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On the p-norm of an Integral Operator in the Half Plane

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Abstract. We give a partial answer to a conjecture of Dostanić on the determination of the norm of a class of integral operators induced by the weighted Bergman projection in the upper half plane.

1 Introduction

Let $\Pi = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. For $1 \le p < \infty$, let $L^p(\Pi)$ be the space of measurable functions on Π with

$$\|f\|_p := \left(\int_\Pi |f(z)|^p dA(z) \right)^{1/p} < \infty,$$

where $dA = (1/\pi) dxdy$ denotes the (normalized) Lebesgue measure on the complex plane.

In [2], Dostanić considered, for $\alpha > -1$, the integral operator

$$\mathcal{K}_\alpha f(z) = 2^\alpha (\alpha + 1) \int_\Pi \frac{(\operatorname{Im} w)^\alpha}{|z - \bar{w}|^{2+\alpha}} f(w) dA(w).$$

This operator appears in a natural way when one considers the orthogonal projection $\mathcal{P}_\alpha$ from the weighted Hilbert space $L^2(\Pi, dA_\alpha)$ onto the weighted Bergman space $\mathscr{A}^2_\alpha(\Pi)$, where

$$dA_\alpha(z) = (\alpha + 1)(2 \operatorname{Im} z)^\alpha dA(z)$$

and $\mathscr{A}^2_\alpha(\Pi)$ is the closed subspace of all analytic functions in $L^2(\Pi, dA_\alpha)$. Explicitly, $\mathcal{P}_\alpha$ is an integral operator on $L^2(\Pi, dA_\alpha)$,

$$\mathcal{P}_\alpha f(z) = \mathrm{i}^{\alpha+2} \int_\Pi \frac{f(w)}{(z - \bar{w})^{2+\alpha}} dA_\alpha(w).$$

It is easy to prove that when $1 < p < \infty$, $\mathcal{K}_\alpha$ is bounded on $L^p(\Pi)$, which immediately implies that the Bergman projection $\mathcal{P}_\alpha$ is also bounded, in view of the obvious relation $\|\mathcal{P}_\alpha\|_p \le \|\mathcal{K}_\alpha\|_p$. Here $\|\mathcal{P}_\alpha\|_p$ and $\|\mathcal{K}_\alpha\|_p$, respectively, denote the operator norms of $\mathcal{P}_\alpha$ and $\mathcal{K}_\alpha$, acting on $L^p(\Pi)$.

Moreover, Dostanić proved the following in [2].

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Theorem 1.1 *Suppose that* $1 < p < \infty$. *Then, for* $\alpha = 2n, n = 0, 1, 2, \dots,$

$$\|\mathcal{K}_\alpha\|_p = \frac{\alpha+1}{\Gamma^2(1+\alpha/2)}\Gamma\Big(\alpha+1-\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big). \tag{1.1}$$

Dostanić conjectured that (1.1) is valid for any $\alpha > -1$, $p > 1$ and $p(\alpha+1) > 1$. In this note we confirm this conjecture under the additional assumption that $p > 3/(\alpha+2)$. More precisely, our main result is the following theorem.

Theorem 1.2 *For any* $\alpha > -1,\ p > 1$ *such that* $p > \max\{1/(\alpha+1), 3/(\alpha+2)\}$, *we have*

$$\|\mathcal{K}_\alpha\|_p = \frac{\alpha+1}{\Gamma^2(1+\alpha/2)}\Gamma\Big(\alpha+1-\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big).$$

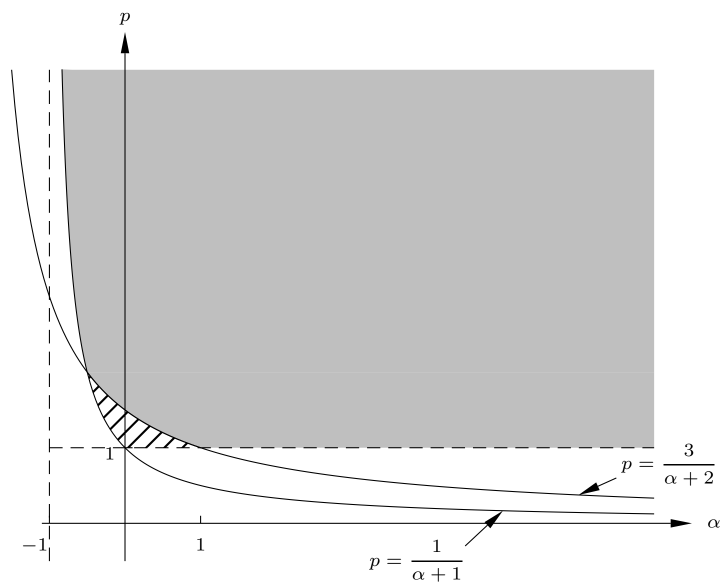

Figure 1: The solid shaded regions indicate the points (α, p) for which $\|\mathcal{K}_\alpha\|_p$ is exactly determined, while the crosshatch regions indicate where the question is still open.

