpha}}$ and $\mathcal{M}_{\boldsymbol{w}}$ described in Theorems 8.2 and 9.1 are identical, it suffices to show the existence of an isomorphism $\mathcal{N}_{\boldsymbol{\alpha}} \cong \mathcal{M}_{\boldsymbol{w}}$ for a stability

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

parameter $\boldsymbol{\alpha}$ satisfying the condition in Proposition 6.1, such that the points $z_i \in \mathcal{N}_{\boldsymbol{\alpha}}$ are mapped to $p_i \in \mathcal{M}_{\boldsymbol{w}}$ for $i = 1, \dots, n-1$. This is clear, since both moduli spaces are $(n-3)$-dimensional projective spaces, and the points are $n-1$ points in general position. ∎

A more general result, which gives an isomorphism between the moduli space of parabolic G-bundles for a simply-connected simple algebraic group G and a GIT quotient of a product of flag varieties, is shown in [Man, Proposition 4.8].

10 Bending Systems on $\mathcal{M}_{\boldsymbol{w}}$

Let $G = SU(2)$, and identify the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ with its dual by the Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Let $T \subset G$ be the maximal torus consisting of diagonal matrices, and take a base

$$x_0 = \begin{pmatrix} 2\pi\sqrt{-1} & 0 \\ 0 & -2\pi\sqrt{-1} \end{pmatrix}$$

of the Lie algebra $\mathfrak{t}$ of T. For $\alpha \in \mathbb{R}_{>0}$, the adjoint orbit $\mathcal{O}_\alpha \subset \mathfrak{g}$ of αx_0 has a natural symplectic form called the Kostant–Kirillov form, as follows. Recall that a tangent vector of $\mathcal{O}_\alpha$ at x can be written as $\operatorname{ad}_\xi(x) = [x, \xi]$ for $\xi \in \mathfrak{g}$. The Kostant–Kirillov form $\omega_{\mathcal{O}_\alpha}$ is given by

$$\omega_{\mathcal{O}_\alpha}\big(\operatorname{ad}_\xi(x), \operatorname{ad}_\eta(x)\big) = \big\langle x, [\xi, \eta] \big\rangle.$$

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}_{>0})^n$, we define $\mathcal{O}_{\boldsymbol{\alpha}} = \prod_i \mathcal{O}_{\alpha_i} \subset \mathfrak{g}^n$ with the i-th projection $\operatorname{pr}_i : \mathcal{O}_{\boldsymbol{\alpha}} \to \mathcal{O}_{\alpha_i}$, $i = 1, \dots, n$. The diagonal G-action on $\mathcal{O}_{\boldsymbol{\alpha}}$ is Hamiltonian with respect to the symplectic form $\sum_i \operatorname{pr}_i^* \omega_{\mathcal{O}_{\alpha_i}}$, and its moment map is given by

$$\mu : \mathcal{O}_{\boldsymbol{\alpha}} \longrightarrow \mathfrak{g}, \quad \boldsymbol{x} = (x_1, \dots, x_n) \longmapsto x_1 + \dots + x_n.$$

From the Kirwan–Kempf–Ness Theorem, the symplectic reduction

$$\mu^{-1}(0)/G = \{\boldsymbol{x} \in \mathcal{O}_{\boldsymbol{\alpha}} \mid x_1 + \dots + x_n = 0\}/G \tag{12}$$

is diffeomorphic to $\mathcal{M}_{\boldsymbol{w}}$ for $\boldsymbol{w} = 2\boldsymbol{\alpha}/|\boldsymbol{\alpha}|$, and the induced symplectic form is compatible with the complex structure (on the smooth locus of $\mathcal{M}_{\boldsymbol{w}}$). In what follows we write this space as $\mathcal{M}_{\boldsymbol{\alpha}}$ to emphasize its symplectic structure $\omega_{\mathcal{M}_{\boldsymbol{\alpha}}}$. Note that $(\mathcal{M}_{k\boldsymbol{\alpha}}, \omega_{\mathcal{M}_{k\boldsymbol{\alpha}}})$ is symplectomorphic to $(\mathcal{M}_{\boldsymbol{\alpha}}, k\omega_{\mathcal{M}_{\boldsymbol{\alpha}}})$ for $k > 0$. The expression (12) shows that $\mathcal{M}_{\boldsymbol{\alpha}}$ parametrizes n-gons in $\mathfrak{g} \cong \mathbb{R}^3$ with fixed side lengths $\alpha_1, \dots, \alpha_n$ modulo Euclidean motions.

Let $e_1, \dots, e_n \in \mathbb{R}^2$ denote side edge vectors of a reference n-gon $P \subset \mathbb{R}^2$, satisfying $e_1 + \dots + e_n = 0$. For a diagonal $d = e_i + e_{i+1} + \dots + e_{i+k}$ of P, we define $\phi_d : \mathcal{M}_{\boldsymbol{\alpha}} \to \mathbb{R}$ as the length function

$$\phi_d(\boldsymbol{x}) = |x_i + x_{i+1} + \dots + x_{i+k}|$$

of the corresponding diagonal in $\boldsymbol{x}$. This function is called a *bending Hamiltonian*, since the Hamiltonian flow of ϕ_d bends n-gons around the diagonal corresponding to d (see [KM96] or [Kly94]).

We fix a triangulation of P given by $n-3$ diagonals $d_1, \dots, d_{n-3}$ that do not intersect in the interior of P, and let Γ denote its dual graph. Note that Γ is a trivalent tree with

n leaves. The bending system associated with Γ is defined by

$$\Phi_\Gamma = (\phi_{d_1}, \dots, \phi_{d_{n-3}}) : \mathcal{M}_{\boldsymbol{\alpha}} \longrightarrow \mathbb{R}^{n-3}.$$

Theorem 10.1 (Kapovich & Millson [KM96], Klyachko [Kly94]) *The $(n-3)$-tuple of functions Φ_Γ is a completely integrable system on $\mathcal{M}_{\boldsymbol{\alpha}}$. The functions ϕ_{d_i} are action variables, and hence define a Hamiltonian torus action on an open dense subset where ϕ_{d_i} are smooth. The image*

$$\Delta_\Gamma(\boldsymbol{\alpha}) := \Phi_\Gamma(\mathcal{M}_{\boldsymbol{\alpha}}) \subset \mathbb{R}^{n-3}$$

is a convex polytope defined by triangle inequalities.

11 Goldman Systems on $\mathcal{N}_{\boldsymbol{\alpha}}$

Let $(X, (z_1, \dots, z_n))$ be a projective line with n marked points. For each marked point $z_i \in X$, we take a small open disk $D_i \subset X$ around z_i such that $\overline{D_i} \cap \overline{D_j} = \varnothing$ for $i \neq j$, and set $\Sigma = X \smallsetminus (D_1 \cup \dots \cup D_n)$. Then the fundamental group of Σ is given by

$$\pi_1(\Sigma) = \langle \gamma_1, \gamma_2, \dots, \gamma_n \mid \gamma_1 \dots \gamma_n = 1 \rangle,$$

where γ_i is the homotopy class representing the i-th boundary component ∂D_i.

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 1/2)^n$, let $\mathcal{C}_{\alpha_i} \subset G$ denote the conjugacy class of $e^{\alpha_i x_0} = \operatorname{diag}(e^{2\pi\sqrt{-1}\alpha_i}, e^{-2\pi\sqrt{-1}\alpha_i})$, and set $\mathcal{C}_{\boldsymbol{\alpha}} = \prod_{i=1}^n \mathcal{C}_{\alpha_i} \subset G^n$. As recalled in Section 5, the moduli space of parabolic $SU(2)$-bundles on X with parabolic weight $\boldsymbol{\alpha}$ can be identified with the moduli space

$$\begin{aligned} \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) &:= \{\rho \in \operatorname{Hom}(\pi_1(\Sigma), G) \mid \rho(\gamma_i) \in \mathcal{C}_{\alpha_i},\ i = 1, \dots, n\}/G \\ &\cong \{\boldsymbol{g} = (g_1, \dots, g_n) \in \mathcal{C}_{\boldsymbol{\alpha}} \mid g_1 \dots g_n = 1\}/G \end{aligned}$$

of G-representations of $\pi_1(\Sigma)$. Since $\mathcal{C}_{\alpha_i}$ is a geodesic sphere around the identity, $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is regarded as a moduli space of n-gons in $G \cong S^3$ with fixed side lengths (*cf. e.g.*, [MP01]).

We recall the description of the symplectic structure on $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ from [GHJW97]. Fix a representation ρ in

$$\widetilde{\mathcal{N}}_{\boldsymbol{\alpha}} = \{\rho \in \operatorname{Hom}(\pi_1(\Sigma), G) \mid \rho(\gamma_i) \in \mathcal{C}_{\alpha_i},\ i = 1, \dots, n\}$$

and let $\mathfrak{g}_\rho$ denote the representation of $\pi_1(\Sigma)$ on $\mathfrak{g}$ given by

$$\pi_1(\Sigma) \xrightarrow{\rho} G \xrightarrow{\operatorname{Ad}} \operatorname{Aut}(\mathfrak{g}).$$

Take a curve ρ_t in $\widetilde{\mathcal{N}}_{\boldsymbol{\alpha}}$ with $\rho_0 = \rho$ and set $u = \frac{d}{dt}\big|_{t=0} \rho_t : \pi_1(\Sigma) \to \mathfrak{g}$. Then ρ_t can be written as

$$\rho_t(\gamma) = \exp\big(tu(\gamma) + O(t^2)\big)\rho(\gamma).$$

The homomorphism condition $\rho_t(\gamma\gamma') = \rho_t(\gamma)\rho_t(\gamma')$ implies that

$$u(\gamma\gamma') = u(\gamma) + \operatorname{Ad}_{\rho(\gamma)} u(\gamma'). \tag{13}$$

From the boundary condition $\rho_t(\gamma_i) \in \mathcal{C}_{\alpha_i}$, we have $\rho_t(\gamma_i) = g_{i,t}^{-1}\rho(\gamma_i)g_{i,t}$ for some $g_{i,t} \in G$. This implies that

$$u(\gamma_i) = \operatorname{Ad}_{\rho(\gamma_i)} \xi_i - \xi_i \tag{14}$$

for each i, where $\xi_i = \frac{d}{dt}\big|_{t=0} g_{i,t} \in \mathfrak{g}$. Namely, $T_\rho \widetilde{\mathcal{N}}_{\boldsymbol{\alpha}}$ is identified with the space of *parabolic 1-cocycles*

$$\begin{aligned} T_\rho \widetilde{\mathcal{N}}_{\boldsymbol{\alpha}} &\cong Z^1_{\mathrm{par}}(\pi_1(\Sigma); \mathfrak{g}_\rho) \\ &= \{u\colon \pi_1(\Sigma) \to \mathfrak{g} \mid u \text{ satisfies (13) and (14)}\}. \end{aligned}$$

Similarly, the tangent space to the G-orbit of ρ is spanned by *parabolic 1-coboundaries*

$$u(\gamma) = \mathrm{Ad}_{\rho(\gamma)}\, \xi - \xi, \qquad \xi \in \mathfrak{g}.$$

Let $B^1_{\mathrm{par}}(\pi_1(\Sigma); \mathfrak{g}_\rho)$ denote the vector space of parabolic 1-coboundaries. Then the tangent space $T_\rho \mathcal{N}_{\boldsymbol{\alpha}}$ at ρ is identified with the *first parabolic cohomology*

$$H^1_{\mathrm{par}}\big(\pi_1(\Sigma); \mathfrak{g}_\rho\big) = Z^1_{\mathrm{par}}\big(\pi_1(\Sigma); \mathfrak{g}_\rho\big) / B^1_{\mathrm{par}}\big(\pi_1(\Sigma); \mathfrak{g}_\rho\big).$$

