Proceedings of the Edinburgh Mathematical Society (2010) **53**, 175–193 ©
DOI:10.1017/S0013091508000187 Printed in the United Kingdom

QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTS AND HARDY TERMS IN $\mathbb{R}^N$

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(Received 3 February 2008)

Abstract We deal with a singular quasilinear elliptic problem, which involves critical Hardy–Sobolev exponents and multiple Hardy terms. Using variational methods and analytic techniques, the existence of ground state solutions to the problem is obtained.

Keywords: quasilinear problem; solution; Hardy–Sobolev inequality; variational method

2000 *Mathematics subject classification:* Primary 35J60; 35J65

1. Introduction

We consider the following quasilinear elliptic problem:

$$\left.\begin{aligned} -\Delta_p u - \sum_{i=1}^{k} \frac{\lambda_i u^{p-1}}{|x-a_i|^p} = \sum_{i=1}^{m} \frac{\gamma_i u^{p^*(s)-1}}{|x-a_i|^s} + \sum_{j=1}^{l} \frac{\mu_j u^{p^*(s)-1}}{|x-b_j|^s}, \quad x \in \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N \setminus \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l\}, \end{aligned}\right\} \tag{1.1}$$

where $a_i, b_j \in \mathbb{R}^N$, $N \geqslant 3$, $1 < p < N$, $k, m, l \geqslant 0$, $k \geqslant m$, $0 < s < p$, $\lambda_i < \bar{\lambda} := ((N-p)/p)^p$, $\gamma_i \geqslant 0$, $\mu_j \geqslant 0$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, l$, and $p^*(s) := p(N-s)/(N-p)$ is the critical Hardy–Sobolev exponent. Without loss of generality, we assume that $a_i \neq b_j$ for any $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$. $k = 0$, $m = 0$ or $l = 0$ means that the corresponding sum term vanishes.

Problem (1.1) is related to the following Hardy–Sobolev inequality [**1**]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(s)}}{|x-a|^s} \, dx \right)^{p/p^*(s)} \leqslant C \int_{\mathbb{R}^N} |\nabla u|^p \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^N), \ a \in \mathbb{R}^N. \tag{1.2}$$

If $s = p$, then $p^*(s) = p$ and the Hardy inequality holds [**11**]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x-a|^p} \, dx \leqslant \frac{1}{\bar{\lambda}} \int_{\mathbb{R}^N} |\nabla u|^p \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^N), \ a \in \mathbb{R}^N,$$

where $\bar{\lambda} = ((N-p)/p)^p$ is the best Hardy constant.

In this paper the space $D^{1,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \|u\|_{D^{1,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x\right)^{1/p}.$$

The function $u \in D^{1,p}(\mathbb{R}^N)$ is said to be a solution of problem (1.1) if $u > 0$ satisfies

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v - \sum_{i=1}^{k} \frac{\lambda_i u^{p-1} v}{|x-a_i|^p} - \sum_{i=1}^{m} \frac{\gamma_i u^{p^*(s)-1} v}{|x-a_i|^s} - \sum_{j=1}^{l} \frac{\mu_j u^{p^*(s)-1} v}{|x-b_j|^s}\right) \mathrm{d}x = 0$$

for all $v \in D^{1,p}(\mathbb{R}^N)$. By the standard elliptic regularity argument, the solution u satisfies

$$u \in C^{1,\alpha}(\mathbb{R}^N \setminus \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l\}).$$

In recent years, much attention has been paid to the singular problems involving the Hardy and the Hardy–Sobolev inequalities (see, for example, [**1**–**14**] and [**17**–**19**] and references therein). Many results were obtained, providing greater insight into these problems. Stimulated by these publications, we will study problem (1.1) in this paper.

For $0 \leqslant s < p$, $-\infty < \lambda < \bar{\lambda}$ and $a \in \mathbb{R}^N$, by the Hardy inequality and the Hardy–Sobolev inequality we can define the best constant $S(\lambda) = S(\lambda, p, s)$ as

$$S(\lambda) := \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p - \lambda |u|^p/|x-a|^p) \,\mathrm{d}x}{(\int_{\mathbb{R}^N} |u|^{p^*(s)}/|x-a|^s \,\mathrm{d}x)^{p/p^*(s)}}. \tag{1.3}$$

Note that the constant $S(\lambda)$ is independent of the singular point a.

