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$\mathbb{Z}[1/p]$-motivic resolution of singularities

M. V. Bondarko

To my wife Olga

Abstract

The main goal of this paper is to deduce (from a recent resolution of singularities result of Gabber) the following fact: (effective) Chow motives with $\mathbb{Z}[1/p]$-coefficients over a perfect field k of characteristic p generate the category $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ (of effective geometric Voevodsky's motives with $\mathbb{Z}[1/p]$-coefficients). It follows that $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ can be endowed with a Chow weight structure w_{Chow} whose heart is $\mathrm{Chow}^{\mathrm{eff}}[1/p]$ (weight structures were introduced in a preceding paper, where the existence of w_{Chow} for $DM^{\mathrm{eff}}_{\mathrm{gm}}\mathbb{Q}$ was also proved). As shown in previous papers, this statement immediately yields the existence of a conservative weight complex functor $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to K^b(\mathrm{Chow}^{\mathrm{eff}}[1/p])$ (which induces an isomorphism on K_0-groups), as well as the existence of canonical and functorial (Chow)-weight spectral sequences and weight filtrations for any cohomology theory on $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$. We also mention a certain Chow t-structure for $DM^{\mathrm{eff}}_{-}[1/p]$ and relate it with unramified cohomology.

Introduction

It is well known that Hironaka's resolution of singularities is very important for the theory of (Voevodsky's) motives over characteristic 0 fields; see [GS96, Voe00] and also [Bon09a, Bon10a].

A recent resolution of singularities result of Gabber (see [Ill08, Theorem 1.3]) could be called '$\mathbb{Z}_{(l)}$-resolution of singularities' over a perfect characteristic p field k (where l is any prime distinct from p). In particular, it implies that Chow motives with $\mathbb{Z}_{(l)}$-coefficients generate the triangulated category of Voevodsky's motives; hence, that this category can be endowed with a *Chow weight structure* (cf. [Bon10a, § 6.6]).

The main goal of this paper is to extend these results to motives with $\mathbb{Z}[1/p]$-coefficients. We also mention several consequences of the main results (they can be easily obtained using the methods of [Bon10a]). In particular, there exists a conservative exact *weight complex* functor $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to K^b(\mathrm{Chow}^{\mathrm{eff}}[1/p])$; $K_0(\mathrm{Chow}^{\mathrm{eff}}[1/p]) \cong K_0(DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p])$. Previously, these results were known to hold only for motives with rational coefficients.

The notion of *weight structures* is central for this paper. Weight structures are natural counterparts of t-structures for triangulated categories, introduced in [Bon10a] (and independently

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in [Pau08]). They were thoroughly studied and applied to motives in [Bon10a, Bon10b] (see also [Bon10d, Heb10] and the survey preprint [Bon09b]). Weight structures allow proving the properties of motives mentioned above; they are also crucial for our proof of the central result (that $\mathrm{Chow}^{\mathrm{eff}}[1/p]$ generates $DM_{\mathrm{gm}}^{\mathrm{eff}}[1/p]$ as a triangulated category).

Now we list the contents of the paper. More details can be found at the beginnings of sections.

In § 1, we recall some basic properties of motives and weight structures. Most of them are just modifications of results of [Bon10a, Voe00]; the only absolutely new result is a new condition for the existence of weight structures. We also recall a recent result on resolution of singularities over characteristic p fields (proved by O. Gabber).

In § 2, we prove our central theorem on the existence of the Chow weight structure for $DM_{\mathrm{gm}}^{\mathrm{eff}}[1/p]$; we deduce this result from its certain $\mathbb{Z}_{(l)}$-version. Next, we deduce the existence of the weight complex functor, and calculate $K_0(DM_{\mathrm{gm}}^{\mathrm{eff}}[1/p])$ and $K_0(DM_{\mathrm{gm}}[1/p])$.

We also list some other applications. $DM_{\mathrm{gm}}[1/p]$ possesses a perfect duality; this allows defining $\mathbb{Z}[1/p]$-motives with compact support for arbitrary smooth varieties. There exist *Chow-weight spectral sequences* for any cohomology of $\mathbb{Z}[1/p]$-motives, as well as a certain *Chow t-structure* for $DM_{-}^{\mathrm{eff}}[1/p]$ (whose heart is $\mathrm{AddFun}(\mathrm{Chow}^{\mathrm{eff}}[1/p]^{\mathrm{op}}, Ab)$). We relate the latter with birational sheaves and unramified cohomology.

The proofs and more details for these applications can be found in the parallel report [Bon10c].

A caution on signs of weights. When the author defined weight structures (in [Bon10a]), he chose $\underline{C}^{w\leqslant 0}$ to be stable with respect to [1] (similarly to the usual convention for t-structures); in particular, this meant that for $\underline{C} = K(B)$ (the homotopy category of cohomological complexes) and for the 'stupid' weight structure for it (see [Bon10a, § 1.1]), a complex C whose only non-zero term is the fifth one was 'of weight 5'. Whereas this convention seems to be quite natural, the author recently realized that for weights of mixed Hodge complexes, mixed Hodge modules (see [Bon10e, Proposition 2.6]), and mixed complexes of sheaves (see [Bon10d, Proposition 3.6.1]) 'classically' exactly the opposite convention is used (so, our C would have weight -5; cf. Proposition 2.3.2(2) below). In the current paper we reverse our convention for signs of weights, to make it compatible with the 'classical' convention (this convention for the Chow weight structure for motives was already used in [Heb10] and in [Wil09]; see [Wil09, Remark 1.2]); so, the signs of weights used below will be opposite to those in [Bon10a, Bon10b], as well as to those in the current versions of [Bon09b, Bon10c, Bon10d, Bon10e].

Notation. For a category C, $A, B \in \mathrm{Obj}\, C$, we denote by $C(A, B)$ the set of C-morphisms from A to B. For categories C, D, we write $D \subset C$ if D is a full subcategory of C.

For a category C, $X, Y \in \mathrm{Obj}\, C$, we say that X is a *retract* of Y if id_X could be factorized through Y. For an additive $D \subset C$, the subcategory D is called *Karoubi-closed* in C if it contains all retracts of its objects in C. The full subcategory of C whose objects are all retracts of objects of D (in C) will be called the Karoubi-closure of D in C.

$X \in \mathrm{Obj}\, C$ will be called compact if the functor $C(X, -)$ respects all small coproducts that exist in C (contrary to tradition, we do not assume that arbitrary coproducts exist).

For $X, Y \in \mathrm{Obj}\, B$, we will write $X \perp Y$ if $B(X, Y) = \{0\}$. For $D, E \subset \mathrm{Obj}\, B$, we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset B$, we will denote by $D^{\perp}$ the class

$$\{Y \in \mathrm{Obj}\, B : X \perp Y\ \forall X \in D\}.$$

Dually, ${}^{\perp}D$ is the class $\{Y \in \mathrm{Obj}\, B : Y \perp X\ \forall X \in D\}$.

$\underline{C}$ below will always denote some triangulated category; usually, it will be endowed with a weight structure w (see Definition 1.3.1 below).

We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a functor that preserves the structures of triangulated categories).

For $f \in \underline{C}(X, Y)$, $X, Y \in \operatorname{Obj} \underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of f. We will often specify a distinguished triangle by two of its morphisms.

For a set of objects $C_i \in \operatorname{Obj} \underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i; for $D \subset \underline{C}$, we will write $\langle D \rangle$ instead of $\langle C : C \in \operatorname{Obj} D \rangle$.