It would be of interest to reformulate Theorem 1.2 as an integral inequality. For simplicity, we consider only the case $\alpha = 0$.

Corollary 1.3 *Let* $1 < p < \infty$ *and* $q := p/(p-1)$ *(the dual exponent). If* $f \in L^p(\Pi)$ *and* $g \in L^q(\Pi)$, *then*

$$\Big|\int_\Pi\int_\Pi \frac{f(z)\overline{g(w)}}{|z-\bar{w}|^2}dA(z)dA(w)\Big| \le \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q.$$

Moreover, when $p > 3/2$, *the constant* $\pi\csc(\pi/p)$ *is sharp.*

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Note the analogy of this result with the classical Hilbert inequality [5, Theorem 316]. If $f \in L^p(0,\infty)$ and $g \in L^q(0,\infty)$, then

$$\Big|\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\Big| \le \frac{\pi}{\sin(\pi/p)}\|f\|_{L^p(0,\infty)}\|g\|_{L^q(0,\infty)},$$

and the constant $\pi\csc(\pi/p)$ is the best possible. So Corollary 1.3 may be thought of as some kind of the "2-dimensional Hilbert inequality".

We now mention other related works. The norm of the Berezin transform on unit disc is calculated in [3]. There is also a nice paper of similar nature by K. Zhu [9], where an asymptotic formula for the norm of the Bergman projection on L^p spaces of the unit ball is given. Also, although not directly related to our results, the determination of the exact L^p norm of singular integral operators has been studied extensively. Results of this type include Pichorides' determination of the p-norm of the Hilbert transform [8] and Iwaniec–Martin's work on the Riesz transform [7]. Also, an outstanding open problem of the past 25 years, known as the Iwaniec conjecture, is the computation of the p-norm of the Beurling–Ahlfors transform [6]. For the present best known estimates on the L^p-norm of the Beurling–Ahlfors transform, see [1] and references therein.

2 Preliminaries

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(a,\, b;\, c;\, z)$ to denote

$${}_2F_1(a,\, b;\, c;\, z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

with $c \neq 0, -1, -2, \dots$, where

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1) \quad \text{for } k \ge 1.$$

We list a few formulas for easy reference (see [4, Chapter II]):

$$\text{(2.1)}\quad {}_2F_1(a,\, b;\, c;\, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \operatorname{Re}(c-a-b) > 0.$$

$$\text{(2.2)}\quad {}_2F_1(a,\, b;\, c;\, z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,\, c-b;\, c;\, z)\,.$$

$$\text{(2.3)}\quad {}_2F_1(a,\, b;\, c;\, z) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\,{}_2F_1(a,\, b;\, \lambda;\, tz)\,dt,$$
$$\operatorname{Re} c > \operatorname{Re}\lambda > 0;\ |\arg(1-z)| < \pi;\ z \neq 1.$$

The following two lemmas are well known; we include the proofs for the reader's convenience.

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Lemma 2.1 *Suppose that $a > 0$, $b > -1$, and $2a - b > 2$. Then*

$$\int_{\Pi} \frac{(\operatorname{Im} w)^b}{|z-\bar{w}|^{2a}} dA(w) = \frac{\Gamma(b+1)\Gamma(2a-b-2)}{2^{2a-2}\Gamma^2(a)} (\operatorname{Im} z)^{2+b-2a}. \tag{2.4}$$

Proof It will be convenient to use real coordinates, so we let $z := x + iy$ and $w := u + iv$. We compute

$$\begin{aligned}
\int_{\Pi} \frac{(\operatorname{Im} w)^b}{|z-\bar{w}|^{2a}} dA(w) &= \frac{1}{\pi}\int_0^\infty \Big\{ \int_{\mathbb{R}} \frac{du}{[(x-u)^2 + (y+v)^2]^a} \Big\} v^b dv \\
&= \frac{1}{\pi}\int_0^\infty \Big\{ \int_{\mathbb{R}} \frac{du}{[u^2 + (y+v)^2]^a} \Big\} v^b dv \\
&= \frac{1}{\pi}\Big\{ \int_{\mathbb{R}} \frac{du}{(1+u^2)^a} \Big\} \Big\{ \int_0^\infty \frac{v^b dv}{(y+v)^{2a-1}} \Big\} \\
&= \frac{1}{\pi}\Big\{ \int_0^\infty \frac{u^{-1/2} du}{(1+u)^a} \Big\} \Big\{ \int_0^\infty \frac{v^b dv}{(1+v)^{2a-1}} \Big\} y^{b-2a+2}.
\end{aligned}$$