The space of 2-chains $C_2(\pi_1(\Sigma); \mathbb{Z})$ is generated by symbols $[\gamma|\gamma']$ for $\gamma, \gamma' \in \pi_1(\Sigma)$, and the cup product

$$\cup\colon H^1_{\mathrm{par}}\big(\pi_1(\Sigma); \mathfrak{g}_\rho\big) \times H^1_{\mathrm{par}}\big(\pi_1(\Sigma); \mathfrak{g}_\rho\big) \longrightarrow H^2\big(\pi_1(\Sigma), \partial\pi_1(\Sigma); \mathbb{R}\big)$$

is given by

$$(u \cup v)\big([\gamma \mid \gamma']\big) = \big\langle u(\gamma), \mathrm{Ad}_{\rho(\gamma)}\, v(\gamma')\big\rangle$$

for 1-cocycles u, v. In what follows we write $\mathrm{Ad}_\gamma = \mathrm{Ad}_{\rho(\gamma)}$ for short. The relative fundamental class in $H_2(\pi_1(\Sigma), \partial\pi_1(\Sigma); \mathbb{Z})$ is represented by

$$\big[\pi_1(\Sigma), \partial\pi_1(\Sigma)\big] = \sum_{i=1}^{n-1} [\gamma_1 \dots \gamma_i \mid \gamma_{i+1}].$$

Theorem 11.1 (Guruprasad et al. [GHJW97, Key Lemma 8.4]) *Let u, v be parabolic 1-cocycles such that $u(\gamma_i) = \mathrm{Ad}_{\gamma_i}\, \xi_i - \xi_i$ and $v(\gamma_i) = \mathrm{Ad}_{\gamma_i}\, \eta_i - \eta_i$, $i = 1, \dots, n$, respectively. Then the symplectic form on $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is given by*

$$\omega_{\mathcal{N}_{\boldsymbol{\alpha}}}(u, v) = (u \cup v)\big([\pi_1, \partial\pi_1]\big) + \frac{1}{2} \sum_{i=1}^{n} \big(\langle \xi_i, \mathrm{Ad}_{\gamma_i}\, \eta_i\rangle - \langle \eta_i, \mathrm{Ad}_{\gamma_i}\, \xi_i\rangle\big). \tag{15}$$

For a later use, we write the first term of (15) more explicitly. By using (13) inductively, we have

$$u(\gamma_1 \dots \gamma_i) = \sum_{k=1}^{i} \mathrm{Ad}_{\gamma_1 \dots \gamma_{k-1}}\, u(\gamma_k) = \sum_{k=1}^{i} \mathrm{Ad}_{\gamma_1 \dots \gamma_{k-1}} (\mathrm{Ad}_{\gamma_k}\, \xi_k - \xi_k).$$

Hence, we obtain

$$\begin{aligned} (u \cup v)([\pi_1, \partial\pi_1]) &= \sum_{i=1}^{n-1} \big\langle u(\gamma_1 \dots \gamma_i), \mathrm{Ad}_{\gamma_1 \dots \gamma_i}\, v(\gamma_{i+1})\big\rangle \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{i} \big\langle \mathrm{Ad}_{\gamma_1 \dots \gamma_{k-1}}\, u(\gamma_k), \mathrm{Ad}_{\gamma_1 \dots \gamma_i}\, v(\gamma_{i+1})\big\rangle \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{i} \big\langle u(\gamma_k), \mathrm{Ad}_{\gamma_k \dots \gamma_i}\, v(\gamma_{i+1})\big\rangle \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{i} \big\langle \mathrm{Ad}_{\gamma_k}\, \xi_k - \xi_k, \mathrm{Ad}_{\gamma_k \dots \gamma_i} (\mathrm{Ad}_{\gamma_{i+1}}\, \eta_{i+1} - \eta_{i+1})\big\rangle. \end{aligned}$$

Next we recall a completely integrable system on $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ introduced by Goldman [Gol86]. For a simple closed curve $C \subset \Sigma$, we write $[C] = \gamma_i \gamma_{i+1} \dots \gamma_{i+k}$ in $\pi_1(\Sigma)$, and define a function $\vartheta_C = \theta_{\boldsymbol{\alpha},C} : \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) \to \mathbb{R}$ by

$$\vartheta_C(\boldsymbol{g}) = \cos^{-1}\Big(\frac{1}{2}\operatorname{tr}(g_i g_{i+1} \dots g_{i+k})\Big).$$

Take a set $C_1, \dots, C_{n-3}$ of simple closed curves defining a pair-of-pants decomposition of Σ. Note that the set of such choices is in one-to-one correspondence with the set of trivalent trees Γ with n-leaves. We then obtain a set of $n-3$ functions

$$\Theta_{\boldsymbol{\alpha},\Gamma} = \Theta_\Gamma = (\vartheta_{C_1}, \dots, \vartheta_{C_{n-3}}) : \mathcal{N}_{\boldsymbol{\alpha}} \longrightarrow \mathbb{R}^{n-3}.$$

Theorem 11.2 (Goldman [Gol86], Jeffrey & Weitsman [JW92]) *For each pair-of-pants decomposition of Σ with dual graph Γ, the set of functions $\Theta_\Gamma : \mathcal{N}_{\boldsymbol{\alpha}} \to \mathbb{R}^{n-3}$ is a completely integrable system. The functions ϑ_{C_i} are action variables, and hence define a Hamiltonian torus action on an open dense subset of $\mathcal{N}_{\boldsymbol{\alpha}}$. The image $\Theta_\Gamma(\mathcal{N}_{\boldsymbol{\alpha}}) \subset \mathbb{R}^{n-3}$ is a convex polytope given by the inequalities*

$$|u_{k_1} - u_{k_2}| \le u_{k_3} \le \min\{u_{k_1} + u_{k_2}, 2 - (u_{k_1} + u_{k_2})\}$$

for each pair-of-pants. In particular, if $|\boldsymbol{\alpha}| < 1$, then the image is given by triangle inequalities, i.e., $\Theta_\Gamma(\mathcal{N}_{\boldsymbol{\alpha}}) = \Delta_\Gamma(\boldsymbol{\alpha})$.

12 Extended Moduli Spaces

Fix base points of ∂D_i for $i = 1, \dots, n$. Let B_i for $i = 1, \dots, n$ be the loop around ∂D_i starting and ending at the base point on ∂D_i, and A_i for $i = 2, \dots, n$ be the path from the base point on ∂D_i to the base point on ∂D_1. Then the generators of $\pi_1(\Sigma)$ are given by $\gamma_1 = [B_1], \gamma_2 = [A_2 B_2 A_2^{-1}], \dots, \gamma_n = [A_n B_n A_n^{-1}]$. Let

$$A_{\mathfrak{t}} = \{\alpha x_0 \in \mathfrak{t} \mid \alpha \in [0, 1/2]\} \subset \mathfrak{t}$$

denote the fundamental alcove.

Definition 12.1 (Jeffrey [Jef94], Hurtubise & Jeffrey [HJ00, Section 2]) The *G-extended moduli space* $\mathcal{N}^G(\Sigma)$ is the space of G-representations of the groupoid generated by $A_2, \dots, A_n$ and $B_1, \dots, B_n$, or equivalently,

$$\mathcal{N}^G(\Sigma) = \big\{ (\boldsymbol{a}, \boldsymbol{b}) \in G^{n-1} \times G^n \mid b_1(a_2 b_2 a_2^{-1}) \dots (a_n b_n a_n^{-1}) = 1 \big\},$$

where $(\boldsymbol{a}, \boldsymbol{b}) = (a_2, \dots, a_n, b_1, \dots, b_n)$. The *$T$-extended moduli space* is defined by

$$\mathcal{N}^T(\Sigma) = \big\{ (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N}^G(\Sigma) \mid b_i \in \exp(A_{\mathfrak{t}}), i = 1, \dots, n \big\} \subset G^{n-1} \times T^n.$$

The $\mathfrak{g}$- and $\mathfrak{t}$-*extended moduli spaces* are defined by

$$\mathcal{N}^{\mathfrak{g}}(\Sigma) = \{(\boldsymbol{a}, \boldsymbol{x}) \in G^{n-1} \times \mathfrak{g}^n \mid e^{x_1}(a_2 e^{x_2} a_2^{-1}) \dots (a_n e^{x_n} a_n^{-1}) = 1\},$$
$$\mathcal{N}^{\mathfrak{t}}(\Sigma) = \{(\boldsymbol{a}, \boldsymbol{x}) \in G^{n-1} \times \mathfrak{t}^n \mid e^{x_1}(a_2 e^{x_2} a_2^{-1}) \dots (a_n e^{x_n} a_n^{-1}) = 1\},$$

respectively, where $(\boldsymbol{a}, \boldsymbol{x}) = (a_2, \dots, a_n, x_1, \dots, x_n)$.

Each a_i and b_i are regarded as holonomies of a flat parabolic connection along A_i and B_i, respectively. Note that we have a natural surjection $\mathcal{N}^{\mathfrak{g}}(\Sigma) \to \mathcal{N}^G(\Sigma)$ given by

$$(a_2, \ldots, a_n, x_1, \ldots, x_n) \longmapsto (a_2, \ldots, a_n, e^{x_1}, \ldots, e^{x_n}).$$

On the other hand, $\mathcal{N}^T(\Sigma)$ is canonically embedded into $\mathcal{N}^{\mathfrak{t}}(\Sigma)$ by

$$(a_2, \ldots, a_n, e^{x_1}, \ldots, e^{x_n}) \longmapsto (a_2, \ldots, a_n, x_1, \ldots, x_n).$$

Proposition 12.2 ([HJ00, Propositions 2.11 and 2.12]) *The space $\mathcal{N}^G(\Sigma)$ is diffeomorphic to $G^{2(n-1)}$ by*

$$\mathcal{N}^G(\Sigma) \to G^{2(n-1)}, \quad (a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) \longmapsto (a_2, \ldots, a_n, b_2, \ldots, b_n),$$

and hence it is smooth. On the other hand, $\mathcal{N}^{\mathfrak{g}}(\Sigma)$ is smooth outside the subset consisting of $(\boldsymbol{a}, \boldsymbol{x})$ satisfying $e^{x_i} = -1$ for all i.

The group G^n acts on $\mathcal{N}^G(\Sigma)$ and $\mathcal{N}^{\mathfrak{g}}(\Sigma)$ by

$$\boldsymbol{\sigma} \cdot (\boldsymbol{a}, \boldsymbol{b}) = (\sigma_1 a_2 \sigma_2^{-1}, \ldots, \sigma_1 a_n \sigma_n^{-1}, \sigma_1 b_1 \sigma_1^{-1}, \ldots, \sigma_n b_n \sigma_n^{-1}),$$
$$\boldsymbol{\sigma} \cdot (\boldsymbol{a}, \boldsymbol{x}) = (\sigma_1 a_2 \sigma_2^{-1}, \ldots, \sigma_1 a_n \sigma_n^{-1}, \mathrm{Ad}_{\sigma_1} x_1, \ldots, \mathrm{Ad}_{\sigma_n} x_n),$$

for $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$. These actions induce T^n-actions

$$\boldsymbol{\sigma} \cdot (\boldsymbol{a}, \boldsymbol{b}) = (\sigma_1 a_2 \sigma_2^{-1}, \ldots, \sigma_1 a_n \sigma_n^{-1}, b_1, \ldots, b_n),$$
$$\boldsymbol{\sigma} \cdot (\boldsymbol{a}, \boldsymbol{x}) = (\sigma_1 a_2 \sigma_2^{-1}, \ldots, \sigma_1 a_n \sigma_n^{-1}, x_1, \ldots, x_n)$$

on $\mathcal{N}^T(\Sigma)$ and $\mathcal{N}^{\mathfrak{t}}(\Sigma)$, respectively.

Proposition 12.3 ([Jef94], [HJ00, Proposition 2.14]) *There exists a closed two-form on $\mathcal{N}^{\mathfrak{g}}(\Sigma)$ that is non-degenerate on an open dense subset, and for which the map*

$$\mu^{\mathfrak{g}} : \mathcal{N}^{\mathfrak{g}}(\Sigma) \longrightarrow \mathfrak{g}^n, \quad (\boldsymbol{a}, \boldsymbol{x}) \longmapsto -\boldsymbol{x} = (-x_1, \ldots, -x_n)$$

is the moment map of the G^n-action. The symplectic reduction $(\mu^{\mathfrak{g}})^{-1}(\mathcal{O}_{\boldsymbol{\alpha}})/G^n$ is symplectomorphic to $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$.

On the other hand, $\mathcal{N}^G(\Sigma)$ admits a structure of quasi-Hamiltonian G^n-space. We briefly recall the notion of quasi-Hamiltonian spaces introduced by Alekseev, Malkin and Meinrenken [AMM98].