We will say that $C_i \in \operatorname{Obj} \underline{C}$ generate $\underline{C}$ if $\underline{C}$ equals $\langle C_i \rangle$. We will say that C_i *weakly generate* $\underline{C}$ if $\{C_i[j], j \in Z\}^{\perp} = \{0\}$.

$D \subset \operatorname{Obj} \underline{C}$ will be called *extension-stable* if, for any distinguished triangle $A \to B \to C$ in $\underline{C}$, we have $A, C \in D \Longrightarrow B \in D$.

k will be our perfect base field of characteristic $p > 0$. Var $\supset$ SmVar $\supset$ SmPrVar will denote the set of all varieties over k, respectively of smooth varieties, respectively of smooth projective varieties.

l below will be some prime number distinct from p (we will assume it to be fixed from time to time).

1. Preliminaries: motives and weight structures

In this section, we recall some basics on motives, weight structures, and resolution of singularities.

In § 1.1, we study Voevodsky's motives with various coefficient rings (following [MVW06, Voe00]).

In § 1.2, we recall a recent result of Gabber on resolution of singularities; we also 'translate it into a motivic form'.

In § 1.3, we recall those basics of the theory of weight structures (developed in [Bon10a]) that will be needed below.

In § 1.4, we prove a certain new criterion for the existence of a weight structure.

1.1 Some basics on motives with various coefficient rings

For motives with integral coefficients, we use the notation of [Voe00]: SmCor, Shv(SmCor) (the category of Nisnevich sheaves with transfers), $\mathrm{Chow}^{\mathrm{eff}} \subset DM^{\mathrm{eff}}_{\mathrm{gm}} \subset DM^{\mathrm{eff}}_{-} \subset D^{-}(\mathrm{Shv}(\mathrm{SmCor}))$; $M_{\mathrm{gm}} : \mathrm{SmVar} \to DM^{\mathrm{eff}}_{\mathrm{gm}}$; $\mathbb{Z}(1)$.

Now recall that (as was shown in [MVW06]) one can do the theory of motives with coefficients in an arbitrary commutative associative ring with a unit R. One should start with the naturally defined category of R-correspondences: $\operatorname{Obj}(\mathrm{SmCor}_R) = \mathrm{SmVar}$; for X, Y in SmVar, we set $\mathrm{SmCor}_R(X, Y) = \bigoplus_U R$ for all integral closed $U \subset X \times Y$ that are finite over X and dominant over a connected component of X. Proceeding as in [Voe00] (i.e. considering the corresponding localization of $K^b(\mathrm{SmCor}_R)$, and complexes of sheaves with transfers with homotopy invariant cohomology), one obtains the theory of motives (i.e. of $DM^{\mathrm{eff}}_{\mathrm{gm}R}$ that lies in $DM_{\mathrm{gm}R}$ and in $DM^{\mathrm{eff}}_{-}{}_R$) that satisfies all basic properties of the 'usual' Voevodsky's motives (i.e. of those with integral coefficients; note that some of the results of [Voe00] were extended to the case char $k > 0$ in [Deg08, HK06]). So, we will apply these properties of motives with R-coefficients without any further mention.

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In this paper, we will mostly consider motives with $\mathbb{Z}[1/p]$- and $\mathbb{Z}_{(l)}$-coefficients. We will denote by $\mathrm{Chow}^{\mathrm{eff}}[1/p] \subset DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \subset DM^{\mathrm{eff}}_{-}[1/p]$, $M_{\mathrm{gm}}[1/p] : \mathrm{SmVar} \to DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ (respectively $\mathrm{Chow}^{\mathrm{eff}}_{(l)} \subset DM^{\mathrm{eff}}_{\mathrm{gm},(l)} \subset DM^{\mathrm{eff}}_{-,(l)}$, $M_{\mathrm{gm},(l)} : \mathrm{SmVar} \to DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p])$ and $\mathrm{Chow}[1/p] \subset DM_{\mathrm{gm}}[1/p]$ the corresponding analogues of Voevodsky's notions (note that we have all of the full embeddings listed, indeed).

We list some of the properties of motivic complexes that we will need below. Recall that DM^{eff}_{-} supports the so-called homotopy t-structure t (coming from $D^{-}(\mathrm{Shv}(\mathrm{SmCor}))$). The heart of t is the category HI of homotopy invariant (Nisnevich) sheaves with transfers. Below, we will denote the hearts of the restrictions of t to $DM^{\mathrm{eff}}_{-}[1/p] \supset DM^{\mathrm{eff}}_{-,(l)}$ by $HI[1/p] \supset HI_{(l)}$.

PROPOSITION 1.1.1. (i) *The functors $DM^{\mathrm{eff}}_{-} \to DM^{\mathrm{eff}}_{-}[1/p]$ (respectively $DM^{\mathrm{eff}}_{-}[1/p] \to DM^{\mathrm{eff}}_{-,(l)}$) given by tensoring sheaves by $\mathbb{Z}[1/p]$ (respectively $\mathbb{Z}[1/p]$-module sheaves by $\mathbb{Z}_{(l)}$) tensor all morphism groups by $\mathbb{Z}[1/p]$ (respectively by $\mathbb{Z}_{(l)}$). The same is true for the (compatible) functors $\mathrm{Chow}^{\mathrm{eff}} \to \mathrm{Chow}^{\mathrm{eff}}[1/p] \to \mathrm{Chow}^{\mathrm{eff}}_{(l)}$ and $DM^{\mathrm{eff}}_{\mathrm{gm}} \to DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$.*

(ii) *The collection of functors $\otimes_{\mathbb{Z}_{(l)}} : DM^{\mathrm{eff}}_{-}[1/p] \to DM^{\mathrm{eff}}_{-,(l)}$ for l running through all primes $\neq p$ is conservative (on $DM^{\mathrm{eff}}_{-}[1/p]$).*

(iii) *The forgetful functors that send a complex of $\mathbb{Z}[1/p]$-module sheaves to the underlying complex of sheaves of abelian groups (respectively a complex of $\mathbb{Z}_{(l)}$-module sheaves to the underlying complex of $\mathbb{Z}[1/p]$-module sheaves) yield full embeddings $DM^{\mathrm{eff}}_{-,(l)} \subset DM^{\mathrm{eff}}_{-}[1/p] \subset DM^{\mathrm{eff}}_{-}$.*

(iv) *For any $U \in \mathrm{SmVar}$, $m \in \mathbb{Z}$, $S \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{-}[1/p]$ (respectively $S \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{-,(l)}$), the mth hypercohomology of U with coefficients in S is naturally isomorphic to $DM^{\mathrm{eff}}_{-}[1/p]$ $(M_{\mathrm{gm}}[1/p](U), S[m])$ (respectively $DM^{\mathrm{eff}}_{-,(l)}(M_{\mathrm{gm},(l)}(U), S[m])$).*

(v) *t can be restricted to $DM^{\mathrm{eff}}_{-}[1/p]$ and $DM^{\mathrm{eff}}_{-,(l)}$; the two functors connecting $DM^{\mathrm{eff}}_{-}[1/p]$ with $DM^{\mathrm{eff}}_{-,(l)}$ (described in the previous assertions) are t-exact with respect to these restrictions.*

(vi) *All objects of $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ are compact in $DM^{\mathrm{eff}}_{-}[1/p]$.*

(vii) *For any $X \in \mathrm{SmVar}$, we have $DM^{\mathrm{eff}}_{-}[1/p](X), DM^{\mathrm{eff}}_{-,(l)}(X) \in DM^{\mathrm{eff}}_{-}[1/p]^{t \leqslant 0}$.*

Proof. (i) It suffices to note that $\mathbb{Z}[1/p]$ is flat over $\mathbb{Z}$, and $\mathbb{Z}_{(l)}$ is flat over $\mathbb{Z}[1/p]$.