Recall the well-known identity

$$\int_0^\infty \frac{t^{p-1} dt}{(1+t)^{p+q}} = B(p,q),$$

where B is the Beta function. We then have

$$\int_{\Pi} \frac{(\operatorname{Im} w)^b}{|z-\bar{w}|^{2a}} dA(w) \ = \ \frac{1}{\pi} B\Big(\frac{1}{2}, a - \frac{1}{2}\Big) B(b+1, 2a-b-2)\, (\operatorname{Im} z)^{b-2a+2}.$$

Finally, an application of the formula

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma\Big(z + \frac{1}{2}\Big)$$

completes the proof. ∎

Lemma 2.2 *For $a \in \mathbb{R}$ and $b > -1$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{2a}} = {}_2F_1\left(a,\ a;\ 1;\ |z|^2\right), \tag{2.5}$$

and

$$\int_D \frac{(1-|w|^2)^b}{|1 - z\bar{w}|^{2a}} dA(w) = \frac{1}{1+b}\, {}_2F_1\left(a,\ a;\ 2+b;\ |z|^2\right). \tag{2.6}$$

Proof We first recall that the binomial expansion

$$(1-\lambda)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \lambda^k$$

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holds for $\lambda \in D$ and $a \in \mathbb{C}$. This, together with the well-known fact

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(m-k)\theta}\,d\theta = \begin{cases} 1, & m = k \\ 0, & m \neq k, \end{cases}$$

allows us to rewrite the left-hand side of (2.5) as

$$\frac{1}{2\pi}\int_0^{2\pi}\Big\{\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k e^{-\mathrm{i}k\theta}\Big\}\Big\{\sum_{m=0}^{\infty}\frac{(a)_m}{m!}\bar{z}^m e^{\mathrm{i}m\theta}\Big\}\,d\theta = \sum_{k=0}^{\infty}\Big\{\frac{(a)_k}{k!}\Big\}^2|z|^{2k},$$

which is exactly the right hand side of (2.5).

Next,

$$\begin{aligned}\int_D \frac{(1-|w|^2)^b}{|1-z\bar{w}|^{2a}}\,dA(w) &= \frac{1}{\pi}\int_0^1 (1-r^2)^b\Big\{\int_0^{2\pi}\frac{d\theta}{|1-ze^{-\mathrm{i}\theta}|^{2a}}\Big\}\,rdr \\ &= \int_0^1 (1-t)^b\,{}_2F_1\left(a,\ a;\ 1;\ t|z|^2\right)dt.\end{aligned}$$

Equation (2.6) then follows from an application of (2.3). ■

The following lemma is crucial.

Lemma 2.3 *For $a \in \mathbb{R}$ and $b > -1$, we have*

$$\int_\Pi \frac{(\operatorname{Im} w)^b\,dA(w)}{|z-\bar{w}|^{2a}|w+\mathrm{i}|^{2b-2a+4}} = \frac{2^{2a-2b-2}}{1+b}|z+\mathrm{i}|^{-2a}\,{}_2F_1\left(a,\ a;\ 2+b;\ \frac{|z-\mathrm{i}|^2}{|z+\mathrm{i}|^2}\right). \tag{2.7}$$

Proof Recall that the inverse Cayley transform

$$\zeta = \phi(z) := \frac{z-\mathrm{i}}{z+\mathrm{i}}$$

maps the upper half plane Π conformally onto the unit disc D. Also, it is easy to check that for any $z, w \in \Pi$,

$$\begin{aligned} \left|1-\phi(z)\overline{\phi(w)}\right|^2 &= \frac{4|z-\bar{w}|^2}{|z+\mathrm{i}|^2|w+\mathrm{i}|^2}; \\ 1-|\phi(w)|^2 &= \frac{4\operatorname{Im} w}{|w+\mathrm{i}|^2}; \\ |\phi'(w)| &= \frac{2}{|w+\mathrm{i}|^2}. \end{aligned} \tag{2.8}$$