Given a compact connected Lie group K with an invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{k}$, let θ (resp. $\overline{\theta}$) be the left-invariant (resp. right-invariant) Maurer–Cartan form, and let

$$\chi = \frac{1}{12}\big\langle \theta, [\theta, \theta] \big\rangle = \frac{1}{12}\big\langle \overline{\theta}, [\overline{\theta}, \overline{\theta}] \big\rangle$$

be the canonical bi-invariant 3-form on K.

Definition 12.4 (Alekseev, Malkin, and Meinrenken [AMM98, Definition 2.2]) A *quasi-Hamiltonian K-space $M = (M, \omega, \mu)$* is a K-manifold M equipped with a K-invariant 2-form ω and K-equivariant map $\mu : M \to K$ such that

(i) $d\omega = -\mu^* \chi$,

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

(ii) $\iota(v_\xi)\omega = (1/2)\mu^*(\theta + \overline{\theta})$ for each $\xi \in \mathfrak{k}$, where v_ξ is the vector field on M given by the infinitesimal action of ξ, and
(iii) $\ker \omega_x = \{v_\xi(x) \mid \xi \in \ker(\mathrm{Ad}_{\mu(x)} + 1)\}$ for each $x \in M$.

We call $\mu: M \to K$ the *K-valued moment map*, or simply the *moment map*.

Example 12.5 (The double [AMM98, Remark 3.2]) Let $D(G) = G \times G$, and define a G^2-action on $D(G)$ by

$$(\sigma_1, \sigma_2) \cdot (a, b) := (\sigma_1 a \sigma_2^{-1}, \mathrm{Ad}_{\sigma_2} b)$$

for $(a, b) \in D(G)$ and $(\sigma_1, \sigma_2) \in G^2$. Then $D(G)$ is a quasi-Hamiltonian G^2-space with the 2-form

$$\omega_D = \frac{1}{2}\langle \mathrm{Ad}_b\, a^*\theta, a^*\theta\rangle + \frac{1}{2}\big\langle a^*\theta, b^*(\theta + \overline{\theta})\big\rangle$$

and the moment map

$$\mu = (\mu_1, \mu_2): D(G) \longrightarrow G^2, \quad (a, b) \longmapsto (\mathrm{Ad}_a\, b, b^{-1}).$$

Theorem 12.6 (Fusion product [AMM98, Theorem 6.1]) *Let (M, ω, μ) be a quasi-Hamiltonian $K \times K \times H$-space, with $\mu = (\mu_1, \mu_2, \mu_3)$, and consider the diagonal embedding $K \times H \hookrightarrow K \times K \times H$, $(k, h) \mapsto (k, k, h)$. Then M is a quasi-Hamiltonian $K \times H$-space with the 2-form*

$$\widetilde{\omega} = \omega + \frac{1}{2}\langle \mu_1^*\theta, \mu_2^*\overline{\theta}\rangle$$

and the moment map

$$\widetilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3): M \longrightarrow K \times H.$$

The product $M_1 \times M_2$ of quasi-Hamiltonian $K \times H_j$-spaces M_j $(j = 1, 2)$ is a quasi-Hamiltonian $K \times H_1 \times K \times H_2$-space. The fusion product $M_1 \circledast M_2$ is a quasi-Hamiltonian $K \times H_1 \times H_2$-space obtained from $M_1 \times M_2$ by fusing K-factors. Note that the fusion product is associative:

$$(M_1 \circledast M_2) \circledast M_3 = M_1 \circledast (M_2 \circledast M_3).$$

We consider $n - 1$ copies of double $D_i = (D(G), \omega_{D_i}, \mu_i)$ $(i = 2, \dots, n)$ with moment map

$$\mu_i = (\mu_{i,1}, \mu_{i,2}): D(G) \longrightarrow G^2, \quad (a_i, b_i) \longmapsto (\mathrm{Ad}_{a_i} b_i, b_i^{-1}).$$

Then the fusion product $D(G)^{\circledast(n-1)} = D_2 \circledast \cdots \circledast D_n$ given by fusing first G-factors is isomorphic to $\mathcal{N}^G(\Sigma)$ as a G^n-manifold, and hence it defines a structure of quasi-Hamiltonian G^n-space on $\mathcal{N}^G(\Sigma)$. Since

$$b_1^{-1} = (\mathrm{Ad}_{a_2} b_2) \dots (\mathrm{Ad}_{a_n} b_n) = \mu_{2,1} \cdot \mu_{3,1} \cdots \mu_{n,1} \tag{16}$$

is a component of the moment map on $D(G)^{\circledast(n-1)} \cong \mathcal{N}^G(\Sigma)$, we have the following theorem.

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

Theorem 12.7 ([AMM98, Section 9]) *There exists a structure of quasi-Hamiltonian G^n-space on $\mathcal{N}^G(\Sigma)$ such that*

$$\mu^G: \mathcal{N}^G(\Sigma) \longrightarrow G^n, \quad (\boldsymbol{a}, \boldsymbol{b}) \longmapsto \boldsymbol{b}^{-1} = (b_1^{-1}, \dots, b_n^{-1})$$

is the moment map. The quasi-Hamiltonian reduction $(\mu^G)^{-1}(\mathcal{C}_{\boldsymbol{\alpha}})/G^n$ is symplectomorphic to $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$.

Remark 12.8 Treloar [Tre02] also shows this fact, and describes the Goldman system as bending Hamiltonians on the moduli space of n-gons in $S^3 \cong SU(2)$.

Set $\mu_{\le i} = \mu_{2,1} \cdot \mu_{3,1} \dots \mu_{i,1}$ for simplicity. Then the 2-form $\omega_{\mathcal{N}^G(\Sigma)}$ on $\mathcal{N}^G(\Sigma)$ is given by

$$\begin{aligned}
\omega_{\mathcal{N}^G(\Sigma)} &= \sum_{i=2}^n \omega_{D_i} + \frac{1}{2} \sum_{i=3}^n \Big\langle (\mu_{\le i-1})^* \theta, (\mu_{i,1})^* \overline{\theta} \Big\rangle \\
&= \sum_{i=2}^n \omega_{D_i} + \frac{1}{2} \sum_{i=3}^n \Big\langle \mathrm{Ad}_{\mu_{i,1}^{-1}} (\mu_{\le i-1})^* \theta, \mathrm{Ad}_{\mu_{i,1}^{-1}} (\mu_{i,1})^* \overline{\theta} \Big\rangle \\
&= \sum_{i=2}^n \omega_{D_i} + \frac{1}{2} \sum_{i=3}^n \Big\langle (\mu_{\le i})^* \theta, (\mu_{i,1})^* \theta \Big\rangle \\
&= \sum_{i=2}^n \Big(\omega_{D_i} + \frac{1}{2} \Big\langle (\mu_{\le i})^* \theta, (\mu_{i,1})^* \theta \Big\rangle \Big).
\end{aligned} \tag{17}$$

Here, we have used

$$\begin{aligned}
\mathrm{Ad}_{\mu_{i,1}^{-1}} \big[(\mu_{\le i-1})^* \theta \big] &= (\mu_{\le i})^* \theta - (\mu_{i,1})^* \theta, \\
\mathrm{Ad}_{\mu_{i,1}^{-1}} (\mu_{i,1})^* \overline{\theta} &= (\mu_{i,1})^* \theta, \\
\langle (\mu_{i,1})^* \theta, (\mu_{i,1})^* \theta \rangle &= 0,
\end{aligned}$$

which follow from

$$\begin{aligned}
g^{-1}(h^{-1} dh) g &= (hg)^{-1} d(hg) - g^{-1} dg, \\
g^{-1}((dg) g^{-1}) g &= g^{-1} dg,
\end{aligned}$$

and the fact that pairing $\langle \cdot, \cdot \rangle$ is symmetric and θ is a one-form.

13 Walls and Quasi-Hamiltonian Reductions

Recall that walls in the space of parabolic weights are given by

$$H_{I,k} = \Big\{ \boldsymbol{\alpha} \in [0, 1/2)^n \;\Big|\; \sum_{j \in J} \alpha_j - \sum_{i \in I} \alpha_i = k \Big\}$$

for $I \subset \{1, \dots, n\}$, $J = \{1, \dots, n\} \smallsetminus I$, and $k \in \mathbb{Z}$. We define $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$ as

$$\epsilon_i = \begin{cases} 1 & i \in J, \\ -1 & i \in I, \end{cases} \tag{18}$$

so that $\sum_{i=1}^n \epsilon_i \alpha_i = k$.

Lemma 13.1 *A parabolic weight $\alpha \in [0,1)^n$ lies on a wall if and only if $\mathcal{C}_{\boldsymbol{\alpha}}$ contains $\boldsymbol{g} = (g_1, \dots, g_n)$ such that $g_1, \dots, g_n$ lie on a common maximal torus and satisfy $g_1 \dots g_n = 1$.*

Proof If $\mathcal{C}_{\boldsymbol{\alpha}}$ contains $\boldsymbol{g} = (g_1, \dots, g_n)$ such that $g_1, \dots, g_n$ lie on a common maximal torus and satisfy $g_1 \cdots g_n = 1$, then one can simultaneously diagonalize $g_1, \dots, g_n$ so that $g_i = \exp(\epsilon_i \alpha_i x_0)$ for some $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$. Then $g_1 \cdots g_n = \exp[(\epsilon_1 \alpha_1 + \cdots + \epsilon_n \alpha_n) x_0] = 1$ implies $\epsilon_1 \alpha_1 + \cdots + \epsilon_n \alpha_n \in \mathbb{Z}$, so that $\boldsymbol{\alpha}$ is on the wall defined by $\boldsymbol{\epsilon}$.

Conversely, if $\boldsymbol{\alpha}$ satisfies $\epsilon_1 \alpha_1 + \cdots + \epsilon_n \alpha_n \in \mathbb{Z}$ for some $\boldsymbol{\epsilon} \in \{\pm 1\}^n$, then $(g_i = \exp(\epsilon_i \alpha_i x_0))_{i=1}^n$ gives an element of $\mathcal{C}_{\boldsymbol{\alpha}}$ contained in the same maximal torus satisfying $g_1 \cdots g_n = 1$. ∎

Since $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is described as the quasi-Hamiltonian reduction $\left(\mu^G\right)^{-1}(\mathcal{C}_{\boldsymbol{\alpha}})/G^n$ by Theorem 12.7, there are two ways for $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ to be singular. One way is for μ^G to have a critical point.

Proposition 13.2 *The critical point set of μ^G consists of $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N}^G(\Sigma)$ such that $b_1, a_2 b_2 a_2^{-1}, \dots, a_n b_n a_n^{-1}$ lie on a common maximal torus.*

Proof Suppose that $\mu^G(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{b}^{-1} \in \mathcal{C}_{\boldsymbol{\alpha}}$. Under the identifications $T_{(\boldsymbol{a},\boldsymbol{b})}\mathcal{N}^G(\Sigma) \cong T_{(\boldsymbol{a},\boldsymbol{b})} G^{2(n-1)} \cong \mathfrak{g}^{2(n-1)}$ and $T_{\boldsymbol{b}^{-1}} G^n \cong \mathfrak{g}^n$ by right translations, $d\mu^G_{(\boldsymbol{a},\boldsymbol{b})} \colon \mathfrak{g}^{2(n-1)} \to \mathfrak{g}^n$ is given by

$$d\mu^G_{(\boldsymbol{a},\boldsymbol{b})}(\xi_2, \dots, \xi_n, \eta_2, \dots, \eta_n) = (-\operatorname{Ad}_{b_1^{-1}} \eta_1, \dots, -\operatorname{Ad}_{b_n^{-1}} \eta_n)$$

with

$$-\operatorname{Ad}_{b_1^{-1}} \eta_1 = \sum_{i=2}^n \operatorname{Ad}_{(a_2 b_2 a_2^{-1}) \dots (a_{i-1} b_{i-1} a_{i-1}^{-1})} \left(\xi_i - \operatorname{Ad}_{a_i b_i a_i^{-1}} \xi_i - \operatorname{Ad}_{a_i} \eta_i \right).$$

Hence, $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \ker d\mu^G_{(\boldsymbol{a},\boldsymbol{b})}$ if and only if $\boldsymbol{\eta} = 0$ and

$$\sum_{i=2}^n \operatorname{Ad}_{(a_2 b_2 a_2^{-1}) \dots (a_{i-1} b_{i-1} a_{i-1}^{-1})} \left(\xi_i - \operatorname{Ad}_{a_i b_i a_i^{-1}} \xi_i \right) = 0. \tag{19}$$