(ii) Immediate from assertion (i).

(iii) Indeed, these functors are one-sided inverses of the functors $DM^{\mathrm{eff}}_{-} \to DM^{\mathrm{eff}}_{-}[1/p] \to DM^{\mathrm{eff}}_{-,(l)}$ described in assertion (i).

(iv) Immediate from [Voe00, Proposition 3.2.3 and Theorem 3.2.6].

(v), (vi) Easy from the previous assertions.

(vii) Immediate from the corresponding fact for $M_{\mathrm{gm}}(X)$, which is obvious given [Voe00, Proposition 3.2.6]. □

Remark 1.1.2. One can also easily see: all the results proved below for $\mathbb{Z}[1/p]$-motives are also valid for motives with coefficients in an arbitrary (unital commutative) $\mathbb{Z}[1/p]$-algebra; to this end, our proofs could be adjusted straightforwardly.

1.2 Gabber's $\mathbb{Z}_{(l)}$-resolution of singularities

Let $l \neq p$ be fixed. The foundation of this paper is the following result (which easily follows from a result of O. Gabber).

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Proposition 1.2.1. *For any $U \in \mathrm{SmVar}$, there exist an open dense subvariety $U' \subset U$ and a finite flat surjective morphism $f : P' \to U'$ (everywhere) of degree prime to l for $P' \in \mathrm{SmVar}$ such that P' has a smooth projective compactification P.*

Proof. We can assume that U is connected.

Let Q' be some compactification of U. Then, by [Ill08, Theorem 1.3], there exist a finite field extension k'/k of degree prime to l (it is separable since k is perfect), a smooth quasi-projective Q/k', and a finite surjective morphism $g : Q \to Q'_{k'}$ of degree prime to l. Since g is proper, Q is actually projective (in our case). We can also assume that g_U is flat (since we can replace U by some U''/k).

Now we restrict scalars from k' to k and denote Q considered as a variety over k by P. We obtain that $P \in \mathrm{SmPrVar}$, and that there exists a finite flat morphism from some $P' \subset P$ to $U' \times \operatorname{Spec} k'$; the degree of this morphism is prime to l. Lastly, it remains to compose this morphism with the natural morphism $U' \times \operatorname{Spec} k' \to U$, whose degree is also prime to l. □

Now we reformulate this statement 'motivically'.

Corollary 1.2.2. *Let $U \in \mathrm{SmVar}$, $\dim U = m$.*

(1) *For U', P' as in Proposition 1.2.1, $M_{\mathrm{gm},(l)}(U')$ is a retract of $M_{\mathrm{gm},(l)}(P')$.*

(2) *There also exist sequences $X_i, Y_i \in \operatorname{Obj} DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$, $0 \leqslant i \leqslant m$, and $f_i \in DM^{\mathrm{eff}}_{\mathrm{gm},(l)}(X_i, X_{i-1})$, $g_i \in DM^{\mathrm{eff}}_{\mathrm{gm},(l)}(Y_i, Y_{i-1})$ (for $1 \leqslant i \leqslant m$), such that $X_0 = M_{\mathrm{gm},(l)}(U)$, $X_m = M_{\mathrm{gm},(l)}(U')$, $Y_0 = M_{\mathrm{gm},(l)}(P)$, $X_m = M_{\mathrm{gm},(l)}(P')$, $\operatorname{Cone} f_i = M_{\mathrm{gm},(l)}(V_i)(i)[2i]$, and $\operatorname{Cone} g_i = M_{\mathrm{gm},(l)}(W_i)(i)[2i]$, for some smooth varieties $V_i, W_i/k$ of dimension $m - i$ (that could be empty).*

Proof. (1) The transpose of the graph of f yields a finite correspondence from U' to P' (in the sense of [Voe00]). Composing it with f and considering as a morphism of motives, we obtain $\deg f \cdot \mathrm{id}_{M_{\mathrm{gm},(l)}(U')}$ (see [SV00, Lemma 2.3.5]). Since $\deg f$ is prime to l, we obtain that $M_{\mathrm{gm},(l)}(U')$ is a retract of $M_{\mathrm{gm},(l)}(P')$ in $DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$.

(2) We recall the Gysin distinguished triangle (see [Deg08, Proposition 4.3] that establishes its existence in the case $\operatorname{char} k > 0$). For a closed embedding $Z \to X$ of smooth varieties, Z is everywhere of codimension c in X, it has the form

$$M_{\mathrm{gm}}(X \backslash Z) \to M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(c)[2c] \to M_{\mathrm{gm}}(X \backslash Z)[1]; \qquad (1)$$

certainly, obvious analogues exist for the functors $M_{\mathrm{gm}}[1/p]$ and $M_{\mathrm{gm},(l)}$.

Hence, in order to prove the assertion, it suffices to choose a sequence of $U_i, P_i \in \mathrm{SmVar}$ such that $U_0 = U' \subset U_1 \subset U_2 \subset \cdots \subset U_m = U$ (respectively $P_0 = P' \subset P_1 \subset P_2 \subset \cdots \subset P_m = P$) and $U_i \backslash U_{i-1}$ is non-singular and has codimension i everywhere in U_i (respectively $P_i \backslash P_{i-1}$ is non-singular and has codimension i everywhere in P_i) for all i. Now, in order to obtain such U_i and P_i, it suffices to consider stratifications of $U \backslash U'$ and $P \backslash P'$. □

1.3 Weight structures: reminder

Now we recall the basics of the theory of weight structures. Note that the signs of weights below are opposite to those in [Bon10a]; see the caution in the Introduction.

Definition 1.3.1. (I) A pair of subclasses $\underline{C}^{w\leqslant 0}, \underline{C}^{w\geqslant 0} \subset \operatorname{Obj} \underline{C}$ will be said to define a weight structure w for $\underline{C}$ if they satisfy the following conditions.

(i) $\underline{C}^{w\geqslant 0}, \underline{C}^{w\leqslant 0}$ are additive and Karoubi-closed in $\underline{C}$.

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(ii) $\underline{C}^{w\leqslant 0} \subset \underline{C}^{w\leqslant 0}[1]$, $\underline{C}^{w\geqslant 0}[1] \subset \underline{C}^{w\geqslant 0}$.

(iii) *Orthogonality.*
$\underline{C}^{w\leqslant 0} \perp \underline{C}^{w\geqslant 0}[1]$.

(iv) *Weight decompositions.*
For any $X \in \operatorname{Obj} \underline{C}$, there exists a distinguished triangle

$$B[-1] \to X \to A \xrightarrow{f} B \tag{2}$$

such that $A \in \underline{C}^{w\geqslant 0}$, $B \in \underline{C}^{w\leqslant 0}$.

(II) The full subcategory $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}^{w=0} = \underline{C}^{w\geqslant 0} \cap \underline{C}^{w\leqslant 0}$ will be called the *heart* of the weight structure w.

(III) $\underline{C}^{w\geqslant i}$ (respectively $\underline{C}^{w\leqslant i}$, respectively $\underline{C}^{w=i}$) will denote $\underline{C}^{w\geqslant 0}[i]$ (respectively $\underline{C}^{w\leqslant 0}[i]$, respectively $\underline{C}^{w=0}[i]$).

(IV) We denote $\underline{C}^{w\geqslant i} \cap \underline{C}^{w\leqslant j}$ by $\underline{C}^{[i,j]}$ (so, it equals $\{0\}$ for $i > j$).