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Now we write the left-hand side of (2.7) as

$$2^{2a-2b-2}|z+\mathrm{i}|^{-2a}\int_{\Pi}\Big\{\frac{|z+\mathrm{i}|^2|w+\mathrm{i}|^2}{4|z-\bar{w}|^2}\Big\}^a\Big\{\frac{4\operatorname{Im} w}{|w+\mathrm{i}|^2}\Big\}^b\Big\{\frac{2}{|w+\mathrm{i}|^2}\Big\}^2 dA(w) = \\ 2^{2a-2b-2}|z+\mathrm{i}|^{-2a}\int_{\Pi}\frac{(1-|\phi(w)|^2)^b}{\big|1-\phi(z)\overline{\phi(w)}\big|^{2a}}|\phi'(w)|^2 dA(w).$$

After the change of the variable $\zeta=\phi(w)$, we get

$$\int_{\Pi}\frac{(\operatorname{Im} w)^b dA(w)}{|z-\bar{w}|^{2a}|w+\mathrm{i}|^{2b-2a+4}} = 2^{2a-2b-2}|z+\mathrm{i}|^{-2a}\int_D\frac{(1-|\zeta|^2)^b}{|1-\phi(z)\bar{\zeta}|^{2a}}dA(\zeta).$$

The lemma then follows from (2.6) and (2.8). ■

3 The Proof of Theorem 1.2

It has been proved in [2, p. 227] that if $\alpha>-1,\ p>1$, and $p(\alpha+1)>1$, then

$$\|\mathcal{K}_\alpha\|_p \le \frac{\alpha+1}{\Gamma^2(1+\alpha/2)}\Gamma\Big(\alpha+1-\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big).$$

So we only need to prove that

$$\|\mathcal{K}_\alpha\|_p \ge \frac{\alpha+1}{\Gamma^2(1+\alpha/2)}\Gamma\Big(\alpha+1-\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big)$$

provided that

$$\alpha>-1,\ p>1 \quad\text{and}\quad p>\max\Big\{\frac{1}{\alpha+1},\frac{3}{\alpha+2}\Big\}. \tag{3.1}$$

Fix $0<\epsilon_0<1$ and for $\epsilon\in(0,\epsilon_0]$, consider the function

$$f_\epsilon(z)=\frac{(\operatorname{Im} z)^{(\epsilon-1)/p}}{|z+\mathrm{i}|^{2+\alpha+2(\epsilon-1)/p}}.$$

Note that our assumption (3.1) (more precisely, the condition $p>3/(\alpha+2)$) guarantees that $f_\epsilon\in L^p(\Pi)$. Moreover, by (2.4), we have

$$\|f_\epsilon\|_p^p=\frac{\Gamma(\epsilon)\Gamma(2p+\alpha p+\epsilon-3)}{2^{2p+p\alpha+2\epsilon-4}\Gamma^2(p+p\alpha/2+\epsilon-1)}. \tag{3.2}$$

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Also, by Lemma 2.3 and (2.2), we have

$$\begin{aligned}\mathcal{K}_\alpha f_\epsilon(z) &= 2^\alpha(\alpha+1)\int_\Pi \frac{(\operatorname{Im} w)^{\alpha+(\epsilon-1)/p}\,dA(w)}{|z-\bar{w}|^{2+\alpha}|w+\mathrm{i}|^{2+\alpha+2(\epsilon-1)/p}}\\
&= \frac{2^{-2(\epsilon-1)/p}(\alpha+1)}{\alpha+1+(\epsilon-1)/p}\,|z+\mathrm{i}|^{-(2+\alpha)}\\
&\quad\times {}_2F_1\left(1+\frac{\alpha}{2},\,1+\frac{\alpha}{2};\,2+\alpha+\frac{\epsilon-1}{p};\,\frac{|z-\mathrm{i}|^2}{|z+\mathrm{i}|^2}\right)\\
&= \frac{2^{-2(\epsilon-1)/p}(\alpha+1)}{\alpha+1+(\epsilon-1)/p}\,|z+\mathrm{i}|^{-(2+\alpha)}\left(1-\frac{|z-\mathrm{i}|^2}{|z+\mathrm{i}|^2}\right)^{(\epsilon-1)/p}\\
&\quad\times {}_2F_1\left(1+\frac{\alpha}{2}+\frac{\epsilon-1}{p},\,1+\frac{\alpha}{2}+\frac{\epsilon-1}{p};\,2+\alpha+\frac{\epsilon-1}{p};\,\frac{|z-\mathrm{i}|^2}{|z+\mathrm{i}|^2}\right).\end{aligned}$$