Since $b_i \in \mathcal{C}_{\alpha_i}$, there exists $g_i \in G$ such that $b_i = g_i e^{\alpha_i x_0} g_i^{-1}$. Setting

$$h_i = (a_2 b_2 a_2^{-1}) \dots (a_{i-1} b_{i-1} a_{i-1}^{-1}) a_i g_i, \qquad \xi_i' = \operatorname{Ad}_{g_i^{-1} a_i^{-1}} \xi_i,$$

the equation (19) is written as

$$\sum_{i=2}^n \operatorname{Ad}_{h_i} \left(\xi_i' - \operatorname{Ad}_{\exp(\alpha_i x_0)} \xi_i' \right) = 0.$$

Since

$$\xi' - \operatorname{Ad}_{\exp(\alpha_i x_0)} \xi' = \begin{pmatrix} 0 & (1 - e^{4\pi\sqrt{-1}\alpha_i}) \xi_{12}' \\ (1 - e^{-4\pi\sqrt{-1}\alpha_i}) \xi_{21}' & 0 \end{pmatrix}$$

for $\xi' = (\xi_{ij}') \in \mathfrak{g}$, the dimension of the image of the map

$$\mathfrak{g}^{n-1} \to \mathfrak{g}, \quad (\xi_2', \dots, \xi_n') \longmapsto \sum_{i=2}^n \operatorname{Ad}_{h_i} \left(\xi_i' - \operatorname{Ad}_{\exp(\alpha_i x_0)} \xi_i' \right)$$

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

is at least two, and is exactly two if and only if there exists some $g \in G$ such that $gh_2, \dots, gh_n$ are diagonal matrices. Note that

$$(gh_i)e^{\alpha_i x_0}(gh_i)^{-1} =$$
$$g\big((a_2b_2a_2^{-1})\dots(a_{i-1}b_{i-1}a_{i-1}^{-1})\big)(a_ib_ia_i^{-1})\big((a_2b_2a_2^{-1})\dots(a_{i-1}b_{i-1}a_{i-1}^{-1})\big)^{-1}g^{-1}.$$

If gh_2 is a diagonal matrix, then so is $ga_2b_2a_2^{-1}g^{-1}$. If h_3 is also a diagonal matrix, then so is $g(a_2b_2a_2^{-1})(a_3b_3a_3^{-1})(a_2b_2a_2^{-1})^{-1}g^{-1}$, and hence $ga_3b_3a_3^{-1}g^{-1}$ is also a diagonal matrix. By continuing the same discussion, one shows that if $gh_2, \dots, gh_n$ are diagonal matrices, then so are $ga_ib_ia_i^{-1}g^{-1}$ for $n = 2, \dots, n$. Then (16) implies that gb_1g^{-1} is also a diagonal matrix. This means that $b_1, a_2b_2a_2^{-1}, \dots, a_nb_na_n^{-1}$ are in the same maximal torus, and Proposition 13.2 is proved. ∎

The other way for $\mathcal{N}_\alpha(\Sigma)$ to be singular is for the G^n-action on the level set $\mathcal{N}^G(\Sigma; \boldsymbol{\alpha}) = (\mu^G)^{-1}(\mathcal{C}_{\boldsymbol{\alpha}})$ to have larger stabilizer than the generic orbit. Note that the generic stabilizer is given by $\{\pm\mathbf{1}\} = \{\pm(1, \dots, 1)\} \subset G^n$.

Proposition 13.3 *The non-free locus of the $G^n/\{\pm\mathbf{1}\}$-action on $\mathcal{N}^G(\Sigma; \boldsymbol{\alpha})$ consists of $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N}^G(\Sigma)$ such that $b_1, a_2b_2a_2^{-1}, \dots, a_nb_na_n^{-1}$ lie on a common maximal torus.*

Proof Suppose that $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in G^n$ fixes $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N}^G(\Sigma; \boldsymbol{\alpha})$, *i.e.*,

$$\sigma_1 a_l \sigma_l^{-1} = a_l, \qquad l = 2, \dots, n, \tag{20}$$
$$\sigma_i b_i \sigma_i^{-1} = b_i, \qquad i = 1, \dots, n. \tag{21}$$

Condition (20) is written as $\sigma_l = a_l^{-1}\sigma_1 a_l$, which means that $\sigma_1, \dots, \sigma_n$ are in the same conjugacy class. By the G^n-action

$$a_l \mapsto g_1 a_l g_l^{-1}, \quad b_i \mapsto g_i b_i g_i^{-1}, \quad \sigma_i \mapsto g_i \sigma_i g_i^{-1},$$

we may assume that $b_i = e^{\alpha_i x_0}$ are diagonal matrices for $i = 1, \dots, n$. Then (21) implies that σ_i is a diagonal matrix if $b_i \neq 1$. We may assume that σ_i is diagonal also in the case $b_i = 1$ by the G^n-action. Since $\sigma_1, \dots, \sigma_n$ are diagonal matrices in the same conjugacy class, one has $\sigma_i = \sigma_1^{\epsilon_i}$ for some diagonal matrix σ_1 and $\epsilon_i \in \{\pm 1\}$. Now we assume that $\boldsymbol{\sigma} \neq \pm\mathbf{1}$. This implies $\sigma_i \neq \pm 1$ for all $i = 1, \dots, n$, since $(\pm 1)^{-1} = \pm 1$. From (20), a_l has the form

$$a_l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{(1-\epsilon_l)/2} \begin{pmatrix} e^{2\pi\sqrt{-1}\tau_l} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\tau_l} \end{pmatrix},$$

and hence

$$a_l b_l a_l^{-1} = \begin{pmatrix} e^{2\pi\sqrt{-1}\epsilon_l\alpha_l} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\epsilon_l\alpha_l} \end{pmatrix}.$$

The condition $b_1(a_2b_2a_2^{-1})\dots(a_nb_na_n^{-1}) = 1$ implies that $\sum_i \epsilon_i\alpha_i = k \in \mathbb{Z}$, which means that $\boldsymbol{\alpha} \in H_{I,k}$ for $I = \{i \mid \epsilon_i = 1\}$.

Conversely, if $\boldsymbol{\alpha} \in H_{I,k}$, then the above argument shows that there exists a set of diagonal matrices $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{N}^G(\Sigma; \boldsymbol{\alpha})$ that has a non-trivial stabilizer. ∎

The proof of Proposition 13.3 shows that any element of the stabilizer of $(\boldsymbol{a},\boldsymbol{b}) \in \mathcal{N}^G(\Sigma;\boldsymbol{\alpha})$ has the form

$$\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n) = (\sigma_1, a_2^{-1}\sigma_1 a_2,\dots,a_n^{-1}\sigma_1 a_n).$$

Note that $b_i = \pm 1$ if and only if $\alpha_i \in \{0,1/2\}$. If $b_i \neq \pm 1$ for some i, then (21) implies that $\sigma_i = a_i^{-1}\sigma_1 a_i$ must be in a maximal torus, and hence the stabilizer is isomorphic to T. On the other hand, if $b_i = \pm 1$ for all i, then the stabilizer is isomorphic to G, since σ_1 can be arbitrary. When $\boldsymbol{\alpha} \in \{0,1/2\}^n$, $\boldsymbol{b} \in \{\pm 1\}^n$ carries no degree of freedom and the $\boldsymbol{a}$-projection induces an isomorphism of $\mathcal{N}^G(\Sigma;\boldsymbol{\alpha})$ with G^{n-1}. The G^n action on G^{n-1} indeed has a stabilizer isomorphic to G, and the quotient $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) = \mathcal{N}^G(\Sigma;\boldsymbol{\alpha})/G^n$ consists of one point.

Propositions 13.2 and 13.3 show that if $\boldsymbol{\alpha}$ lies on some $H_{I,k}$, then the singular locus of $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is given by $[g_1,\dots,g_n] \in \mathcal{C}_{\boldsymbol{\alpha}}/G$ such that $g_1,\dots,g_n$ lie on a common maximal torus. Then one can diagonalize $g_1,\dots,g_n$ simultaneously, so that $g_i = \exp(\epsilon_i\alpha_i x_0)$, where $\boldsymbol{\epsilon}$ are given in (18). If $\boldsymbol{\alpha}$ lies on k walls, then $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ has k isolated singularities, each of which is given by $[\exp(\epsilon_1\alpha_1 x_0),\dots,\exp(\epsilon_n\alpha_n x_0)]$.

Corollary 13.4 *Suppose that $\boldsymbol{\alpha}$ is a weight lying on some $H_{I,k}$. Let $(\boldsymbol{a},\boldsymbol{b}) \in \mathcal{N}^G(\Sigma;\boldsymbol{\alpha})$ be a critical point of μ^G, and $\boldsymbol{g} \in \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ be the corresponding singular point. Then there exists an open neighborhood $U \subset \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ of $\boldsymbol{g}$ such that $\mathcal{N}^G(\Sigma;\boldsymbol{\alpha})$ is locally homeomorphic to $(\mathfrak{g}^n/\mathfrak{t}) \times \big((U\times T)\big/(\{\boldsymbol{g}\}\times T)\big)$. In particular, $\mathcal{N}^G(\Sigma;\boldsymbol{\alpha})$ admits a G^n-invariant Whitney stratification.*

Here, $(U\times T)\big/(\{\boldsymbol{g}\}\times T)$ is the topological space obtained from $U\times T$ by contracting the subset $\{\boldsymbol{g}\}\times T \subset U \times T$ to a point, and $\mathfrak{g}^n/\mathfrak{t}$ is the quotient vector space.

Propositions 13.2 and 13.3 imply the following corollary.

Corollary 13.5 *If $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are in the same chamber, then $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is diffeomorphic to $\mathcal{N}_{\boldsymbol{\alpha}'}(\Sigma)$.*

Let

$$\mu^T = \mu^G|_{\mathcal{N}^T(\Sigma)} \colon \mathcal{N}^T(\Sigma) \longrightarrow T^n, \quad (\boldsymbol{a},\boldsymbol{b}) \longmapsto \boldsymbol{b}^{-1}$$

be the restriction of the group-valued moment map. Then

$$(\mu^G)^{-1}(\mathcal{C}_{\boldsymbol{\alpha}}) \cap \mathcal{N}^T(\Sigma) = (\mu^T)^{-1}(e^{-\alpha_1 x_0},\dots,e^{-\alpha_n x_0}).$$

Corollary 13.6 *If $\boldsymbol{\alpha} \notin \{0,1/2\}^n$, then the diffeomorphism $(\mu^G)^{-1}(\mathcal{C}_{\boldsymbol{\alpha}})/G^n \cong \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ induces*

$$(\mu^T)^{-1}(e^{-\alpha_1 x_0},\dots,e^{-\alpha_n x_0})/T^n \cong \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma).$$

14 Gluing and Goldman Systems

In this section, we see the Goldman's functions via gluing of Riemann surfaces, following the idea of Hurtubise and Jeffrey [HJ00].