(V) We will say that $(\underline{C}, w)$ is *bounded below* if $\bigcup_{i\in\mathbb{Z}} \underline{C}^{w\geqslant i} = \operatorname{Obj} \underline{C}$.

(VI) We will say that $(\underline{C}, w)$ is *bounded* if $\bigcup_{i\in\mathbb{Z}} \underline{C}^{w\leqslant i} = \operatorname{Obj} \underline{C} = \bigcup_{i\in\mathbb{Z}} \underline{C}^{w\geqslant i}$.

(VII) Let H be a full subcategory of a triangulated $\underline{C}$.
We will say that H is *negative* if $\operatorname{Obj} H \perp (\bigcup_{i>0} \operatorname{Obj}(H[i]))$.

(VIII) We will say that a triangulated category $\underline{C}$ is *bounded with respect to* some $H \subset \operatorname{Obj} \underline{C}$ if, for any $X \in \operatorname{Obj} \underline{C}$, there exist $j_X, q_X \in \mathbb{Z}$ such that

$$\bigcup_{i<q_X} \operatorname{Obj} H[i] \perp X \quad \text{and} \quad X \perp \bigcup_{i>j_X} \operatorname{Obj} H[i]. \tag{3}$$

Now we recall those properties of weight structures that will be needed below (and that can be easily formulated), and prove one new assertion. We will not mention more complicated matters (weight complexes and K_0) here; instead, we will just formulate the corresponding 'motivic' results below.

PROPOSITION 1.3.2. *Let $\underline{C}$ be a triangulated category; w will be a weight structure for $\underline{C}$ everywhere except for assertions (iv) and (v).*

(i) *$\underline{C}^{w\leqslant 0}$, $\underline{C}^{w\geqslant 0}$, and $\underline{C}^{w=0}$ are extension-stable.*

(ii) *For any $q, r \in \mathbb{Z}$, $X \in \underline{C}^{[q,r]}$, there exist $X^r \in \underline{C}^{w=0}$ and $f \in \underline{C}(X, X^r[r])$ such that* $\operatorname{Cone} f \in \underline{C}^{w\leqslant r}$.

(iii) *For any $i \leqslant j \in \mathbb{Z}$, we have that $\underline{C}^{[i,j]}$ is the smallest extension-stable subclass of $\operatorname{Obj} \underline{C}$ containing $\bigcup_{i\leqslant l\leqslant j} \underline{C}^{w=l}$. In particular, if w (for $\underline{C}$) is bounded, then $\underline{C} = \langle \underline{Hw} \rangle$.*

(iv) *Let $\underline{C}$ be triangulated and idempotent complete; let $H \subset \operatorname{Obj} \underline{C}$ be negative and additive. Then there exists a unique bounded weight structure w on the Karoubi-closure T of $\langle H \rangle$ in $\underline{C}$ such that $H \subset T^{w=0}$. Its heart is the Karoubi-closure of H in $\underline{C}$.*

(v) *Let $\underline{D}$ be a triangulated category that is weakly generated by some additive set $H \subset D$ of compact objects; suppose that there exists an extension-stable $D \subset \operatorname{Obj} \underline{D}$ such that $H \cup D[1] \subset D$ and arbitrary (small) coproducts exist in D. Denote by H' the Karoubi-closure of the category of all (small) coproducts of objects of H in $\underline{D}$; denote by $\underline{E}$ the triangulated subcategory of $\underline{D}$ whose objects are characterized by the following part of (3): there exists a $q_Y \in \mathbb{Z}$ such that $\bigcup_{i<q_Y} \operatorname{Obj} H[i] \perp X$.*

Then there exists a bounded below weight structure w' for $\underline{E}$ such that $\underline{Hw'}=H'$.

Besides, a compact $X\in \operatorname{Obj}\underline{D}$ belongs to $\underline{E}^{[q,j]}$ (for $q\leqslant j\in\mathbb{Z}$) whenever it satisfies (3) with $j_X=j$ and $q_X=q$.

Proof. (i) This is [Bon10a, Proposition 1.3.3(3)].

(ii) Immediate from the distinguished triangle $A\to B\to X[1]$ and the previous assertion.

(iii) A weight decomposition of $X[-r]$ yields a distinguished triangle $X\to A'\xrightarrow{f'} B'\to X[1]$ for $A'\in\underline{C}^{w\geqslant r}$, $B'\in\underline{C}^{w\leqslant r}$. Assertion (i) implies that $A'\in\underline{C}^{w=r}$. Hence, we can take $X^r=A'[-r]$, $f=f'$.

(iv) Easy from [Bon10a, Proposition 1.5.6(2)].

(v) By [Bon10a, Theorem 4.3.2(II1)], there exists a unique bounded weight structure on $\langle H\rangle$ such that $D\subset\langle H\rangle^{w=0}$. Next, [Bon10a, Proposition 5.2.2] yields that w can be extended to the whole T; along with [Bon10a, Theorem 4.3.2(II2)], it also allows calculating $T^{w=0}$ in this case.

(vi) The existence of w' is immediate from [Bon10a, Theorem 4.3.2(III), version (ii)]. The second part of the assertion is given by [Bon10a, Theorem 4.3.2(III), part V2] (cf. [Bon10a, Definition 4.2.1]).

1.4 The 'main weight structure lemma'

The main part of the proof of the central theorem is a certain weight structure statement (not contained in [Bon10a]). We formulate and prove it here, since it could be used independently from motives (so, it could be useful even if in the future the resolution of singularities will be fully established over fields of arbitrary characteristic).

Theorem 1.4.1. *Let $\underline{D},D,H$ be as in Proposition 1.3.2(v). Let $\underline{C}\subset\underline{D}$ be an idempotent complete triangulated subcategory such that all objects of $\underline{C}$ are compact in $\underline{D}$, $H\subset\underline{C}$, and $\underline{C}$ is bounded with respect to H.*

Then the following statements are valid.

(1) *$\underline{C}$ is contained in the Karoubi-closure I of $\langle H\rangle$ in $\underline{D}$.*

(2) *There exists a bounded weight structure w for $\underline{C}$ such that $\underline{Hw}$ is the Karoubi-closure of H in $\underline{C}$.*

(3) *For $X\in\operatorname{Obj}\underline{C}$, we have $X\in\underline{C}^{[q,j]}$ whenever one can take j for j_X and q for q_X in (3).*

Proof. We adopt the notation of Proposition 1.3.2(v).

We have $\underline{C}\subset\underline{E}$ (by the definition of the latter). Besides (as proved in Proposition 1.3.2(v)), the analogue of assertion (3) with w' instead of w and with $\underline{E}^{[q,j]}$ instead of $\underline{C}^{[q,j]}$ is valid.

Now we prove assertion (1). We denote $\operatorname{Obj} I$ by G.

We should prove that

$$X\in\operatorname{Obj}\underline{C}\ \cap\ \underline{E}^{[r,q]}\Longrightarrow X\in G \tag{4}$$

for any $r\leqslant q\in\mathbb{Z}$.

First, let $q=r$. Then $X[-q]$ is a retract of $\coprod_{i\in I}H_i$ for some set I and $H_i\in\operatorname{Obj}H$. So, $\mathrm{id}_{X[-q]}$ factorizes through $\coprod_{i\in I}H_i$. Since $X[-q]$ is compact, $\underline{D}(X[-q],\coprod H_i)=\bigoplus\underline{D}(X[-q],H_i)$; so, $\mathrm{id}_{X[-q]}$ also can be factorized through $\coprod_{i\in J}H_i$ for some finite $J\subset I$. Hence, $X[-q]$ is a retract of $\coprod_{i\in J}H_i$; so, $X\in G$.