A challenging problem related to (1.1) and (1.3) is to investigate the extremal functions by which the best constant $S(\lambda)$ is achieved. Here we recall the result in [**13**], where the following limiting problem was investigated for $\lambda \in [0, \bar{\lambda})$:

$$\left.\begin{aligned} -\Delta_p u - \lambda \frac{u^{p-1}}{|x-a|^p} &= \frac{u^{p^*(s)-1}}{|x-a|^s} \quad \text{in } \mathbb{R}^N \setminus \{a\}, \\ u \in D^{1,p}(\mathbb{R}^N), &\quad u > 0 \text{ in } \mathbb{R}^N \setminus \{a\}. \end{aligned}\right\} \tag{1.4}$$

The author proved that for $0 \leqslant \lambda < \bar{\lambda}$, $1 < p < N$ and $0 \leqslant s < p$ the problem (1.4) has radially symmetric ground states

$$V^a_{p,\lambda,\varepsilon}(x) = \varepsilon^{(p-N)/p} U_{p,\lambda}\left(\frac{x-a}{\varepsilon}\right) = \varepsilon^{(p-N)/p} U_{p,\lambda}\left(\frac{|x-a|}{\varepsilon}\right) \quad \forall \varepsilon > 0 \tag{1.5}$$

that satisfy

$$\int_{\mathbb{R}^N} \left(|\nabla V^a_{p,\lambda,\varepsilon}(x)|^p - \lambda \frac{|V^a_{p,\lambda,\varepsilon}(x)|^p}{|x-a|^p}\right) \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|V^a_{p,\lambda,\varepsilon}(x)|^{p^*(s)}}{|x-a|^s} \,\mathrm{d}x = S(\lambda)^{p^*(s)/(p^*(s)-p)}. \tag{1.6}$$

The function $U_{p,\lambda}$ is the unique radial solution of (1.4) satisfying

$$U_{p,\lambda}(1) = \left(\frac{(N-s)(\bar{\lambda}-\lambda)}{N-p}\right)^{1/(p^*(s)-p)},$$

$$\left.\begin{aligned} \lim_{r\to 0^+} r^{a(\lambda)}U_{p,\lambda}(r) = C_1 > 0, \quad \lim_{r\to 0^+} r^{a(\lambda)+1}|U'_{p,\lambda}(r)| = C_1 a(\lambda) \geqslant 0,\\ \lim_{r\to +\infty} r^{b(\lambda)}U_{p,\lambda}(r) = C_2 > 0, \quad \lim_{r\to +\infty} r^{b(\lambda)+1}|U'_{p,\lambda}(r)| = C_2 b(\lambda) > 0, \end{aligned}\right\} \tag{1.7}$$

where C_1 and C_2 are positive constants depending on λ, p and N, and $a(\lambda)$ and $b(\lambda)$ are zeros of the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \lambda, \quad t \geqslant 0, \tag{1.8}$$

that satisfy

$$0 \leqslant a(\lambda) < \delta < b(\lambda), \quad \delta := \frac{N-p}{p}. \tag{1.9}$$

Furthermore, there exist positive constants $C_3(\lambda)$ and $C_4(\lambda)$ such that

$$0 < C_3(\lambda) \leqslant U_{p,\lambda}(x)(|x|^{a(\lambda)/\delta} + |x|^{b(\lambda)/\delta})^{\delta} \leqslant C_4(\lambda). \tag{1.10}$$

When $\lambda = 0$ in (1.3), the best constant $S(0)$ is achieved by the following explicit extremal functions [**9**]:

$$V^a_{p,0,\varepsilon}(x) = \varepsilon^{(p-N)/p}U_{p,0}\left(\frac{x-a}{\varepsilon}\right) = C_5\varepsilon^{(p-N)/p}\left(1+\left|\frac{x-a}{\varepsilon}\right|^{(p-s)/(p-1)}\right)^{(p-N)/(p-s)}, \tag{1.11}$$

where $C_5 > 0$ is a particular constant and $\varepsilon > 0$ is an arbitrary constant.

The above results are useful and crucial for the study of problem (1.1).

In this paper we need the following assumptions, where in the case when $k = 0$, $m = 0$ or $l = 0$ we just mean that the corresponding terms vanish and the corresponding assumptions in $(\mathcal{H}_1)$–$(\mathcal{H}_7)$ are trivially satisfied.

$(\mathcal{H}_1)$ $0 \leqslant m \leqslant k$, $0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_m$.

$(\mathcal{H}_2)$ $l \geqslant 0$, $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_l$.

$(\mathcal{H}_3)$ $k \geqslant 0$, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k$ and there exists $k_0 \in \{0, 1, 2, \dots, k-1\}$ such that $\lambda_{k_0} \leqslant 0 < \lambda_{k_0+1} \leqslant \lambda_{k_0+2} \leqslant \cdots \leqslant \lambda_k$, ($\lambda_{k_0} = 0$ if $k_0 = 0$) and $\sum_{i=k_0+1}^{k} \lambda_i < \bar{\lambda}$.