Fix a simple closed curve C in Σ and consider a decomposition $\Sigma = \Sigma^+ \cup_C \Sigma^-$ into two surfaces by cutting Σ along C. We may assume that the boundary components of Σ^+ (resp. Σ^-) are $B_1^+ = B_1,\dots,B_{m+1}^+ = B_{m+1}$, and $B_{m+2}^+ = C$ (resp. $B_1^- = C, B_2^- =$

$B_{m+2}, \dots, B^-_{n-m} = B_n$). Then $\mathcal{N}^G(\Sigma^+)$ (resp. $\mathcal{N}^G(\Sigma^-)$) has the action of $G^{m+2} = G_1^+ \times \cdots \times G^+_{m+2}$ (resp. $G^{n-m} = G_1^- \times \cdots \times G^-_{n-m}$) corresponding to the boundary components. We write the moment maps $\mu^G_\pm$ on $\mathcal{N}^G(\Sigma^\pm)$ as

$$\mu^G_+ = (\mu^G_{B_1^+}, \dots, \mu^G_{B^+_{m+2}}) = (\mu^+_{\le m+2}, \mu^+_{2,2}, \dots, \mu^+_{m+2,2}),$$
$$\mu^G_- = (\mu^G_{B_1^-}, \dots, \mu^G_{B^-_{n-m}}) = (\mu^-_{\le n-m}, \mu^-_{2,2}, \dots, \mu^-_{n-m,2}).$$

For the diagonal subgroup $G_C \subset G^+_{m+2} \times G_1^-$, the moment map of the G_C-action on the fusion product $\mathcal{N}^G(\Sigma^+ \amalg \Sigma^-) := \mathcal{N}^G(\Sigma^+) \circledast \mathcal{N}^G(\Sigma^-)$ is given by

$$\nu^G_C = \mu^G_{B^+_{m+2}} \cdot \mu^G_{B_1^-} \colon \mathcal{N}^G(\Sigma^+ \amalg \Sigma^-) \longrightarrow G, \quad \big((\boldsymbol{a}^+, \boldsymbol{b}^+), (\boldsymbol{a}^-, \boldsymbol{b}^-)\big) \longmapsto (b_1^- b^+_{m+2})^{-1}.$$

We define the "gluing map" $\pi^G_C \colon \big(\nu^G_C\big)^{-1}(1) \to \mathcal{N}^G(\Sigma)$ by

$$\pi_C\big((\boldsymbol{a}^+, \boldsymbol{b}^+), (\boldsymbol{a}^-, \boldsymbol{b}^-)\big) = (a_2^+, \dots, a^+_{m+1}, a^+_{m+2}a_2^-, \dots, a^+_{m+2}a^-_{n-m}; b_1^+, \dots, b^+_{m+1}, b_2^-, \dots, b^-_{n-m}).$$

(See Figure 2.)

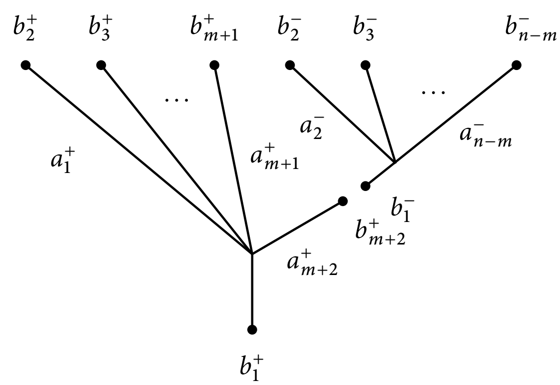

Figure 2: The dual graph of $\Sigma^+ \amalg \Sigma^-$.

Then we have the following proposition.

Proposition 14.1 *The map* $\pi_C \colon \big(\nu^G_C\big)^{-1}(1) \to \mathcal{N}^G(\Sigma)$ *induces an isomorphism*

$$\big(\nu^G_C\big)^{-1}(1)/G_C \cong \mathcal{N}^G(\Sigma)$$

of quasi-Hamiltonian G^n*-space.*

Proof It is easy to see that π_C is well defined and surjective. To see that the induced map is injective, suppose that

$$\pi_C\big((\boldsymbol{a}^+, \boldsymbol{b}^+), (\boldsymbol{a}^-, \boldsymbol{b}^-)\big) = \pi_C\big((\boldsymbol{c}^+, \boldsymbol{d}^+), (\boldsymbol{c}^-, \boldsymbol{d}^-)\big)$$

for $((\boldsymbol{a}^+,\boldsymbol{b}^+),(\boldsymbol{a}^-,\boldsymbol{b}^-)),((\boldsymbol{c}^+,\boldsymbol{d}^+),(\boldsymbol{c}^-,\boldsymbol{d}^-))\in(\nu_C^G)^{-1}(1)$. Then we have

$$
\begin{aligned}
a_i^+ &= c_i^+, && i=2,\dots,m+1,\\
a_{m+2}^+a_j^- &= c_{m+2}^+c_j^-, && j=2,\dots,n-m,\\
b_i^+ &= d_i^+, && i=1,\dots,m+1,\\
b_j^- &= d_j^-, && j=2,\dots,n-m.
\end{aligned}
\tag{22}
$$

Note that $b_{m+2}^+, b_1^-, d_{m+2}^+, d_1^-$ are determined by

$$
\begin{aligned}
b_1^+\big(a_2^+b_2^+(a_2^+)^{-1}\big)\dots\big(a_{m+2}^+b_{m+2}^+(a_{m+2}^+)^{-1}\big) &= 1,\\
b_1^-\big(a_2^-b_2^-(a_2^-)^{-1}\big)\dots\big(a_{n-m}^+b_{n-m}^+(a_{n-m}^+)^{-1}\big) &= 1,\\
d_1^+\big(c_2^+d_2^+(c_2^+)^{-1}\big)\dots\big(c_{m+2}^+d_{m+2}^+(c_{m+2}^+)^{-1}\big) &= 1,\\
d_1^-\big(c_2^-d_2^-(c_2^-)^{-1}\big)\dots\big(c_{n-m}^+d_{n-m}^+(c_{n-m}^+)^{-1}\big) &= 1.
\end{aligned}
$$

Setting $\sigma=(c_{m+2}^+)^{-1}a_{m+2}^+\in G=G_C$, condition (22) is written as

$$c_j^-=\sigma a_j^-,\quad j=2,\dots,n-m.$$

This implies that $((\boldsymbol{c}^+,\boldsymbol{d}^+),(\boldsymbol{c}^-,\boldsymbol{d}^-))=\sigma\cdot((\boldsymbol{a}^+,\boldsymbol{b}^+),(\boldsymbol{a}^-,\boldsymbol{b}^-))$. Hence the induced map $(\nu_C^G)^{-1}(1)/G\to\mathcal{N}^G(\Sigma)$ is injective.

It remains to check that $\iota^*\omega_{\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)}=\pi_C^*\omega_{\mathcal{N}^G(\Sigma)}$, where

$$\iota\colon(\nu_C^G)^{-1}(1)\hookrightarrow\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)$$

is the inclusion. From (17), the 2-form $\omega_{\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)}$ on $\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)$ is given by

$$
\begin{aligned}
\omega_{\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)} &= \omega_{\mathcal{N}^G(\Sigma^+)}+\omega_{\mathcal{N}^G(\Sigma^-)}+\frac12\big\langle(\mu_{m+2,2}^+)^*\theta,(\mu_{\le n-m}^-)^*\overline{\theta}\big\rangle\\
&=\sum_{i=2}^{m+2}\Big(\omega_{D_i^+}+\frac12\big\langle(\mu_{\le i}^+)^*\theta,(\mu_{i,1}^+)^*\theta\big\rangle\Big)\\
&\quad+\sum_{j=2}^{n-m}\Big(\omega_{D_i^-}+\frac12\big\langle(\mu_{\le j}^-)^*\theta,(\mu_{j,1}^-)^*\theta\big\rangle\Big)\\
&\quad+\frac12\big\langle(\mu_{m+2,2}^+)^*\theta,(\mu_{\le n-m}^-)^*\overline{\theta}\big\rangle.
\end{aligned}
$$

Since $(\mu_{m+2,2}^+)^{-1}=b_{m+2}^+=(b_1^-)^{-1}=\mu_{\le n-m}^-$ and $\mu_{m+1,1}^+=\mathrm{Ad}_{a_{m+2}^+}\mu_{\le n-m}^-$ on $(\nu_C^G)^{-1}(1)$, we have

$$\iota^*\omega_{D_{m+2}^+}=\big\langle(a_{m+2}^+)^*,(\mu_{\le n-m}^-)^*(\theta+\overline{\theta})-\mathrm{Ad}_{\mu_{\le n-m}^-}(a_{m+2}^+)^*\theta\big\rangle$$

and

$$
\begin{aligned}
&\iota^*\langle(\mu_{\le m+2}^+)^*\theta,(\mu_{m+2,1}^+)^*\theta\rangle=\big\langle(\mu_{\le m+1}^+)^*\theta,(\mathrm{Ad}_{a_{m+2}^+}\mu_{\le n-m}^-)^*\overline{\theta}\big\rangle\\
&\iota^*\langle(\mu_{m+2,2}^+)^*\theta,(\mu_{\le n-m}^-)^*\overline{\theta}\rangle=0.
\end{aligned}
$$

Then the restriction $\iota^* \omega_{\mathcal{N}^G(\Sigma^+ \amalg \Sigma^-)}$ is given by

$$\begin{aligned}
\iota^* \omega_{\mathcal{N}^G(\Sigma^+ \amalg \Sigma^-)} = & \sum_{i=2}^{m+1} \Big(\omega_{D_i^+} + \frac{1}{2} \langle (\mu_{\le i}^+)^* \theta, (\mu_{i,1}^+)^* \theta \rangle \Big) \\
& + \sum_{j=2}^{n-m} \Big(\omega_{D_i^-} + \frac{1}{2} \langle (\mu_{\le j}^-)^* \theta, (\mu_{j,1}^-)^* \theta \rangle \Big) \\
& + \frac{1}{2} \Big\langle (\mu_{\le m+1}^+)^* \theta, (\mathrm{Ad}_{a_{m+2}^+} \mu_{\le n-m}^-)^* \overline{\theta} \Big\rangle \\
& + \frac{1}{2} \Big\langle (a_{m+2}^+)^* \theta, (\mu_{\le n-m}^-)^* (\theta + \overline{\theta}) - \mathrm{Ad}_{\mu_{\le n-m}^-} (a_{m+2}^+)^* \theta \Big\rangle .
\end{aligned}$$

On the other hand, the pull-back of $\omega_{\mathcal{N}^G(\Sigma)}$ is given by

$$\begin{aligned}
\pi_C^* \omega_{\mathcal{N}^G(\Sigma)} = & \pi_C^* \sum_{i=2}^{m+1} \Big(\omega_{D_i} + \frac{1}{2} \langle (\mu_{\le i})^* \theta, (\mu_{i,1})^* \theta \rangle \Big) \\
& + \pi_C^* \sum_{j=2}^{n-m} \Big(\omega_{D_{m+j}} + \frac{1}{2} \langle (\mu_{\le m+j})^* \theta, (\mu_{m+j,1})^* \theta \rangle \Big)
\end{aligned}$$

with

$$\pi_C^* \Big(\omega_{D_i} + \frac{1}{2} \langle (\mu_{\le i})^* \theta, (\mu_{i,1})^* \theta \rangle \Big) = \omega_{D_i^+} + \frac{1}{2} \langle (\mu_{\le i}^+)^* \theta, (\mu_{i,1}^+)^* \theta \rangle \tag{23}$$

for $i = 2, \dots, m+1$. By using

$$\pi_C^* a_{m+j}^* \theta = (a_{m+2}^+ a_j^-)^* \theta = \mathrm{Ad}_{(a_j^-)^{-1}} (a_{m+2}^+)^* \theta + (a_j^-)^* \theta$$

for $j = 2, \dots, n-m$ and formulae

$$(\mathrm{Ad}_a\, b)^* \theta = \mathrm{Ad}_{ab^{-1}}\, a^* \theta + \mathrm{Ad}_a\, b^* \theta - a^* \overline{\theta}, \tag{24}$$

$$(\mathrm{Ad}_a\, b)^* \overline{\theta} = a^* \overline{\theta} + \mathrm{Ad}_a\, b^* \overline{\theta} - \mathrm{Ad}_{ab}\, a^* \theta, \tag{25}$$

we have

$$\pi_C^* \omega_{D_{m+j}} = \omega_{D_j^-} + \frac{1}{2} \Big\langle (a_{m+2}^+)^* \theta, (\mu_{j,1}^-)^* (\theta + \overline{\theta}) - \mathrm{Ad}_{\mu_{j,1}^-} (a_{m+2}^+)^* \theta \Big\rangle . \tag{26}$$

Similarly,

$$\begin{aligned}
& \pi_C^* \langle (\mu_{\le m+j})^* \theta, (\mu_{m+j,1})^* \theta \rangle \\
& = \Big\langle (\mu_{\le m+1}^+ \cdot \mathrm{Ad}_{a_{m+2}^+} \mu_{\le j}^-)^* \theta, (\mathrm{Ad}_{a_{m+2}^+} \mu_{j,1}^-)^* \theta \Big\rangle \\
& = \Big\langle \mathrm{Ad}_{(\mathrm{Ad}_{a_{m+2}^+} \mu_{\le j}^-)^{-1}} (\mu_{\le m+1}^+)^* \theta + (\mathrm{Ad}_{a_{m+2}^+} \mu_{\le j}^-)^* \theta, (\mathrm{Ad}_{a_{m+2}^+} \mu_{j,1}^-)^* \theta \Big\rangle \\
& = \Big\langle (\mu_{\le m+1}^+)^* \theta, \mathrm{Ad}_{\mathrm{Ad}_{a_{m+2}^+} \mu_{\le j}^-} (\mathrm{Ad}_{a_{m+2}^+} \mu_{j,1}^-)^* \theta \Big\rangle \\
& \qquad + \Big\langle (\mathrm{Ad}_{a_{m+2}^+} \mu_{\le j}^-)^* \theta, (\mathrm{Ad}_{a_{m+2}^+} \mu_{j,1}^-)^* \theta \Big\rangle ,
\end{aligned}$$