Note that the condition $p > 1/(\alpha+1)$ is sufficient for the application of Lemma 2.3; the condition $p > 3/(\alpha+2)$ is not necessary here. We write this in the form

$$\mathcal{K}_\alpha f_\epsilon(z) = \frac{\alpha+1}{\alpha+1+(\epsilon-1)/p}\,\frac{(\operatorname{Im} z)^{(\epsilon-1)/p}}{|z+\mathrm{i}|^{2+\alpha+2(\epsilon-1)/p}}\,\Psi\left(\epsilon,\,\frac{|z-\mathrm{i}|^2}{|z+\mathrm{i}|^2}\right), \tag{3.3}$$

where

$$\Psi(\epsilon,\lambda) = {}_2F_1\left(1+\frac{\alpha}{2}+\frac{\epsilon-1}{p},\,1+\frac{\alpha}{2}+\frac{\epsilon-1}{p};\,2+\alpha+\frac{\epsilon-1}{p};\,\lambda\right).$$

Lemma 3.1 *The function*

$$y \mapsto \Psi\left(\epsilon,\,\frac{|x+\mathrm{i}y-\mathrm{i}|^2}{|x+\mathrm{i}y+\mathrm{i}|^2}\right)$$

is a decreasing function on $(0,1)$ *when* $x\in\mathbb{R}$ *and* $\epsilon\in[0,\epsilon_0]$ *are fixed.*

Proof We first notice that the hypergeometric function $\Psi(\epsilon,t)$ is an increasing function of t on the interval $[0,1)$, since all its Taylor coefficients are positive. Next, an easy calculation shows

$$\frac{\partial}{\partial y}\left(\frac{|x+\mathrm{i}y-\mathrm{i}|^2}{|x+\mathrm{i}y+\mathrm{i}|^2}\right) = \frac{-4x^2+4(y-1)}{|x+\mathrm{i}y+\mathrm{i}|^4},$$

which implies that the function $y\mapsto |x+\mathrm{i}y-\mathrm{i}|^2/|x+\mathrm{i}y+\mathrm{i}|^2$ is decreasing on $[0,1)$ for any fixed $x\in\mathbb{R}$. The lemma is proved. ∎

Define

$$\Psi^*(\epsilon) := \frac{\Gamma(2+\alpha+(\epsilon-1)/p)\Gamma((-\epsilon+1)/p)}{\Gamma^2(1+\alpha/2)}.$$

Lemma 3.2 *For each $\epsilon \in (0, \epsilon_0]$ and $x \in \mathbb{R}$,*

$$\lim_{y \to 0^+} \Psi\Big(\epsilon, \frac{|x + iy - i|^2}{|x + iy + i|^2}\Big) = \Psi^*(\epsilon).$$

Moreover, the convergence is uniform on $[0, \epsilon_0]$.

Proof The first statement follows immediately from (2.1). To prove the second assertion, we view $\Psi(\epsilon, |x+iy-i|^2/|x+iy+i|^2)$ as a family of continuous functions of ϵ on $[0, \epsilon_0]$ indexed by y. By Lemma 3.1 and the first assertion, we know that these functions tend monotonically to Ψ^* pointwise as $y \searrow 0$. Besides, Ψ^* is continuous on $[0, \epsilon_0]$. Therefore, the convergence is uniform by Dini's theorem. ∎

Lemma 3.2 implies that for any $\eta > 0$, there exists a $\delta \in (0, 1)$, independent of ϵ, such that

$$\Psi\Big(\epsilon, \frac{|x + iy - i|^2}{|x + iy + i|^2}\Big) \geq (1 - \eta) \frac{\Gamma(2 + \alpha + (\epsilon - 1)/p)\Gamma((-\epsilon + 1)/p)}{\Gamma^2(1 + \alpha/2)},$$

whenever $0 < y < \delta$. Combining this with (3.3), we conclude that

$$\mathcal{K}_\alpha f_\epsilon(z) \geq (1 - \eta) \frac{(\alpha + 1)}{\Gamma^2(1 + \alpha/2)} \Gamma\Big(1 + \alpha + \frac{\epsilon - 1}{p}\Big) \Gamma\Big(\frac{1 - \epsilon}{p}\Big) f_\epsilon(z) \chi_E.$$

where $E := \{z : 0 < \operatorname{Im} z < \delta\}$. Since $\|\mathcal{K}_\alpha\|_p \geq \|\mathcal{K}_\alpha f_\epsilon\|_p / \|f_\epsilon\|_p$, we have

$$\|\mathcal{K}_\alpha\|_p \geq (1 - \eta) \frac{(\alpha + 1)}{\Gamma^2(1 + \alpha/2)} \Gamma\Big(1 + \alpha + \frac{\epsilon - 1}{p}\Big) \Gamma\Big(\frac{1 - \epsilon}{p}\Big) \times \Big(1 - \|f_\epsilon\|_p^{-p} \int_{\Pi \setminus E} |f_\epsilon(z)|^p \, dA(z)\Big)^{1/p}$$