Now we prove (4) in the general case by induction on $q-r$.

Suppose that it is fulfilled for all q, r such that $q-r \leqslant m$ for some $m \geqslant 0$. We prove (4) for some fixed $X \in \operatorname{Obj} \underline{C} \cap \underline{E}^{[s,t]}$, where $t-s=m+1$. By Proposition 1.3.2(ii), there exist $X^t \in \operatorname{Obj} H'$ and $f \in \underline{D}(X, X^t[t])$ such that $\operatorname{Cone} f \in \underline{E}^{w' \leqslant t}$. By the definition of H', X^t is a retract of some $\coprod_{i \in I} H_i$, $H_i \in \operatorname{Obj} H$. Since $\operatorname{Cone} f \in \underline{E}^{w' \leqslant t}$, a cone of the induced morphism $X \to \coprod_{i \in I} H_i[t]$ also belongs to $\underline{E}^{w' \leqslant t}$ (since it is the direct sum of $\operatorname{Cone} f$ with the 'complement' of $X^t[t]$ to $\coprod_{i \in I} H_i[t]$). So, we assume that $X^t = \coprod_{i \in I} H_i$. Now, since $\underline{D}(X, \coprod H_i[t]) = \bigoplus \underline{D}(X, H_i[t])$, f can be factorized through $\coprod_{i \in J} H_i[t]$ (for some finite J). Then $\operatorname{Cone} f = \operatorname{Cone}(f' : X \to \bigoplus_{i \in J} H_i[t]) \bigoplus \coprod_{i \in I \setminus J} X_i[t]$, where f' is the morphism 'induced' by f. So, $\operatorname{Cone} f' \in \underline{E}^{w' \leqslant t}$; it also belongs to $\underline{E}^{w' \geqslant s}$ by Proposition 1.3.2(i). Hence, $\operatorname{Cone} f' \in G$. Since $\bigoplus_{i \in J} H_i[t] \in G$, we obtain that $X \in G$.

Now, Proposition 1.3.2(iv) implies that w' can be restricted to $\underline{C}$ and the weight structure w obtained is the one required for assertion (2). Besides, the reasoning above also proves assertion (3) (by Proposition 1.3.2(i)). $\square$

2. Motivic resolution of singularities

In § 2.1, we prove (almost) a $\mathbb{Z}_{(l)}$-version of our main result. Then Theorem 1.4.1 allows us to deduce our central theorem (in § 2.2).

Next (in § 2.3), we recall (following [Bon10a]) that the existence of w_{Chow} implies the existence of the weight complex functor ($DM_{\mathrm{gm}}[1/p] \to K^b(\mathrm{Chow}[1/p])$; it is exact and conservative); we also compute certain K_0-groups of $DM_{\mathrm{gm}}[1/p]$ (and $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$).

We also mention several other applications. $DM_{\mathrm{gm}}[1/p]$ is a perfect triangulated category; hence, for any smooth variety, there exists a 'reasonable' motif with compact support for it (in $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$). For any cohomology H defined on $DM_{\mathrm{gm}}[1/p]$, *Chow-weight* spectral sequences (that relate H-cohomology of Voevodsky's motives with that of Chow ones) exist and are $DM_{\mathrm{gm}}[1/p]$-functorial starting from E_2. Besides, there exists a certain *Chow t-structure* t_{Chow} for $DM^{\mathrm{eff}}_{-}[1/p]$ whose heart is $\mathrm{AddFun}(\mathrm{Chow}^{\mathrm{eff}}[1/p]^{\mathrm{op}}, Ab)$. A homotopy invariant sheaf with transfers belongs to the heart of t_{Chow} whenever it is birational; we express unramified cohomology in terms of t_{Chow}. We also mention *birational motives* and birational homotopy invariant sheaves with transfers (as defined in [KS]).

2.1 $\mathbb{Z}_{(l)}$-version of the central theorem

We fix some $l (\in \mathbb{P} \setminus \{p\})$.

We prove a statement that is essentially the $\mathbb{Z}_{(l)}$-version of our main result. We do not formulate it in this way since our goal is just to prepare for the proof of Theorem 2.2.1. Yet the notation ${DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-m,0]}$ certainly comes from weight structures.

PROPOSITION 2.1.1. (1) *$DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ is the idempotent completion of $\langle M_{\mathrm{gm},(l)}(P),\ P \in \mathrm{SmPrVar}\rangle$.*

(2) *Let $U \in \mathrm{SmVar}$, $\dim U = m$; let $P \in \mathrm{SmPrVar}$. Then $DM^{\mathrm{eff}}_{-,(l)}(M_{\mathrm{gm},(l)}(U), M_{\mathrm{gm},(l)}(P)[i]) = \{0\}$ for $i > 0$; $DM^{\mathrm{eff}}_{-,(l)}(M_{\mathrm{gm},(l)}(P), M_{\mathrm{gm},(l)}(U)[i]) = \{0\}$ for $i > m$.*

Proof. First, we note that by [Deg08, Theorem 5.23] the subcategory $H_{DM^{\mathrm{eff}}_{\mathrm{gm}}}$ of $DM^{\mathrm{eff}}_{\mathrm{gm}}$ whose objects are $\{M_{\mathrm{gm},(l)}(P),\ P \in \mathrm{SmPrVar}\}$ is negative (here we use the isomorphism of $(M_{\mathrm{gm}}(X, \mathbb{Z}(i)[j]))$ with higher Chow groups). Hence, $\{M_{\mathrm{gm},(l)}(P),\ P \in \mathrm{SmPrVar}\}$ is negative in $DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ also; we denote this category by H.

We define ${DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[r,0]} \subset \operatorname{Obj} DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ for $r \leqslant 0$ as the smallest extension-stable Karoubi-closed subclass of $\operatorname{Obj} DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ that contains $M_{\mathrm{gm},(l)}(P)[s]$ for all $P \in \mathrm{SmPrVar}$, $r \leqslant s \leqslant 0$.

Since $DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ is the idempotent completion of $\langle M_{\mathrm{gm},(l)}(U), U \in \mathrm{SmVar}\rangle$ (in $DM^{\mathrm{eff}}_{-,(l)}$) by definition, in order to prove assertion (1) it suffices to verify: in $DM^{\mathrm{eff}}_{-,(l)}$ the Karoubi-closure of $\langle M_{\mathrm{gm},(l)}(P), P \in \mathrm{SmPrVar}\rangle$ contains all $M_{\mathrm{gm},(l)}(U)$ for $U \in \mathrm{SmVar}$. Hence, the negativity of H easily implies: in order to prove both of our assertions, it suffices to verify that $M_{\mathrm{gm},(l)}(U) \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-m,0]}$ for any U as in assertion (2).

The latter statement is obvious for $m = 0$. We prove it in general by induction on m.

First, we note that ${DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-m,0]}(1)[2] \subset {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-m,0]}$ for any m, since $M_{\mathrm{gm},(l)}(P)(1)[2]$ is a retract of $M_{\mathrm{gm},(l)}(P \times \mathbb{P}^1)$ (for $P \in \mathrm{SmVar}$). Hence, $M_{\mathrm{gm},(l)}(Z)(c)[2c] \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[1-n,0]}$ for any Z of dimension $<n$ and any $c \geqslant 0$.

Suppose now that our assertion is true for all $m < n$. We verify it for a U of dimension n.