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

and formulae (24) and (25) imply

$$
\begin{aligned}
(27)\quad & \pi_C^*\langle(\mu_{\le m+j})^*\theta,(\mu_{m+j,1})^*\theta\rangle\\
&= \langle(\mu^-_{\le j})^*\theta,(\mu^-_{j,1})^*\theta\rangle+\left\langle(a^+_{m+2})^*\theta,\mathrm{Ad}_{\mu^-_{j,1}}(a^+_{m+2})^*\theta-(\mu^-_{j,1})^*(\theta+\overline{\theta})\right\rangle\\
&\quad -\left\langle(a^+_{m+2})^*\theta,\mathrm{Ad}_{\mu^-_{\le j}}(a^+_{m+2})^*\theta-\mathrm{Ad}_{\mu^-_{\le j-1}}(a^+_{m+2})^*\theta\right\rangle\\
&\quad +\langle(a^+_{m+2})^*\theta,(\mu^-_{\le j})^*(\theta+\overline{\theta})-(\mu^-_{\le j-1})^*(\theta+\overline{\theta})\rangle\\
&\quad +\left\langle(\mu^+_{\le m+1})^*\theta,(\mathrm{Ad}_{a^+_{m+2}}\mu^-_{\le j})^*\overline{\theta}-(\mathrm{Ad}_{a^+_{m+2}}\mu^-_{\le j-1})^*\overline{\theta}\right\rangle,
\end{aligned}
$$

where we assume that $\mu^-_{\le 1}=1$ is a constant map. Combining (23), (26), and (27), we have $\pi_C^*\omega_{\mathcal{N}^G(\Sigma)}=\iota^*\omega_{\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)}$. ∎

We consider the action of $G^+_{m+2}=G^+_{m+2}\times\{1\}\subset G^+_{m+2}\times G^-_1$ with moment map

$$\mu^G_C=\mu^G_{B^+_{m+2}}:\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)\longrightarrow G,\quad \mu^G_C(\boldsymbol{a}^\pm,\boldsymbol{b}^\pm)=(b^+_{m+2})^{-1}.$$

Since G^{n-3} acts on the b^+_{m+2}-component by conjugation, the function

$$\tilde{\vartheta}_C=\cos^{-1}\Big(\frac{1}{2}\operatorname{tr}\mu^G_C\Big):\mathcal{N}^G(\Sigma^+\amalg\Sigma^-)\longrightarrow\mathbb{R}$$

descends to $\mathcal{N}^G(\Sigma)$, and induces a Goldman's function $\vartheta_C:\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)\to\mathbb{R}$. Let $\nu^T_C=\nu^G_C|_{\mathcal{N}^T(\Sigma^+\amalg\Sigma^-)}$ be the restriction of the moment map to $\mathcal{N}^T(\Sigma^+\amalg\Sigma^-)=\mathcal{N}^T(\Sigma^+)\times\mathcal{N}^T(\Sigma^-)$. Then $(\nu^T_C)^{-1}(1)\subset(\nu^G_C)^{-1}(1)$ is preserved under the action of the maximal torus $T^+_{m+2}\times T^-_1\subset G^+_{m+2}\times G^-_1$. The Hamiltonian torus action of ϑ_C is induced from the action of $T^+_{m+2}\times\{1\}\subset T^+_{m+2}\times T^-_1$ on $(\nu^T_C)^{-1}(1)$ (see [HJ00]).

Now we fix a pair-of-pants decomposition $\Sigma=\bigcup_{i=1}^{n-2}\Sigma_i$ given by $n-3$ simple closed curves $C_1,\dots,C_{n-3}$ with dual graph Γ, and let C_i^+, C_i^- denote the copies of C_i in the disjoint union $\coprod_i\Sigma_i$. Then the fusion product $\mathcal{N}^G(\coprod_i\Sigma_i):=\mathcal{N}^G(\Sigma_1)\circledast\cdots\circledast\mathcal{N}^G(\Sigma_{n-2})$ has the actions of diagonal subgroups $G^{n-3}=\prod_i G_{C_i}$ in $G^{2(n-3)}=\prod_i G_{C_i^+}\times G_{C_i^-}$ with moment map $\nu^G_\Gamma:\mathcal{N}^G(\coprod_i\Sigma_i)\to G^{n-3}$. We can define the gluing map $\pi_\Gamma:(\nu^G_\Gamma)^{-1}(\mathbf{1})\to\mathcal{N}^G(\Sigma)$ in a similar manner.

Corollary 14.2 *The map $\pi_\Gamma:\big(\nu^G_\Gamma\big)^{-1}(\mathbf{1})\to\mathcal{N}^G(\Sigma)$ induces an isomorphism*

$$\big(\nu^G_\Gamma\big)^{-1}(\mathbf{1})/G^{n-3}\cong\mathcal{N}^G(\Sigma)$$

of quasi-Hamiltonian G^n-spaces. The functions $\tilde{\vartheta}_{C_1},\dots,\tilde{\vartheta}_{C_{n-3}}$ induces the Goldman system

$$\Theta_{\boldsymbol{\alpha},\Gamma}=(\vartheta_{C_1},\dots,\vartheta_{C_{n-3}}):\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)\longrightarrow\mathbb{R}^{n-3}.$$

The Hamiltonian torus action of $\Theta_{\boldsymbol{\alpha},\Gamma}$ is given by the action of the maximal torus $\prod_{i=1}^{n-3}T_{C_i^+}\subset\prod_{i=1}^{n-3}(G_{C_i^+}\times\{1\})$ on $\big(\nu^T_\Gamma\big)^{-1}(\mathbf{1})\subset\mathcal{N}^T(\coprod_i\Sigma_i)$.

Remark 14.3 The reduction $(\nu^T_\Gamma)^{-1}(\mathbf{1})/T^{n-3}$ of the T-extended moduli space $\mathcal{N}^T(\coprod_i\Sigma_i)$ is not homeomorphic to $\mathcal{N}^T(\Sigma)$ on the locus where holonomies along any components of $\partial\Sigma_i$ are central for some i.

15 Isomorphisms of Goldman Systems

Fix a generic parabolic weight $\boldsymbol{\alpha} \in (0,1/2)^n$ such that $|\boldsymbol{\alpha}| < 1$. Then $t\boldsymbol{\alpha} = (t\alpha_1, \dots, t\alpha_n)$ and $\boldsymbol{\alpha}$ are in the same chamber for $t \in (0,1]$, and hence $\mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma)$ is diffeomorphic to $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ for $t \in (0,1]$. Note that the images $\Delta_\Gamma(t\boldsymbol{\alpha}) = \Theta_{t\boldsymbol{\alpha},\Gamma}(\mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma))$ of the Goldman systems are related by scalings $\Delta_\Gamma(t\boldsymbol{\alpha}) = t\Delta_\Gamma(\boldsymbol{\alpha})$. In this section we prove the following theorem.

Theorem 15.1 *Suppose that $\boldsymbol{\alpha}$ satisfies the above condition. Then for each Γ, there exists a family of symplectomorphism*

$$\psi_t : \big(\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma), \omega_{\mathcal{N}_{\boldsymbol{\alpha}}}\big) \longrightarrow \big(\mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma), (1/t)\omega_{\mathcal{N}_{t\boldsymbol{\alpha}}}\big)$$

such that $(1/t)\psi_t^* \Theta_{t\boldsymbol{\alpha},\Gamma} = \Theta_{\boldsymbol{\alpha},\Gamma}$. Namely,

$$\mathfrak{N}(\Sigma) = \bigcup_{t\in(0,1]} \mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma) \longrightarrow (0,1] \tag{28}$$

is trivial as a family of symplectic manifolds equipped with completely integrable systems.

We first consider a decomposition $\Sigma = \Sigma^+ \cup_C \Sigma^-$ given by a single simple closed curve as in Section 14.

Lemma 15.2 *For $t \in (0,1]$, there exists a diffeomorphism $\psi_t : \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) \to \mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma)$ such that $(1/t)\psi_t^* \vartheta_{t\boldsymbol{\alpha},C} = \vartheta_{\boldsymbol{\alpha},C}$.*

Proof Let $\mathfrak{C} = \bigcup_{t\in(0,1]} \mathcal{C}_{t\boldsymbol{\alpha}} \subset G^n$ be the family of conjugacy classes with projection $\pi_{\mathfrak{C}} : \mathfrak{C} \to (0,1]$. Then the total space $\mathfrak{N}(\Sigma)$ of the family (28) is given by

$$\mathfrak{N}(\Sigma) = (\mu^G)^{-1}(\mathfrak{C})/G^n,$$

where $\mu^G : \mathcal{N}^G(\Sigma) \to G^n$ is the moment map. Since $|\boldsymbol{\alpha}| < 1$, the family $\mathfrak{C}$ is trivialized by

$$\mathcal{C}_{\boldsymbol{\alpha}} \longrightarrow \mathcal{C}_{t\boldsymbol{\alpha}}, \quad \boldsymbol{c} = (c_1, \dots, c_n) \longmapsto \boldsymbol{c}^t = \big((c_1)^t, \dots, (c_n)^t\big), \tag{29}$$

where $c^t = g e^{tx} g^{-1}$ for $c = g e^x g^{-1} \in \mathcal{C}_\alpha$ with $x \in A_t$.

Let

$$\mu^G_{\partial\Sigma} = \big(\mu^G_{B_1^+}, \dots, \mu^G_{B_{m+1}^+}, \mu^G_{B_2^-}, \dots, \mu^G_{B_{n-m}^+}\big) : \mathcal{N}^G(\Sigma^+ \amalg \Sigma^-) \longrightarrow G^n$$

be the moment map corresponding to the boundary components of Σ, and set

$$\begin{aligned}\widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-) &= (\mu^G_{\partial\Sigma})^{-1}(\mathfrak{C})\\ &= \bigcup_{t\in(0,1]} \Big(\bigcup_{\alpha^+_{m+2},\, \alpha^-_1} \mathcal{N}^G(\Sigma^+; t\boldsymbol{\alpha}^+) \times \mathcal{N}^G(\Sigma^-; t\boldsymbol{\alpha}^-)\Big),\end{aligned}$$

where $\boldsymbol{\alpha}^+ = (\alpha^+_1, \dots, \alpha^+_{m+2})$, $\boldsymbol{\alpha}^- = (\alpha^-_1, \dots, \alpha^-_{n-m})$ with

$$(\alpha^+_1, \dots, \alpha^+_{m+1}) = (\alpha_1, \dots, \alpha_{m+1}), \quad (\alpha^-_2, \dots, \alpha^-_{n-m}) = (\alpha_{m+2}, \dots, \alpha_n).$$

This space has an action of $G^{n+2} = \prod_{i=1}^{m+2} G_i^+ \times \prod_{i=1}^{n-m} G_i^-$ and a G^{n+2}-invariant stratification induced from those on $\mathcal{N}^G(\Sigma^\pm; \boldsymbol{\alpha}^\pm)$. Note that the lower dimensional strata

https://doi.org/10.4153/CJM-2015-004-2 Published online by Cambridge University Press

of $\widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-)$ has the form

$$\bigcup_{t\in(0,1]} \mathcal{N}^G(\Sigma^+; t\boldsymbol{\alpha}^+) \times \operatorname{Sing}\mathcal{N}^G(\Sigma^-; t\boldsymbol{\alpha}^-)$$

with $\alpha_1^- = \sum_{i=2}^{n-m} \epsilon_i \alpha_i^-$, or

$$\bigcup_{t\in(0,1]} \operatorname{Sing}\mathcal{N}^G(\Sigma^+; t\boldsymbol{\alpha}^+) \times \mathcal{N}^G(\Sigma^-; t\boldsymbol{\alpha}^-)$$

with $\alpha_{m+2}^+ = \sum_{i=1}^{m+1} \epsilon_i \alpha_i^+$ for some $\epsilon_i \in \{\pm 1\}$. From Proposition 13.2 and $|\boldsymbol{\alpha}^\pm| < 1$, trivialization (29) lifts to that on $\bigcup_{t\in(0,1]} \operatorname{Sing}\mathcal{N}^G(\Sigma^\pm; t\boldsymbol{\alpha}^\pm)$ given by

$$\operatorname{Sing}\mathcal{N}^G(\Sigma^\pm; \boldsymbol{\alpha}^\pm) \longrightarrow \operatorname{Sing}\mathcal{N}^G(\Sigma^\pm; t\boldsymbol{\alpha}^\pm), \quad (\boldsymbol{a}, \boldsymbol{b}) \longmapsto (\boldsymbol{a}, \boldsymbol{b}^t).$$

From Corollary 13.4, the space $\widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-)$ is locally homeomorphic to $V \times C(L)$ for some open set V in a strata and a cone $C(L) = ([0,\infty) \times L)/(\{0\} \times L)$ over a submanifold L. We fix a G^{n+2}-invariant Riemannian metric on $\widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-)$ such that it has the form $g_V + dr^2 + r^2 g_L$ on each neighborhood $V \times C(L)$ of the singular locus, where g_V and g_L are G^{n+2}-invariant Riemannian metrics on V and L, respectively, and $r \in [0,\infty)$.