$(\mathcal{H}_4)$ $\mu_l^{-p/p^*(s)}S(0) \leqslant (\sum_{i=1}^m \gamma_i + \sum_{j=1}^l \mu_j)^{-p/p^*(s)}S(\sum_{i=1}^k \lambda_i)$, $\sum_{i=1}^k \lambda_i < 0$.

$(\mathcal{H}_5)$ $\gamma_m^{-p/p^*(s)}S(\lambda_m) \leqslant (\sum_{i=1}^m \gamma_i + \sum_{j=1}^l \mu_j)^{-p/(p^*(s))}S(\sum_{i=1}^k \lambda_i)$, $\sum_{i\neq m,i=1}^k \lambda_i < 0$.

$(\mathcal{H}_6)$ $\mu_l^{-p/p^*(s)}S(0) \leqslant \gamma_m^{-p/p^*(s)}S(\lambda_m)$.

$(\mathcal{H}_7)$ $\gamma_m^{-p/p^*(s)}S(\lambda_m) \leqslant \mu_l^{-p/p^*(s)}S(0)$.

Under the above assumptions, the following form $Q(u)$ is well defined:

$$Q(u) := \int_{\mathbb{R}^N} \left(|\nabla u|^p - \sum_{i=1}^{k} \lambda_i \frac{|u|^p}{|x-a_i|^p} \right) \mathrm{d}x.$$

From $(\mathcal{H}_3)$ and by the Hardy inequality, $Q(u)$ is positive definite:

$$Q(u) \geqslant \left(1 - \frac{1}{\bar{\lambda}} \sum_{i=k_0+1}^{k} \lambda_i \right) \|u\|^p \quad \forall u \in D^{1,p}(\mathbb{R}^N).$$

Then we can define the best constant $A = A(p, s, \lambda_1, \dots, \lambda_k, \gamma_1, \dots, \gamma_k, \mu_1, \dots, \mu_l)$:

$$A := \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p - \sum_{i=1}^{k} \lambda_i |u|^p / |x-a_i|^p)\, \mathrm{d}x}{(\int_{\mathbb{R}^N} (\sum_{i=1}^{m} \gamma_i |u|^{p^*(s)}/|x-a_i|^s + \sum_{j=1}^{l} \mu_j |u|^{p^*(s)}/|x-b_j|^s)^{p/p^*(s)})}. \tag{1.12}$$

In this paper, we will investigate the solutions to problem (1.1). Among all possible solutions of problem (1.1), we are interested in those having the smallest energy, termed ground states. These solutions minimize the Rayleigh quotient in (1.12). To state clearly the conclusions of this paper, some notation needs to be explained. For $N > \max\{p^2, p+1\}$ and $\lambda_i, \lambda \in (0, \bar{\lambda})$ we always set

$$\beta_i = b(\lambda_i) - \delta, \quad \delta = \frac{N-p}{p}, \; i = 1, 2, \dots, k. \tag{1.13}$$

$$\lambda_* = \frac{1}{p^p}(N-p+s)^{p-1}(N-p-(p-1)s), \tag{1.14}$$

$$\theta(\lambda) = C_3(\lambda) \left(\int_{\mathbb{R}^N} \frac{|U_{p,\lambda}(x)|^{p^*(s)}}{|x|^s} \right)^{-1/p^*(s)}, \tag{1.15}$$

$$\bar{\theta}(\lambda) = C_4(\lambda) \left(\int_{\mathbb{R}^N} \frac{|U_{p,\lambda}(x)|^{p^*(s)}}{|x|^s} \right)^{-1/p^*(s)}, \tag{1.16}$$

$$\Theta_j = \sum_{i=1}^{k} \frac{\lambda_i}{|a_i - b_j|^p}, \quad 1 \leqslant j \leqslant l. \tag{1.17}$$

$$\Lambda_i = \sum_{j=1}^{k_0} \lambda_j \frac{(\bar{\theta}(\lambda_i))^p}{|a_j - a_i|^{p\beta_i}} + \sum_{j \neq i,\, j=k_0+1}^{k} \lambda_j \frac{(\theta(\lambda_i))^p}{|a_j - a_i|^{p\beta_i}}, \quad k_0 + 1 \leqslant i \leqslant k. \tag{1.18}$$

where k_0 is defined as in $(\mathcal{H}_3)$, $C_3(\lambda)$ and $C_4(\lambda)$ are the constants in (1.9) and $b(\lambda_i)$ are defined as in (1.5)–(1.9) by replacing λ with λ_i, $i = 1, 2, \dots, k$.