We apply Corollary 1.2.2(2). In the notation of Corollary 1.2.2(2) (for $m = n$), we obtain for any $i > 0$: $X_{i-1} \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-n,0]}$ whenever $X_i \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-n,0]}$, and $Y_{i-1} \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-n,0]}$ whenever $Y_i \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-n,0]}$. Since $Y_0 \in {DM^{\mathrm{eff}}_{\mathrm{gm},(l)}}^{[-n,0]}$, the same is true for Y_n and hence also for X_n and for $X_0 = M_{\mathrm{gm},(l)}(U)$. □

2.2 The main result: 'motivic $\mathbb{Z}[1/p]$-resolution of singularities'

Theorem 2.2.1. (1) *$DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ is the idempotent completion of $\langle M_{\mathrm{gm}}[1/p](P), P \in \mathrm{SmPrVar}\rangle$.*

(2) *There exists a bounded weight structure w_{Chow} for $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ such that $\underline{Hw}_{\mathrm{Chow}} = \mathrm{Chow}^{\mathrm{eff}}[1/p]$.*

(3) *For $U \in \mathrm{SmVar}$, $\dim U = m$, we have $M_{\mathrm{gm}}[1/p](U) \in DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]^{[-m,0]}$.*

(4) *For any open dense embedding $U \to V$, for $U, V \in \mathrm{SmVar}$, we have $\mathrm{Cone}(M_{\mathrm{gm}}(U) \to M_{\mathrm{gm}}(V)) \in DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]^{w_{\mathrm{Chow}} \leqslant 0}$.*

Proof. We set $H = \{M_{\mathrm{gm}}[1/p](P), P \in \mathrm{SmPrVar}\}$, $\underline{C} = DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$, and $\underline{D} = DM^{\mathrm{eff}}_{-}[1/p]$, $D = DM^{\mathrm{eff}}_{-}[1/p]^{t \leqslant 0}$, and verify that the assumptions of Theorem 1.4.1 are fulfilled.

By Proposition 1.1.1(vi), all objects of $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ are compact in $DM^{\mathrm{eff}}_{-}[1/p]$. We have $H \subset D$ by Proposition 1.1.1(vii). Besides, D is extension-stable, contains $D[1] = DM^{\mathrm{eff}}_{-}[1/p]^{t \leqslant -1}$, and admits arbitrary coproducts.

Using [Deg08, Theorem 5.23], we obtain (similarly to the proof of Proposition 2.1.1) that H is negative.

By Proposition 2.1.1, for any $l (\neq p)$ the image of $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ in $DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$ is bounded with respect to the image of H in $DM^{\mathrm{eff}}_{\mathrm{gm},(l)}$. Hence, $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ is bounded with respect to H.

It remains to verify that for any $S \in \operatorname{Obj} DM^{\mathrm{eff}}_{-,(l)}$, $S \neq 0$, there exist $P \in \mathrm{SmPrVar}$ and $j \in \mathbb{Z}$ such that $DM^{\mathrm{eff}}_{-}[1/p](M_{\mathrm{gm}}[1/p](P), S[j]) \neq \{0\}$.

Recall that $DM^{\mathrm{eff}}_{-}[1/p]$ is a full subcategory of $D^{-}(\mathrm{Shv}(\mathrm{SmCor}))$. So, there exist some $U \in \mathrm{SmVar}$ and $m \in \mathbb{Z}$ such that the mth hypercohomology of S at U is non-zero. We choose some $l \neq p$ such that this hypercohomology group is not l-torsion. Then the mth hypercohomology at U of S_l is non-zero also, where S_l is the image of S in $DM^{\mathrm{eff}}_{-,(l)}$. Now, by Proposition 1.1.1(iv) this group is exactly $DM^{\mathrm{eff}}_{-,(l)}(M_{\mathrm{gm},(l)}(U), S_l[m])$. Then Proposition 2.1.1(1) easily implies: there exist $P \in \mathrm{SmPrVar}$ and $j \in \mathbb{Z}$ such that $DM^{\mathrm{eff}}_{-,(l)}(M_{\mathrm{gm},(l)}(P), S_l[j]) \neq \{0\}$. Hence, $DM^{\mathrm{eff}}_{-}[1/p](M_{\mathrm{gm}}[1/p](P), S[j]) \neq \{0\}$ also.

Now we can apply Theorem 1.4.1; it yields assertions (1) and (2) immediately. Applying Proposition 2.1.1(2) for all $l \neq p$ simultaneously along with Theorem 1.4.1(3), we prove assertion (3).

Assertion (4) can be easily deduced from assertion (3) by induction. To this end, we choose a sequence of $U_i \in \mathrm{SmVar}$ such that: $U_0 = U \subset U_1 \subset U_2 \subset \cdots U_m = V$ (for some $m \in \mathbb{Z}$) and $U_{i+1} \backslash U_i$ is non-singular and has some codimension c_i everywhere in U_{i+1} for all i. Then, applying (1) repeatedly, we obtain the result; cf. the proof of Proposition 2.1.1. □

Remark 2.2.2. (1) Our 'globalization' argument (i.e. passing from $\mathbb{Z}_{(l)}$-coefficients to $\mathbb{Z}[1/p]$-ones) certainly can be applied in other situations; it only requires some of 'formal' properties of motives (with $\mathbb{Z}[1/p]$ and $\mathbb{Z}_{(l)}$-coefficients) to be fulfilled.

One could even pass to integral coefficients if similar $\mathbb{Z}_{(p)}$-information is available also.

(2) A category of relative Voevodsky's motives could be an example of a setup of this sort. This means: one should consider (some) Voevodsky's motives over a base scheme S; note that in [CD09] a rational coefficient version of such a category was thoroughly studied and called the category of Beilinson motives, whereas in [Bon10d, Heb10] a certain Chow weight structure for this category was introduced. Unfortunately, currently we do not know much about S-motives with $\mathbb{Z}_{(l)}$-coefficients.

(3) We will mention several implications from our result below. Now we will only note that any $X \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ has a 'filtration' (that can be easily described in terms of weight decompositions of $X[i]$, $i \in \mathbb{Z}$) whose 'factors' are objects of $\mathrm{Chow}^{\mathrm{eff}}[1/p]$ (this is a *weight Postnikov tower* of X; see [Bon10a, Definition 1.5.8]). In particular, for any $U \in \mathrm{SmVar}$, $X = M_{\mathrm{gm}}[1/p](U)$, there exist $X^0 \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}[1/p]$ and $f \in DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p](X, X^0)$ such that $\mathrm{Cone}\, f \in DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]^{w_{\mathrm{Chow}} \leqslant 0}$. Note here that $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p](X, X^0)$ can be described in terms of SmCor; one can assume that $X^0 = M_{\mathrm{gm}}[1/p](P)$ for some $P \in \mathrm{SmPrVar}$.

Now, if U admits a smooth compactification P, then $M_{\mathrm{gm}}[1/p](P)$ is one of the possible choices of X^0 (see part (4) of the theorem). So, our results yield the existence of a certain 'motivic' analogue of a smooth compactification of U; this justifies the title of the paper. Moreover, for motives with $\mathbb{Z}_{(l)}$-coefficients one could try to find some X^0 using Gabber's resolution of singularities of results. Yet with $\mathbb{Z}[1/p]$-coefficients this result seems to be very far from being obvious from 'geometry'; it is also not clear how to look for a 'geometric' candidate for X^0 in the absence of a $\mathbb{Z}[1/p]$-analogue of Proposition 1.2.1. □

2.3 The weight complex functor, $K_0(DM_{\mathrm{gm}}[1/p])$, and other applications

In order to study $DM_{\mathrm{gm}}[1/p]$, we will need the following lemma (that is immediate from Theorem 2.2.1).