Lemma 3.3

$$\lim_{\epsilon \to 0+} \|f_\epsilon\|_p^{-p} \int_{\Pi \setminus E} |f_\epsilon(z)|^p \, dA(z) = 0. \tag{3.4}$$

Proof A similar calculation to Lemma 2.1 leads to

$$\begin{aligned}\int_{\Pi \setminus E} |f_\epsilon(z)|^p \, dz &= \Big\{\int_0^\infty \frac{x^{-1/2}}{(1 + x)^{p + p\alpha/2 + \epsilon - 1}} dx\Big\} \Big\{\int_\delta^\infty \frac{y^{\epsilon - 1}}{(1 + y)^{2p + p\alpha + 2\epsilon - 3}} dy\Big\} \\ &= \frac{\Gamma(1/2)\Gamma(p + p\alpha/2 + \epsilon - 3/2)}{\Gamma(p + p\alpha/2 + \epsilon - 1)} \int_\delta^\infty \frac{y^{\epsilon - 1}}{(1 + y)^{2p + p\alpha + 2\epsilon - 3}} dy.\end{aligned}$$

Also,

$$\begin{aligned}\int_\delta^\infty \frac{y^{\epsilon - 1}}{(1 + y)^{2p + p\alpha + 2\epsilon - 3}} dy &= \Big\{\int_\delta^1 + \int_1^\infty\Big\} \frac{y^{\epsilon - 1}}{(1 + y)^{2p + p\alpha + 2\epsilon - 3}} dy \\ &\leq \frac{1 - \delta}{\delta} + \int_1^\infty \frac{dy}{y^{2p + p\alpha - 2}} = \frac{1 - \delta}{\delta} + \frac{1}{2p + p\alpha - 3}.\end{aligned}$$

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Note that δ is independent on ϵ (by Lemma 3.2). Thus,

$$\sup_{\epsilon \in [0,\epsilon_0]} \int_{\Pi \setminus E} |f_\epsilon(z)|^p dz < \infty,$$

but in view of (3.2), $\lim_{\epsilon \to 0^+} \|f_\epsilon\|_p^{-p} = 0$, which proves (3.4). ∎

Now, letting $\epsilon \to 0^+$, we have

$$\|\mathcal{K}_\alpha\|_p \geq (1-\eta)\frac{(\alpha+1)}{\Gamma^2(1+\alpha/2)}\Gamma\Big(1+\alpha-\frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big).$$

Since η is arbitrary, this completes the proof of Theorem 1.2.

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References

[1] R. Bañuelos and P. Janakiraman, *L^p-bounds for the Beurling-Ahlfors transform.* Trans. Amer. Math. Soc. **360**(2008), no. 7, 3603–3612. http://dx.doi.org/10.1090/S0002-9947-08-04537-6

[2] M. Dostanić, *Integral operators induced by Bergman type kernels in the half plane.* Asymptot. Anal. **67**(2010), no. 3–4, 217–228.

[3] ———, *Norm of the Berezin transform on L^p spaces.* J. Anal. Math. **104**(2008), 13–23. http://dx.doi.org/10.1007/s11854-008-0014-8

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions.* Vol. I, McGraw-Hill, New York, 1953.

[5] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities.* Second ed., Cambridge University Press, Cambridge, 1952.

[6] T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings.* Z. Anal. Anwendungen **1**(1982), no. 6, 1–16.

[7] T. Iwaniec and G. Martin, *Riesz transforms and related singular integrals.* J. Reine Angew. Math. **473**(1996), 25–57.

[8] S. K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov.* Studia Math. **44**(1972), 165–179.

[9] K. Zhu, *A sharp norm estimate of the Bergman projection in L^p spaces.* In: Bergman spaces and related topices in complex analysis, Contemp. Math., 404, American Mathematical Society, Providence, RI, 2006, pp. 199–205.

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