The main results of this paper can be summarized in the following theorems. We can verify that the intervals for λ_m in Theorems 1.1 and 1.2 are not empty.

Theorem 1.1. *Suppose that $N > \max\{p^2, p+1\}$ and $(\mathcal{H}_1)$–$(\mathcal{H}_5)$ hold. Assume that one of the following conditions holds:*

(i) $k_0 = 0$, $k = m = 1$, $l \geqslant 1$;

(ii) $k \geqslant 2$, $m \geqslant k_0 + 1$, $l \geqslant 1$, $0 < \lambda_m \leqslant \lambda_*$;

(iii) $k \geqslant 2$, $m \geqslant k_0 + 1$, $l \geqslant 1$, $\lambda_* < \lambda_m < \bar{\lambda}$, $\Lambda_m > 0$.

Then the infimum in (1.12) is achieved and problem (1.1) has one ground state.

Theorem 1.2. *Suppose that* $N > \max\{p^2, p+1\}$ *and* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$, $(\mathcal{H}_5)$ *and* $(\mathcal{H}_7)$ *hold. Assume that one of the following conditions holds:*

(i) $k_0 = 0$, $k = m = 1$, $l \geqslant 1$;

(ii) $k \geqslant 2$, $m \geqslant k_0 + 1$, $m + l \geqslant 2$, $0 < \lambda_m \leqslant \lambda_*$;

(iii) $k \geqslant 2$, $m \geqslant k_0 + 1$, $l \geqslant 0$, $\lambda_* < \lambda_m < \bar{\lambda}$, $\Lambda_m > 0$.

Then the infimum in (1.12) is achieved and problem (1.1) has a ground state.

Theorem 1.3. *Suppose that* $N > \max\{p^2, p+1\}$, $1 \leqslant m \leqslant k$ *and* $l \geqslant 1$. *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ *and* $(\mathcal{H}_6)$ *hold. Then the infimum in (1.12) is achieved and problem (1.1) has one ground state.*

Theorem 1.4. *Suppose that* $N > \max\{p^2, p+1\}$ *and* $(\mathcal{H}_2)$–$(\mathcal{H}_4)$ *hold. Assume that one of the following conditions holds:*

(i) $k \geqslant 0$, $m = 0$ *and* $l \geqslant 2$;

(ii) $k \geqslant 1$, $m = 0$, $l = 1$ *and* $\Theta_1 > 0$.

Then the infimum in (1.12) is achieved. Moreover, problem (1.1) has a ground state.

Remark 1.5. The assumptions $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ are compatible if we choose suitable parameters λ_i, γ_i and $\mu_j, 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l$. For the same reason, $(\mathcal{H}_4)$ and $(\mathcal{H}_6)$, or $(\mathcal{H}_5)$ and $(\mathcal{H}_7)$, can also be compatible. If $1 \leqslant m \leqslant k_0$ and $\lambda_m < 0$, $S(\lambda_m)$ cannot be achieved. In this case, from Theorem 1.3 we obtain the existence of ground state if $(\mathcal{H}_6)$ replaces $(\mathcal{H}_5)$.

This paper is organized as follows. In § 2 we study the Palais–Smale condition by the concentration compactness principle. In § 3, the asymptotic properties of the extremals for $S(\lambda)$ are investigated. Section 4 is devoted to the proofs of Theorems 1.1–1.4.

To end this section, we explain some of the notation used in this paper. $D^{-1,p}(\mathbb{R}^N)$ is the dual space of $D^{1,p}(\mathbb{R}^N)$, $L^q(\mathbb{R}^N, |x - a_i|^\tau)$ means the weighted $L^q(\mathbb{R}^N)$ space with the weight $|x - a_i|^\tau$. For $t > 0$, $O(\varepsilon^t)$ denotes any quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leqslant C$ and $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$. By $o(1)$ we denote a generic infinitesimal value. In the following argument, we employ C to denote the positive constants and omit $\mathrm{d}x$ in integrals for convenience.

2. The Palais–Smale condition

We define the following functional in the space $D^{1,p}(\mathbb{R}^N)$:

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p - \sum_{i=1}^{k}\frac{\lambda_i|u|^p}{|x-a_i|^p}\right) - \frac{A}{p^*(s)}\left(\int_{\mathbb{R}^N}\sum_{i=1}^{m}\frac{\gamma_i|u|^{p^*(s)}}{|x-a_i|^s} + \int_{\mathbb{R}^N}\sum_{j=1}^{l}\frac{\mu_j|u|^{p^*(s)}}{|x-b_j|^s}\right). \tag{2.1}$$

Note that if $u>0$ is a critical point of J, then $v = A^{1/(p^*(s)-p)}u$ is a solution of problem (1.1). To continue, we recall the following standard definition.