LEMMA 2.3.1. (1) $DM_{\mathrm{gm}}[1/p] = \langle \mathrm{Chow}[1/p] \rangle$.

(2) *There exists a weight structure on* $DM_{\mathrm{gm}}[1/p]$ *extending* w_{Chow} *for* $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$, *whose heart is* $\mathrm{Chow}[1/p]$.

Proof. Since $- \otimes \mathbb{Z}(1)[2]$ is a full embedding of $DM^{\mathrm{eff}}_{\mathrm{gm}}$ into itself (see [Voe10]), the same is true for $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$. Now we note that $DM_{\mathrm{gm}}[1/p] = \bigcup_{i \in \mathbb{Z}} DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p](i)[2i]$, whereas $\mathrm{Chow}[1/p] = \bigcup_{i \in \mathbb{Z}} \mathrm{Chow}^{\mathrm{eff}}[1/p](i)[2i]$. The result follows immediately. □

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This yields the following result.

Proposition 2.3.2. (1) *There exists an exact conservative weight complex functor* $t: DM_{\mathrm{gm}}[1/p] \to K^b(\mathrm{Chow}[1/p])$ *which restricts to an (exact conservative) functor* $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to K^b(\mathrm{Chow}^{\mathrm{eff}}[1/p])$.

(2) *For* $X \in \mathrm{Obj}\, DM_{\mathrm{gm}}[1/p]$, $i, j \in \mathbb{Z}$, *we have* $X \in DM_{\mathrm{gm}}[1/p]^{[i,j]}$ *whenever* $t(X)$ *is (homotopy equivalent to a complex) concentrated in degrees* $[-j, -i]$.

Proof. (1) By [Bon10a, Proposition 5.3.3], this follows from the existence of bounded Chow-weight structures for $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \subset DM_{\mathrm{gm}}[1/p]$ along with the fact that these categories admit differential graded enhancements (see [Bon10a, Definition 6.1.2 and § 7.3]).

(2) Immediate from [Bon10a, Theorem 3.3.1(IV)]. □

The 'first ancestor' of weight complex functors (the 'current' one and that for general triangulated categories with weight structures were introduced in [Bon10a]) was defined by Gillet and Soulé in [GS96]. To a variety X over a characteristic 0 field, they (essentially) assigned $t(M^c_{\mathrm{gm}}(X))$; see [Bon09a, §§ 6.5–6.6] and § 2.3 below. Yet for char $k > 0$ their methods only yield the existence of weight complexes with values either in $K^b(\mathrm{Chow}^{\mathrm{eff}}\mathbb{Q})$ or in $K(\mathrm{Chow}^{\mathrm{eff}}_{(l)})$ (i.e. they do not prove that $\mathbb{Z}_{(l)}$-weight complexes are always homotopy equivalent to bounded ones; see [GS09, § 5]).

The existence of t allows us to calculate certain K_0-groups of $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \subset DM_{\mathrm{gm}}[1/p]$ (since the induced homomorphism $K_0(DM_{\mathrm{gm}}[1/p]) \to K_0(K^b(\mathrm{Chow}[1/p])) \cong K^0(\mathrm{Chow}[1/p])$ inverts the homomorphism $i: K_0(\mathrm{Chow}[1/p]) \to K_0(DM_{\mathrm{gm}}[1/p])$; the only problem here is to prove that $\langle \mathrm{Chow}[1/p] \rangle = DM_{\mathrm{gm}}[1/p]$, so that i is surjective).

Proposition 2.3.3. *We define* $K_0(\mathrm{Chow}^{\mathrm{eff}}[1/p])$ *(respectively* $K_0(\mathrm{Chow}[1/p])$*) as the groups whose generators are* $[X]$, $X \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}[1/p]$ *(respectively* $X \in \mathrm{Obj}\,\mathrm{Chow}[1/p]$*), and the relations are:* $[Z] = [X] + [Y]$ *for* $X, Y, Z \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}[1/p]$ *(respectively* $X, Y, Z \in \mathrm{Obj}\,\mathrm{Chow}[1/p]$*) such that* $Z \cong X \bigoplus Y$. *For* $K_0(DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p])$ *(respectively* $K_0(DM_{\mathrm{gm}}[1/p])$*), we take similar generators and set* $[B] = [A] + [C]$ *if* $A \to B \to C \to A[1]$ *is a distinguished triangle.*

Then the embeddings $\mathrm{Chow}^{\mathrm{eff}}[1/p] \to DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ *and* $\mathrm{Chow}[1/p] \to DM_{\mathrm{gm}}[1/p]$ *yield isomorphisms* $K_0(\mathrm{Chow}^{\mathrm{eff}}[1/p]) \cong K_0(DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p])$ *and* $K_0(\mathrm{Chow}[1/p]) \cong K_0(DM_{\mathrm{gm}}[1/p])$.

Proof. Immediate from Lemma 2.3.1 and [Bon10a, Proposition 5.3.3(3)].

Here we use the fact that $DM_{\mathrm{gm}}[1/p]$ is idempotent complete since $DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ is. □

Remark 2.3.4. Certainly, we have similar isomorphisms for motives with coefficients in any commutative $\mathbb{Z}[1/p]$-algebra. Besides, all these isomorphisms are actually ring isomorphisms.

Now we list some other applications of the main result. More details can be found in [Bon10c, § 3].

Remark 2.3.5. (i) Applying an argument of M. Levine described in [HK06, Appendix B], we obtain that the full subcategory of $DM_{\mathrm{gm}}[1/p]$ generated by $\mathrm{Chow}[1/p]$ (i.e. the whole $DM_{\mathrm{gm}}[1/p]$) enjoys a perfect duality such that the dual of $M_{\mathrm{gm}}[1/p](P)$ for $P \in \mathrm{SmPrVar}$ is $M_{\mathrm{gm}}[1/p](P)(-m)[-2m]$ if P is purely of dimension m. The dual of $DM_{\mathrm{gm}}[1/p]^{w_{\mathrm{Chow}} \leqslant 0}$ with respect to this duality is $DM_{\mathrm{gm}}[1/p]^{w_{\mathrm{Chow}} \geqslant 0}$, and vice versa.

As explained in [HK06, Appendix B], using duality one can define reasonable motives with compact support over k (note that the method of [Voe00, § 4] is not known to work without Hironaka's resolution of singularities): for $U \in \mathrm{SmVar}$ purely of dimension m, we set $M_{\mathrm{gm}}[1/p]^c(U) = \widehat{M_{\mathrm{gm}}[1/p]}(U)(m)[2m] \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$. We also obtain that $M^c_{\mathrm{gm}}(U) \in DM^{\mathrm{eff}}_{\mathrm{gm}}[1/p]^{[0,\dim U]}$.

(ii) Now recall: any weight structure yields certain weight spectral sequences for any cohomology theory (see [Bon10a, Theorem 2.4.2]). So, we obtain certain *Chow-weight* spectral sequences $T(H,X)$ (that relate the cohomology of $X \in \mathrm{Obj}\, DM_{\mathrm{gm}}[1/p]$ with that of the terms of its weight complex, i.e. of Chow motives) for any cohomological functor $H : DM_{\mathrm{gm}}[1/p] \to \underline{A}$ ($\underline{A}$ is abelian); $T(H,X)$ is $DM_{\mathrm{gm}}[1/p]$-functorial in X starting from E_2. $T(H,X)$ induces a certain (Chow)-weight filtration on $H^*(X)$; this filtration is $DM_{\mathrm{gm}}[1/p]$-functorial, and can be (easily) described in terms of weight decompositions (only); see [Bon10a, § 2.1].