Let $\nu_C^G : \mathcal{N}^G(\Sigma^+ \amalg \Sigma^-) \to G_C$ be the moment map of the action of the diagonal subgroup $G_C \subset G_{m+2}^+ \times G_1^-$, and define

$$\mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-) = (\mu_{\partial\Sigma}^G, \nu_C^G)^{-1}(\mathfrak{C} \times \{1\}) = (\nu_C^G)^{-1}(1) \cap \widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-)$$

so that the family $\mathfrak{N}(\Sigma) \to (0,1]$ is given by

$$\pi_{\mathfrak{C}} \circ \mu_{\partial\Sigma}^G : \mathfrak{N}(\Sigma) \cong \mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-)/(G^n \times G_C) \longrightarrow \mathfrak{C} \longrightarrow (0,1].$$

Then the horizontal lift of the trivialization (29) of $\mathfrak{C} \to (0,1]$ gives a G^{n+1}-equivariant trivialization

$$\begin{aligned} &\psi_t : \mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-)_1 \longrightarrow \mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-)_t, \\ &\big((\boldsymbol{a}^+, e^{\boldsymbol{x}^+}), (\boldsymbol{a}^-, e^{\boldsymbol{x}^-})\big) \longmapsto \big((\boldsymbol{c}^+(\boldsymbol{a}, \boldsymbol{x}, t), e^{t\boldsymbol{x}^+}), (\boldsymbol{c}^-(\boldsymbol{a}, \boldsymbol{x}, t), e^{t\boldsymbol{x}^-})\big) \end{aligned} \tag{30}$$

of the family $\widetilde{\mathfrak{N}}(\Sigma^+ \amalg \Sigma^-) \to (0,1]$ preserving the stratification, where

$$\mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-)_t = (\mu_{\partial\Sigma}^G, \nu_C^G)^{-1}\big(\mathcal{C}_{t\alpha} \times \{1\}\big)$$

is the fiber over $t \in (0,1]$. Since ψ_t is G^{n+1}-equivariant, it descends to a diffeomorphism $\psi_t : \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) \to \mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma)$. From the construction of ψ_t, we have

$$\frac{1}{t}\psi_t^* \widetilde{\vartheta}_{t,\boldsymbol{\alpha}C} = \frac{1}{t}\psi_t^* \cos^{-1}\Big(\frac{1}{2}\operatorname{tr} e^{x_{m+2}^+}\Big) = \frac{1}{t}\cos^{-1}\Big(\frac{1}{2}\operatorname{tr} e^{t x_{m+2}^+}\Big) = \widetilde{\vartheta}_{\boldsymbol{\alpha},C},$$

which completes the proof. ∎

Remark 15.3 From (30), the flow ψ_t preserves the subfamily

$$\begin{aligned} \mathfrak{N}^T(\Sigma^+ \amalg \Sigma^-) &= \bigcup_{t\in(0,1]} (\mu_{\partial\Sigma}^T, \nu_T^G)^{-1}(e^{-t\alpha_1 x_0}, \dots, e^{-t\alpha_1 x_0}, 1) \\ &= \mathfrak{N}^G(\Sigma^+ \amalg \Sigma^-) \cap \mathcal{N}^T(\Sigma^+ \amalg \Sigma^-) \end{aligned}$$

of $\mathcal{N}^G(\Sigma^+ \amalg \Sigma^-)$. The flow ψ_t restricted to $\mathfrak{N}^T(\Sigma^+ \amalg \Sigma^-)$ is also equivariant under the action of $T_{m+2}^+ \times \{1\} \subset G_{m+2}^+ \times G_1^-$, and hence $\psi_t : \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) \to \mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma)$ is equivariant under the action of the Hamiltonian S^1-action of ϑ_C.

Proof of Theorem 15.1 Let $\Sigma = \bigcup_{i=1}^{n-2} \Sigma_i$ be the pair-of-pants decomposition given by Γ. For the group valued moment map

$$\mu^G = (\mu^G_{\partial\Sigma}, \nu^G_{C_1}, \dots, \nu^G_{C_{n-3}}) : \mathcal{N}^G(\Sigma_1 \amalg \cdots \amalg \Sigma_{n-2}) \longrightarrow G^n \times G^{n-3}$$

of the $G^n \times G^{n-3}$-action, we define

$$\mathfrak{N}^G\big(\coprod_i \Sigma_i\big) = (\mu^G)^{-1}(\mathfrak{C} \times \{\mathbf{1}\}) \quad \text{and} \quad \mathfrak{N}^T\big(\coprod_i \Sigma_i\big) = \mathfrak{N}^G\big(\coprod_i \Sigma_i\big) \cap \mathcal{N}^T\big(\coprod_i \Sigma_i\big).$$

By applying the above argument, we obtain a trivialization $\psi_t : \mathfrak{N}^G(\coprod_i \Sigma_i)_1 \to \mathfrak{N}^G(\coprod_i \Sigma_i)_t$ of $\mathfrak{N}^G(\coprod_i \Sigma_i)$ that induces trivializations of $\mathfrak{N}^T(\coprod_i \Sigma_i)$ and $\mathfrak{N}(\Sigma)$ and satisfies

$$\frac{1}{t}\psi_t^* \Theta_{t\boldsymbol{\alpha},\Gamma} = \Theta_{\boldsymbol{\alpha},\Gamma}.$$

In particular, ψ_i preserves the action variables on $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$.

The Hamiltonian torus action of the Goldman system, which is defined on an open dense subset $U \subset \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$, is induced from the action of a maximal torus $\prod_{i=1}^{n-3}(T_{C_i^+} \times \{1\})$ in $\prod_i (G_{C_i^+} \times \{1\}) \subset \prod_i (G_{C_i^+} \times G_{C_i^-})$. Since the trivialization $\psi_t : \mathfrak{N}^T(\coprod_i \Sigma_i)_1 \to \mathfrak{N}^T(\coprod_i \Sigma_i)_t$ is $\prod_i (T_{C_i^+} \times T_{C_i^-})$-equivariant, the Hamiltonian torus action of the Goldman systems are preserved by ψ_t. This means that $\psi_t : \mathcal{N}_{\boldsymbol{\alpha}}(\Sigma) \to \mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma)$ preserves angle variables. Hence, $(1/t)\psi_t^* \omega_{\mathcal{N}_{t\alpha}}$ coincides with $\omega_{\mathcal{N}_\alpha}$ on U. Since ψ_t is a diffeomorphism and U is dense, we have $(1/t)\psi_t^* \omega_{\mathcal{N}_{t\alpha}} = \omega_{\mathcal{N}_\alpha}$ on $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$. ■

16 Goldman Systems and Bending Systems

We see in Theorem 9.3 that $\mathcal{N}_{\boldsymbol{\alpha}}$ is isomorphic to $\mathcal{M}_{\boldsymbol{\alpha}}$ as complex manifolds if $|\boldsymbol{\alpha}| < 1$. On the other hand, Jeffrey [Jef94] proved the following by using the $\mathfrak{g}$-extended moduli space.

Proposition 16.1 (Jeffrey [Jef94, Theorem 6.6]) *For sufficiently small $\boldsymbol{\alpha} \in (0, 1/2)^n$, the moduli space $\mathcal{N}_{\boldsymbol{\alpha}}(\Sigma)$ is symplectomorphic to $\mathcal{M}_{\boldsymbol{\alpha}}$.*

Outline of the proof The proposition is proved by using a canonical local model of Hamiltonian spaces called the Marle–Guillemin–Sternberg form [GS84, Mar85]. Recall that the moment map of the G^n-action on the $\mathfrak{g}$-extended moduli space $\mathcal{N}^{\mathfrak{g}}(\Sigma)$ is given by

$$\mu^{\mathfrak{g}} : \mathcal{N}^{\mathfrak{g}}(\Sigma) \longrightarrow \mathfrak{g}^n, \quad (\boldsymbol{a}, \boldsymbol{x}) \longmapsto -\boldsymbol{x}.$$

Since the stabilizer of $(\mathbf{1}, \mathbf{0}) \in (\mu^{\mathfrak{g}})^{-1}(\mathbf{0})$ is the diagonal subgroup $G \subset G^n$, the fiber $(\mu^{\mathfrak{g}})^{-1}(\mathbf{0})$ is identified with G^n/G by

$$G^n/G \longrightarrow (\mu^{\mathfrak{g}})^{-1}(\mathbf{0}), \quad [\sigma_1, \sigma_2, \dots, \sigma_n] \longrightarrow (\sigma_1\sigma_2^{-1}, \dots, \sigma_1\sigma_n^{-1}).$$

Then the Marle–Guillemin–Sternberg form of a neighborhood of $(\mu^{\mathfrak{g}})^{-1}(\mathbf{0})$ is a neighborhood of the zero section of the vector bundle $G^n \times_G (\mathfrak{g}^n/\mathfrak{g})^* \to G^n/G$ equipped with the moment map

$$\mu_{\mathrm{MGS}} : G^n \times_G (\mathfrak{g}^n/\mathfrak{g})^* \longrightarrow \mathfrak{g}^n, \quad [\boldsymbol{\sigma}, \boldsymbol{y}] \longrightarrow (\mathrm{Ad}(\sigma_i) y_i)_i.$$

This implies that

$$\begin{aligned}(\mu^{\mathfrak{g}})^{-1}(\mathcal{O}_{\boldsymbol{\alpha}})/G^n &= (\mu_{\mathrm{MGS}})^{-1}(\mathcal{O}_{\boldsymbol{\alpha}})/G^n \\ &= \{[\boldsymbol{\sigma},\boldsymbol{y}] \in G^n \times_G (\mathfrak{g}^n/\mathfrak{g})^* \mid \mathrm{Ad}(\sigma_i)y_i \in \mathcal{O}_{\alpha_i}, i=1,\dots,n\}/G^n \\ &\cong (\mathcal{O}_{\boldsymbol{\alpha}} \cap \{(x,\dots,x) \in \mathfrak{g}^n \mid x \in \mathfrak{g}\}^{\perp})/G \\ &= \{(x_1,\dots,x_n) \in \mathcal{O}_{\boldsymbol{\alpha}} \mid x_1+\cdots+x_n = 0\}/G = \mathcal{M}_{\boldsymbol{\alpha}}. \qquad \blacksquare\end{aligned}$$

Fix $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| < 1$, and consider the family

$$f\colon \mathfrak{N}(\Sigma) = \bigcup_{t\in(0,1]} \big(\mathcal{N}_{t\boldsymbol{\alpha}}(\Sigma), (1/t)\omega_{\mathcal{N}_{t\alpha}}\big) \longrightarrow (0,1]$$

of symplectic manifolds. From Proposition 16.1, a fiber $(\mathcal{N}_{t\boldsymbol{\alpha}}, \omega_{\mathcal{N}_{t\alpha}})$ over sufficiently small $t \in (0,1]$ is symplectomorphic to $(\mathcal{M}_{t\boldsymbol{\alpha}}, \omega_{\mathcal{M}_{t\alpha}})$. Since $(\mathcal{M}_{\boldsymbol{\alpha}}, \omega_{\mathcal{M}_{\alpha}})$ is symplectomorphic to $(\mathcal{M}_{t\boldsymbol{\alpha}}, (1/t)\omega_{\mathcal{M}_{t\alpha}})$ by scaling $\boldsymbol{x} \longmapsto t\boldsymbol{x}$, we can extend the family $f\colon \mathfrak{N}(\Sigma) \to (0,1]$ to a family over $[0,1]$ by setting $f^{-1}(0) = (\mathcal{M}_{\boldsymbol{\alpha}}, \omega_{\mathcal{M}_{\alpha}})$.