Definition 2.1. Let X be a Banach space and let X^{-1} be the dual space of X. The functional $I\in C^1(X,\mathbb{R})$ is said to satisfy the Palais–Smale condition at level c (abbreviated to $(\mathrm{PS})_c$), if any sequence $\{u_n\}\subset X$ satisfying $I(u_n)\to c$ and $I'(u_n)\to 0$ in X^{-1} as $n\to\infty$ contains a subsequence converging strongly in X to a critical point of I.

We take $I = J$ and $X = D^{1,p}(\mathbb{R}^N)$. The following lemma provides a local Palais–Smale condition for J.

Lemma 2.2. *Suppose that $k, m, l \geqslant 1$ and $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold. Then the functional J satisfies $(PS)_c$ for all $c < c^*$ with*

$$c^* := \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)A^{-p/(p^*(s)-p)}(B^*)^{p^*(s)/(p^*(s)-p)},$$

where

$$B^* := \min\left\{\gamma_m^{-p/p^*(s)}S(\lambda_m), \mu_l^{-p/p^*(s)}S(0), \left(\sum_{i=1}^{m}\gamma_i + \sum_{j=1}^{l}\mu_j\right)^{-p/p^*(s)} S\left(\sum_{i=1}^{k}\lambda_i\right)\right\}.$$

Proof. Suppose that the sequence $\{u_n\}\subset D^{1,p}(\mathbb{R}^N)$ satisfies $J(u_n)\to c < c^*$ and $J'(u_n)\to 0$ in $D^{-1,p}(\mathbb{R}^N)$. Then from $(\mathcal{H}_1)$ it follows that $\{u_n\}$ is a bounded sequence in $D^{1,p}(\mathbb{R}^N)$. Up to a subsequence and for some $u_0\in D^{1,p}(\mathbb{R}^N)$ we have $u_n \rightharpoonup u_0$ weakly in $D^{1,p}(\mathbb{R}^N)$, $u_n\to u_0$ almost everywhere in $\mathbb{R}^N$, and $u_n\to u_0$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N, |x-b_j|^{-s})$ for all $t\in[1,p^*(s))$, $j = 1,2,\dots,l$. Then by the concentration compactness theorem [**15**,**16**] and up to a subsequence if necessary, there exist real numbers τ_{a_i}, τ_{b_j}, γ_{a_i}, ν_{a_i}, ω_{b_j}, $i = 1,2,\dots,k$, $j = 1,2,\dots,l$, such that the following convergences hold in the sense of measures:

$$|\nabla u_n|^p \rightharpoonup \mathrm{d}\tau \geqslant |\nabla u_0|^p + \sum_{i=1}^{k}\tau_{a_i}\delta_{a_i} + \sum_{j=1}^{l}\tau_{b_j}\delta_{b_j},$$

$$\frac{\lambda_i|u_n|^p}{|x-a_i|^p} \rightharpoonup \mathrm{d}\gamma_i = \frac{\lambda_i|u_0|^p}{|x-a_i|^p} + \gamma_{a_i}\delta_{a_i}, \quad i = 1,2,\dots,k,$$

https://doi.org/10.1017/S0013091508000187 Published online by Cambridge University Press

$$\frac{|u_n|^{p^*(s)}}{|x-a_i|^s} \rightharpoonup \mathrm{d}\nu_i = \frac{|u_0|^{p^*(s)}}{|x-a_i|^s} + \nu_{a_i}\delta_{a_i}, \quad i = 1, 2, \dots, m,$$
$$\frac{|u_n|^{p^*(s)}}{|x-b_j|^s} \rightharpoonup \mathrm{d}\omega_j = \frac{|u_0|^{p^*(s)}}{|x-b_j|^s} + \omega_{b_j}\delta_{b_j}, \quad j = 1, 2, \dots, l,$$

where δ_x is the Dirac mass at x.

To study the concentration at infinity, we set

$$\tau_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^p,$$
$$\gamma_\infty = \left(\sum_{i=1}^{k} \lambda_i\right) \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{|u_n|^p}{|x|^p},$$
$$\nu_\infty = \left(\sum_{i=1}^{m} \gamma_i + \sum_{j=1}^{l} \mu_j\right) \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{|u_n|^{p^*(s)}}{|x|^s}.$$

Then, by arguments similar to those in [**7**] and [**13**], we can verify the following claims. We omit their proofs.