In particular, one can take H being $\mathbb{Z}_l$-étale or $\mathbb{Z}[1/p]$-motivic cohomology of motives. For $\mathbb{Q}_l$-étale cohomology of smooth varieties, we get Deligne's weight spectral sequences in this way (see [Bon10a, Remark 2.4.3]). Note here: it certainly suffices to have the Chow weight structure for $DM_{\mathrm{gm},(l)}$ in order to have Chow-weight spectral sequences for $H \otimes \mathbb{Z}_{(l)}$; yet without a $\mathbb{Z}[1/p]$-weight structure it would not be clear at all that the whole collection of these spectral sequences (for all $l \neq p$) can be chosen to come from a single *weight Postnikov tower* for X (see [Bon10a, Definition 1.5.8]). In particular, it is not (really) important whether we use the $\mathbb{Z}[1/p]$-Chow-weight structure or the $\mathbb{Z}_{(l)}$-one in order to construct the weight spectral sequences for $\mathbb{Z}_l$-étale cohomology if we fix l; yet $\mathbb{Z}[1/p]$-weight structure yields certain 'relations' between these spectral sequences for various l, as well as with $\mathbb{Z}[1/p]$-motivic cohomology.

Lastly, recall that for motivic cohomology we obtain quite new spectral sequences (yet a certain easy partial case can be obtained from Bloch's long exact localization sequence for higher Chow groups of varieties) that do not have to degenerate at any fixed level (even rationally; see [Bon10a, Remark 2.4.3]).

(iii) Theorem 2.2.1 along with [Bon10a, Theorem 4.5.2(I1)] immediately yields: there exists a t-structure t_{Chow} for $DM^{\mathrm{eff}}_{-}[1/p]$ whose heart is isomorphic to $\mathrm{AddFun}(\mathrm{Chow}^{\mathrm{eff}}[1/p]^{\mathrm{op}}, Ab)$; this isomorphism is given by restricting $DM^{\mathrm{eff}}_{-}[1/p](-,Y)$ to $\mathrm{Chow}^{\mathrm{eff}}[1/p] \subset DM^{\mathrm{eff}}_{-}[1/p]$ for $Y \in \mathrm{Obj}\, \underline{Ht}_{\mathrm{Chow}} \subset \mathrm{Obj}\, DM^{\mathrm{eff}}_{-}[1/p]$.

(iv) Our methods easily yield certain properties of birational motives and sheaves (some of them were already proved in [KS]; yet note that we extend them to motives with $\mathbb{Z}[1/p]$-coefficients for char $k = p$); see [Bon10c, § 3.3]. In particular, we obtain a certain Chow weight structure for the category $DM_{\mathrm{gm}}[1/p]^0$ defined as in [KS], and calculate its heart.

Here we call $S \in HI[1/p]$ *birational* if $S(f)$ is an isomorphism for any open dense embedding f in SmVar. In [Bon10c, § 3.3], it is proved that $S \in HI[1/p]$ belongs to $\underline{Ht}_{\mathrm{Chow}}$ whenever it is birational. Moreover, $S^0 = H^0_{t_{\mathrm{Chow}}}(S)$ is the maximal birational subsheaf for any $S \in \mathrm{Obj}\, HI[1/p]$). Besides, if $V \in \mathrm{SmVar}$ possesses a smooth projective compactification P, then the image of $S^0(V)$ in $S(V)$ equals the image of $S(P)$ in $S(V)$.

(v) Using this, in [Bon10c, § 3.4] t_{Chow} was related with unramified cohomology: for $C \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{-}[1/p]$ and $X \in \mathrm{SmVar}$, $i \in \mathbb{Z}$, it was proved that $H^i_{\mathrm{un}}(X,C) \cong H^0_{t_{\mathrm{Chow}}}(H^i_t(C))(X)$.

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References

Bon09a M. V. Bondarko, *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura*, J. Inst. Math. Jussieu **8** (2009), 39–97, see also http://arxiv.org/abs/math.AG/0601713.

Bon09b M. V. Bondarko, *Weight structures and motives; comotives, coniveau and Chow-weight spectral sequences: a survey*, Preprint, http://arxiv.org/abs/0903.0091.

Bon10a M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), 387–504, see also http://arxiv.org/abs/0704.4003.

Bon10b M. V. Bondarko, *Motivically functorial coniveau spectral sequences; direct summands of cohomology of function fields*, Doc. Math., extra volume: Andrei Suslin's Sixtieth Birthday (2010), 33–117.

Bon10c M. V. Bondarko, *$\mathbb{Z}[1/p]$-motivic resolution of singularities, and applications*, Preprint, http://arxiv.org/abs/1002.2651.

Bon10d M. V. Bondarko, *Weights for relative motives; relation with mixed complexes of sheaves*, Preprint, http://arxiv.org/abs/1007.4543.

Bon10e M. V. Bondarko, *Weights and t-structures: in general triangulated categories, for 1-motives, mixed motives, and for mixed Hodge complexes and modules*, Preprint, http://arxiv.org/abs/1011.3507.

CD09 D. Cisinski and F. Deglise, *Triangulated categories of mixed motives*, Preprint, http://arxiv.org/abs/0912.2110.

Deg08 F. Deglise, *Around the Gysin triangle II*, Doc. Math. **13** (2008), 613–675.

GS96 H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. Math. **478** (1996), 127–176.

GS09 H. Gillet and C. Soulé, *Motivic weight complexes for arithmetic varieties*, J. Algebra **322** (2009), 3088–3141.

Heb10 D. Hébert, *Structures de poids à la Bondarko sur les motifs de Beilinson*, Preprint, http://arxiv4.library.cornell.edu/abs/1007.0219.

HK06 A. Huber and B. Kahn, *The slice filtration and mixed Tate motives*, Compositio Math. **142** (2006), 907–936.

Ill08 L. Illusie, *On Gabber's refined uniformization*, Talks at the University of Tokyo, 17, 22, 31 January, 7 February, 2008, http://www.math.u-psud.fr/~illusie/refined_uniformization3.pdf.

KS B. Kahn and R. Sujatha, *Birational motives, I*, Preprint, http://www.math.uiuc.edu/K-theory/0596/.

MVW06 C. Mazza, V. Voevodsky and Ch. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2 (American Mathematical Society, Providence, RI, 2006).

Pau08 D. Pauksztello, *Compact cochain objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. **6** (2008), 25–42.

SV00 A. Suslin and V. Voevodsky, *Relative cycles and Chow sheaves*, in *Cycles, transfers and motivic homology theories*, Annals of Mathematics Studies, vol. 143, eds V. Voevodsky, A. Suslin and E. Friedlander (Princeton University Press, Princeton, NJ, 2000), 10–86.

Voe00 V. Voevodsky, *Triangulated category of motives*, in *Cycles, transfers and motivic homology theories*, Annals of Mathematics Studies, vol. 143, eds V. Voevodsky, A. Suslin and E. Friedlander (Princeton University Press, Princeton, NJ, 2000), 188–238.

Voe10 V. Voevodsky, *Cancellation theorem*, Doc. Math., extra volume: Andrei Suslin's Sixtieth Birthday (2010), 671–685.

Wil09 J. Wildeshaus, *Chow motives without projectivity*, Compositio Math. **145** (2009), 1196–1226.

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