Proposition 16.2 *The symplectic trivialization $\{\psi_t\}$ of $\mathfrak{N}(\Sigma) \to (0,1]$ given in Theorem* 15.1 *extends to the family over $[0,1]$. Moreover, this trivialization identifies Goldman systems $(1/t)\Theta_{t\boldsymbol{\alpha},\Gamma}\colon \mathcal{N}_{t\boldsymbol{\alpha}} \to \mathbb{R}^{n-3}$ and the bending system $\Phi_\Gamma\colon \mathcal{M}_{\boldsymbol{\alpha}} \to \mathbb{R}^{n-3}$.*

Proof Fix $\boldsymbol{g} \in \mathcal{N}_{\boldsymbol{\alpha}}$ and let

$$\boldsymbol{g}(t) = \big(g_1(t),\dots,g_n(t)\big) = \big(e^{x_1(t)},\dots,e^{x_n(t)}\big) := \psi_t(\boldsymbol{g}) \in \mathcal{N}_{t\boldsymbol{\alpha}}$$

be the trajectory of ψ_t starting from $\boldsymbol{g}$. Then $\boldsymbol{x}(t) = (x_1(t),\dots,x_n(t))$ is a smooth curve in $\bigcup_t \mathcal{O}_{t\boldsymbol{\alpha}}$ of the form $x_i(t) = tx_i + O(t^2)$. Since $g_1(t)\dots g_n(t) = 1 + t(x_1+\cdots+x_n) + O(t^2)$, the point $\boldsymbol{x} = (x_1,\dots,x_n)$ lies in $\mathcal{M}_{\boldsymbol{\alpha}}$. We also take smooth families of tangent vectors $u(t), v(t) \in T_{\boldsymbol{g}(t)}\mathcal{N}_{t\boldsymbol{\alpha}}$ such that $d\psi_t(u(1)) = u(t)$ and $d\psi_t(v(1)) = v(t)$. Then

$$\frac{1}{t}\omega_{\mathcal{N}_{t\alpha}}\big(u(t),v(t)\big) = \omega_{\mathcal{N}_{\alpha}}\big(u(1),v(1)\big)$$

for all $t \in (0,1]$. Let $\boldsymbol{\xi}(t) = \boldsymbol{\xi} + O(t)$, $\boldsymbol{\eta}(t) = \boldsymbol{\eta} + O(t)$ be smooth curves in $\mathfrak{g}^n$ such that

$$u(t)(\gamma_i) = \mathrm{Ad}_{g_i(t)}\,\xi_i(t) - \xi_i(t), \quad v(t)(\gamma_i) = \mathrm{Ad}_{g_i(t)}\,\eta_i(t) - \eta_i(t).$$

Since

$$\mathrm{Ad}_{g_i(t)}\,\xi_i(t) - \xi_i(t) = t[x_i,\xi_i] + O(t^2), \tag{31}$$

$\boldsymbol{\xi} = (\xi_1,\dots,\xi_n)$, $\boldsymbol{\eta} = (\eta_1,\dots,\eta_n) \in \mathfrak{g}^n$ give tangent vectors of $\mathcal{M}_{\boldsymbol{\alpha}}$ at $\boldsymbol{x}$. Note that (31) also implies that $\big(u(t) \cup v(t)\big)\big[\pi_1(\Sigma), \partial\pi_1(\Sigma)\big]\big) = O(t^2)$. On the other hand, the second term of $\omega_{\mathcal{N}_{t\alpha}}$ in (15) has the form

$$\frac{1}{2}\sum_{i=1}^{n}\Big(\big\langle \xi_i(t), \mathrm{Ad}_{g_i(t)}\,\eta_i(t)\big\rangle - \big\langle \eta_i(t), \mathrm{Ad}_{g_i(t)}\,\xi_i(t)\big\rangle\Big) = t\sum_{i=1}^{n}\big\langle x_i, [\xi_i,\eta_i]\big\rangle + O(t^2).$$

Thus we have

$$\frac{1}{t}\omega_{\mathcal{N}_{t\alpha}}\big(u(t),v(t)\big) = \omega_{\mathcal{M}_{\alpha}}(\boldsymbol{\xi},\boldsymbol{\eta}) + O(t).$$

Since the left-hand side is independent of t, we have $\frac{1}{t}\omega_{\mathcal{N}_{t\alpha}}\big(u(t), v(t)\big) = \omega_{\mathcal{M}_{\alpha}}(\boldsymbol{\xi}, \boldsymbol{\eta})$, or equivalently $\psi_0^* \omega_{\mathcal{M}_{\alpha}} = \omega_{\mathcal{N}_{\alpha}}$.

Next we show that the integrable systems are identified. Suppose that the k-th boundary component C_k is given by $[C_k] = \gamma_{i_k} \dots \gamma_{i_k+n_k}$. If we write

$$g_{i_k}(t) \dots g_{i_k+n_k}(t) = e^{y_k(t)}$$

for $y_k(t) \in \mathfrak{g}$, then $y_k(t)$ has eigenvalues $\pm\vartheta_{t\alpha, C_k}(\boldsymbol{g}(t))$. Since $y_k(t) = t(x_{i_k} + \cdots + x_{i_k+n_k}) + O(t^2)$ and the eigenvalues of $x_{i_k} + \cdots + x_{i_k+n_k}$ are $\pm\phi_{d_k}(\boldsymbol{x})$, we have

$$\frac{1}{t}\vartheta_{t\alpha, C_k}(\boldsymbol{g}(t)) = \phi_{d_k}(\boldsymbol{x}) + O(t).$$

Theorem 15.1 implies that the left-hand side is also independent of t, and hence $\frac{1}{t}\vartheta_{t\alpha, C_k}$ is identified with ϕ_{d_k} by the trivialization. ∎

Acknowledgment We thank Shinnosuke Okawa for valuable discussions, and the anonymous referee for suggesting improvements. A part of this paper was written while K. U. was visiting the Korea Institute for Advanced Study, whose hospitality and financial support are gratefully acknowledged.

References

[AMM98] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*. J. Differential Geom. **48**(1998), no. 3, 445–495.

[Bau91] S. Bauer, *Parabolic bundles, elliptic surfaces and* SU(2)*-representation spaces of genus zero Fuchsian groups*. Math. Ann. **290**(1991), no. 3, 509–526. http://dx.doi.org/10.1007/BF01459257

[DH98] I. V. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients*. Inst. Hautes Études Sci. Publ. Math. **87**(1998), 5–56.

[FH05] P. Foth and Y. Hu, *Toric degenerations of weight varieties and applications*. In: Travaux mathématiques. Fasc. XVI, Trav. Math., XVI, Univ. Luxemb., Luxembourg, 2005, pp. 87–105.

[GHJW97] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. **89** (1997), no. 2, 377–412. http://dx.doi.org/10.1215/S0012-7094-97-08917-1

[GL87] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*. In: Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., 1273, Springer, Berlin, 1987, pp. 265–297.

[Gol86] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*. Invent. Math. **85**(1986), no. 2, 263–302. http://dx.doi.org/10.1007/BF01389091

[GS84] V. Guillemin and S. Sternberg, *A normal form for the moment map*. In: Differential geometric methods in mathematical physics (Jerusalem, 1982), Math. Phys. Stud., 6, Reidel, Dordrecht, 1984, pp. 161–175.

[HJ00] J. C. Hurtubise and L. C. Jeffrey, *Representations with weighted frames and framed parabolic bundles*. Canad. J. Math. **52**(2000), no. 6, 1235–1268. http://dx.doi.org/10.4153/CJM-2000-052-4

[HK97] J.-C. Hausmann and A. Knutson, *Polygon spaces and Grassmannians*. Enseign. Math. (2) **43**(1997), no. 1–2, 173–198.

[HMM11] B. Howard, C. Manon, and J. Millson, *The toric geometry of triangulated polygons in Euclidean space*. Canad. J. Math. **63**(2011), no. 4, 878–937. http://dx.doi.org/10.4153/CJM-2011-021-0

[Jef94] L. C. Jeffrey, *Extended moduli spaces of flat connections on Riemann surfaces*. Math. Ann. **298**(1994), no. 4, 667–692. http://dx.doi.org/10.1007/BF01459756

[JW92] L. C. Jeffrey and J. Weitsman, *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula*. Comm. Math. Phys. **150**(1992), no. 3, 593–630. http://dx.doi.org/10.1007/BF02096964

[JW94] ______, *Toric structures on the moduli space of flat connections on a Riemann surface: volumes and the moment map*. Adv. Math. **106**(1994), no. 2, 151–168. http://dx.doi.org/10.1006/aima.1994.1054

[JW97] ______, *Toric structures on the moduli space of flat connections on a Riemann surface. II. Inductive decomposition of the moduli space*. Math. Ann. **307**(1997), no. 1, 93–108. http://dx.doi.org/10.1007/s002080050024

[Kly94] A. A. Klyachko, *Spatial polygons and stable configurations of points in the projective line*. In: Algebraic geometry and its applications (Yaroslavl, 1992), Aspects Math., E25, Vieweg, Braunschweig, 1994, pp. 67–84.

[KM96] M. Kapovich and J. J. Millson, *The symplectic geometry of polygons in Euclidean space*. J. Differential Geom. **44**(1996), no. 3, 479–513.

[KY02] Y. Kamiyama and T. Yoshida, *Symplectic toric space associated to triangle inequalities*. Geom. Dedicata **93**(2002), 25–36. http://dx.doi.org/10.1023/A:1020393910472

[Len11] H. Lenzing, *Weighted projective lines and applications*. In: Representations of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 153–187.

[Man] C. A. Manon, *The algebra of conformal blocks*. arxiv:0910.0577

[Mar85] C.-M. Marle, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*. Rend. Sem. Mat. Univ. Politec. Torino **43**(1985), no. 2, 227–251.

[MP01] J. J. Millson and J. A. Poritz, *Around polygons in* $\mathbb{R}^3$ *and* S^3. Comm. Math. Phys. **218**(2001), no. 2, 315–331. http://dx.doi.org/10.1007/PL00005557

[MS80] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*. Math. Ann. **248**(1980), no. 3, 205–239. http://dx.doi.org/10.1007/BF01420526

[NNU10] T. Nishinou, Y. Nohara, and K. Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions*. Adv. Math. **224**(2010), no. 2, 648–706. http://dx.doi.org/10.1016/j.aim.2009.12.012

[NU14] Y. Nohara and K. Ueda, *Toric degenerations of integrable systems on Grassmannians and polygon spaces*. Nagoya Math. J. **214**(2014), 125–168. http://dx.doi.org/10.1215/00277630-2643839

[SS04] D. Speyer and B. Sturmfels, *The tropical Grassmannian*. Adv. Geom. **4**(2004), no. 3, 389–411.

[Tha96] M. Thaddeus, *Geometric invariant theory and flips*. J. Amer. Math. Soc. **9**(1996), no. 3, 691–723. http://dx.doi.org/10.1090/S0894-0347-96-00204-4

[Tre02] T. Treloar, *The symplectic geometry of polygons in the 3-sphere*. Canad. J. Math. **54**(2002), no. 1, 30–54. http://dx.doi.org/10.4153/CJM-2002-002-1

[Wei92] J. Weitsman, *Real polarization of the moduli space of flat connections on a Riemann surface*. Comm. Math. Phys. **145**(1992), no. 3, 425–433. http://dx.doi.org/10.1007/BF02099391

Faculty of Education, Kagawa University, Saiwai-cho 1-1, Takamatsu, Kagawa, 760-8522, Japan
e-mail: nohara@ed.kagawa-u.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
e-mail: kazushi@ms.u-tokyo.ac.jp