Claim 2.3. *For $i = 1, 2, \dots, m$, either*

$$\nu_{a_i} = 0 \quad \text{or} \quad \nu_{a_i} \geqslant \left(\frac{S(\lambda_i)}{\gamma_i A}\right)^{p^*(s)/(p^*(s)-p)}.$$

Claim 2.4. *For $j = 1, 2, \dots, l$, either*

$$\omega_{b_j} = 0 \quad \text{or} \quad \omega_{b_j} \geqslant \left(\frac{S(0)}{\mu_j A}\right)^{p^*(s)/(p^*(s)-p)}.$$

Claim 2.5. *We claim that either*

$$\nu_\infty = 0 \quad \text{or} \quad \nu_\infty \geqslant \left(\sum_{i=1}^{m} \gamma_i + \sum_{j=1}^{l} \mu_j\right)^{-p/(p^*(s)-p)} \left(\frac{S(\sum_{i=1}^{k} \lambda_i)}{A}\right)^{p^*(s)/(p^*(s)-p)}.$$

By $(\mathcal{H}_1)$ we deduce that

$$\gamma_m^{-p/p^*(s)} S(\lambda_m) = \min\{\gamma_i^{-p/p^*(s)} S(\lambda_i),\ i = 1, 2, \dots, m\},$$
$$\mu_l^{-p/p^*(s)} S(0) = \min\{\mu_j^{-p/p^*(s)} S(0),\ j = 1, 2, \dots, l\}.$$

Then from Claims 2.3–2.5 it follows that

$$c = J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle + o(1)$$
$$= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) A \int_{\mathbb{R}^N} \left(\sum_{i=1}^{m} \frac{\gamma_i |u_n|^{p^*(s)}}{|x-a_i|^s} + \sum_{j=1}^{l} \frac{\mu_j |u_n|^{p^*(s)}}{|x-b_j|^s}\right) + o(1)$$

$$= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) A\left(\int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{\gamma_i |u_0|^{p^*(s)}}{|x-a_i|^s} + \int_{\mathbb{R}^N} \sum_{j=1}^{l} \frac{\mu_j |u_0|^{p^*(s)}}{|x-b_j|^s}\right) + \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) A\left(\sum_{i=1}^{m} \gamma_i \nu_{a_i} + \sum_{j=1}^{l} \mu_j \omega_{b_j} + \nu_\infty\right).$$

From the definition of c^* it follows that $\nu_\infty = 0$, $\nu_{a_i} = 0$, $i = 1, 2, \dots, m$ and $\omega_{b_j} = 0$, $j = 1, 2, \dots, l$. Up to a subsequence, we obtain that $u_n \to u_0$ in $D^{1,p}(\mathbb{R}^N)$.

Hence, the proof of Lemma 2.2 is completed. □

Arguing as in Lemma 2.2, we also obtain the following lemmas.

Lemma 2.6. *Suppose that $m \geqslant 1$ and $l = 0$. Then the functional J satisfies $(PS)_c$ for all $c < c_1^*$, where*

$$c_1^* := \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) A^{-p/(p^*(s)-p)} \times \left(\min\left\{\gamma_m^{-p/p^*(s)} S(\lambda_m), \left(\sum_{i=1}^{m} \gamma_i\right)^{-p/p^*(s)} S\left(\sum_{i=1}^{k} \lambda_i\right)\right\}\right)^{p^*(s)/(p^*(s)-p)}.$$

Lemma 2.7. *Suppose that $l \geqslant 1$ and $m = 0$. Then the functional J satisfies $(PS)_c$ for all $c < c_2^*$, where*

$$c_2^* := \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) A^{-p/(p^*(s)-p)} \times \left(\min\left\{\mu_l^{-p/p^*(s)} S(0), \left(\sum_{j=1}^{l} \mu_j\right)^{-p/p^*(s)} S\left(\sum_{i=1}^{k} \lambda_i\right)\right\}\right)^{p^*(s)/(p^*(s)-p)}.$$

3. Asymptotic behaviours of the extremal function

In order to prove Theorems 1.1–1.4, we first establish several lemmas. Consider the minimizers in (1.3) and set

$$z_\varepsilon^\lambda(x) = \varepsilon^{-\delta} U_{p,\lambda}\left(\frac{|x|}{\varepsilon}\right) \left(\int_{\mathbb{R}^N} \frac{|U_{p,\lambda}(|x|)|^{p^*(s)}}{|x|^s}\right)^{-1/p^*(s)}.$$

Then we need to investigate the asymptotic properties of $z_\varepsilon^\lambda(x)$ in the case when $\varepsilon \to 0$. These properties play an important role in the proofs of Theorems 1.1–1.4.

In addition to the parameters λ_*, β_i, Λ_i, $\theta(\lambda)$, $\bar{\theta}(\lambda)$ and Θ_j defined in (1.13)–(1.18), we use the following notation:

$$\beta = b(\lambda) - \delta, \qquad \delta = \frac{N-p}{p},$$

$$\bar{\Lambda}_i = \sum_{j \neq i,\, j=1}^{k} \frac{\lambda_j}{|a_i - a_j|^p}, \quad 1 \leqslant i \leqslant k,$$

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$$\lambda^* = \frac{(N-p^2)N^{p-1}}{p^p}, \qquad \lambda^{**} = \frac{(N-ps)N^{p-1}}{(p^*(s))^p},$$
$$\delta_{\lambda,s} = \int_{\mathbb{R}^N} \frac{\mathrm{d}x}{|x|^s|x-e_1|^{p^*(s)b(\lambda)}}, \quad e_1 = (1,0,\dots,0) \in \mathbb{R}^N.$$

Lemma 3.1. *Suppose $\xi \in \mathbb{R}^N \setminus \{0\}$, $0 < s \leqslant p$ and $N > \max\{ps, s+1\}$. Then as $\varepsilon \to 0^+$ we have*

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x = \frac{\varepsilon^s}{|\xi|^s}\int_{\mathbb{R}^N} |z_1^\lambda|^{p^*(s)} + o(\varepsilon^s) \qquad \textit{for } 0 \leqslant \lambda < \lambda^{**},$$
$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \geqslant \theta(\lambda)^{p^*(s)}\frac{\varepsilon^s|\ln\varepsilon|}{|\xi|^s} + o(\varepsilon^s|\ln\varepsilon|) \qquad \textit{for } \lambda = \lambda^{**},$$
$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \leqslant \bar\theta(\lambda)^{p^*(s)}\frac{\varepsilon^s|\ln\varepsilon|}{|\xi|^s} + o(\varepsilon^s|\ln\varepsilon|) \qquad \textit{for } \lambda = \lambda^{**},$$
$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \geqslant \theta(\lambda)^{p^*(s)}(\delta_{\lambda,s})\frac{\varepsilon^{p^*(s)\beta}}{|\xi|^{p^*(s)\beta}} + o(\varepsilon^{p^*(s)\beta}) \quad \textit{for } \lambda^{**} < \lambda < \bar\lambda,$$
$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \leqslant \bar\theta(\lambda)^{p^*(s)}(\delta_{\lambda,s})\frac{\varepsilon^{p^*(s)\beta}}{|\xi|^{p^*(s)\beta}} + o(\varepsilon^{p^*(s)\beta}) \quad \textit{for } \lambda^{**} < \lambda < \bar\lambda.$$

Proof. The above estimates can be applied to

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda(x)|^p}{|x+\xi|^p}\,\mathrm{d}x.$$

Since $p^*(s) = p$, $\delta_{\lambda,s} = \delta_{\lambda,p}$ and $\lambda^{**} = \lambda^*$ as $s = p$, we can obtain the similar estimates for the above expression just by replacing s, $p^*(s)$, $\delta_{\lambda,s}$ and λ^{**} by p, p, $\delta_{\lambda,p}$ and λ^*, respectively.

The proof follows a similar line to that in [**7**]. Here we need to use the property (1.7) of the parameters $a(\lambda)$ and $b(\lambda)$. Then we have

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x = \varepsilon^s \int_{|x|<\xi/2\varepsilon} \frac{|z_1^\lambda|^{p^*(s)}}{|\varepsilon x+\xi|^s}\,\mathrm{d}x + \varepsilon^s \int_{|x|\geqslant\xi/2\varepsilon} \frac{|z_1^\lambda|^{p^*(s)}}{|\varepsilon x+\xi|^s}\,\mathrm{d}x. \tag{3.1}$$

For the first part, from (1.9) we deduce that

$$\begin{aligned}
&\varepsilon^s\bigg|\int_{|x|<|\xi|/2\varepsilon} |z_1^\lambda(x)|^{p^*(s)}\bigg(\frac{1}{|\varepsilon x+\xi|^s} - \frac{1}{|\xi|^s}\bigg)\,\mathrm{d}x\bigg| \\
&\qquad\leqslant C\varepsilon^s \int_{|x|<|\xi|/2\varepsilon} (|x|^{a(\lambda)/\delta} + |x|^{b(\lambda)/\delta})^{-(N-s)}\bigg|\frac{1}{|\varepsilon x+\xi|^s} - \frac{1}{|\xi|^s}\bigg|\,\mathrm{d}x \\
&\qquad\leqslant C\varepsilon^s \int_0^{|\xi|/2\varepsilon} \frac{r^{N-1}\,\mathrm{d}r}{(r^{a(\lambda)/\delta} + r^{b(\lambda)/\delta})^{N-s}} \\
&\qquad= O(\varepsilon^{p^*(s)(b(\lambda)-\delta)}) \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{3.2}$$

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where we have employed the following fact: if $|x| < |\xi|/2\varepsilon$, then there exists some constant $C(\xi) > 0$ depending only on ξ such that

$$\left|\frac{1}{|\varepsilon x+\xi|^s} - \frac{1}{|\xi|^s}\right| \leqslant C(\xi).$$

On the other hand, from (1.9) it also follows that

$$\begin{aligned}
&\varepsilon^s \int_{|x|\geqslant\xi/2\varepsilon} \frac{|z_1^\lambda|^{p^*(s)}}{|\varepsilon x+\xi|^s}\,\mathrm{d}x \\
&\quad\leqslant C\varepsilon^{p^*(s)(b(\lambda)-\delta)} \int_{|x-\xi|\geqslant|\xi|/2} \frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta}+|x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{-(N-s)}}{|x|^s|x-\xi|^{p^*(s)a(\lambda)}}\,\mathrm{d}x \\
&\quad= C\varepsilon^{p^*(s)(b(\lambda)-\delta)}\biggl(\int_{|x-\xi|\geqslant|\xi|/2,\,|x|<2|\xi|} + \int_{|x-\xi|\geqslant|\xi|/2,\,|x|\geqslant2|\xi|}\biggr) \\
&\quad\leqslant C\varepsilon^{p^*(s)(b(\lambda)-\delta)}\biggl(\int_0^{2|\xi|} r^{N-s-1}\,\mathrm{d}r + \int_{2|\xi|}^{+\infty} \frac{\mathrm{d}r}{r^{1+p^*(s)(b(\lambda)-\delta)}}\biggr) \\
&\quad= O(\varepsilon^{p^*(s)(b(\lambda)-\delta)}) \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned} \tag{3.3}$$

To proceed, we need to investigate the properties of the function in (1.6):

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \lambda, \quad t\in[0,+\infty).$$

It is easy to verify that $f(t)$ has a unique minimum at $\delta = (N-p)/p$ and is increasing on the interval $(\delta,+\infty)$. Since $N/p^*(s), b(\lambda)\in(\delta,+\infty)$, for $N > ps$ and $\lambda\geqslant 0$ we thus have that

$$\begin{aligned}
\frac{N}{p^*(s)} < b(\lambda) \quad&\iff\quad f\biggl(\frac{N}{p^*(s)}\biggr) < f(b(\lambda)) = 0 \quad\iff\quad \lambda<\lambda^{**},\\
\frac{N}{p^*(s)} = b(\lambda) \quad&\iff\quad f\biggl(\frac{N}{p^*(s)}\biggr) = f(b(\lambda)) = 0 \quad\iff\quad \lambda=\lambda^{**},\\
\frac{N}{p^*(s)} > b(\lambda) \quad&\iff\quad f\biggl(\frac{N}{p^*(s)}\biggr) > f(b(\lambda)) = 0 \quad\iff\quad \lambda>\lambda^{**}.
\end{aligned}$$

On the other hand, the equation

$$(p-1)t^p - (N-p)t^{p-1} + \lambda = 0, \quad \lambda\in(0,\bar\lambda),$$

determines the implicit functions

$$t_1 = a(\lambda) : (0,\bar\lambda)\mapsto(0,\delta), \qquad t_2 = b(\lambda) : (0,\bar\lambda)\mapsto\biggl(\delta,\frac{N-p}{p-1}\biggr).$$

By direct calculations we see that

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{(p-1)(N-p-pt)t^{p-2}}.$$

Hence, $a(\lambda)$ is strictly increasing on $[0,\bar\lambda)$ and $b(\lambda)$ is strictly decreasing on $[0,\bar\lambda)$.

(i) If $N > ps$, $0 \leqslant \lambda < \lambda^{**}$, then $b(\lambda) > N/p^*(s)$, $p^*(s)(b(\lambda)-\delta) > s$ and $z_1^\lambda \in L^{p^*(s)}(\mathbb{R}^N)$. From (3.1)–(3.3) it follows that

$$\begin{aligned}\varepsilon^s \int_{|x|<|\xi|/2\varepsilon} |z_1^\lambda(x)|^{p^*(s)} \frac{1}{|\varepsilon x+\xi|^s}\,\mathrm{d}x &= \frac{\varepsilon^s}{|\xi|^s}\int_{|x|<|\xi|/2\varepsilon} |z_1^\lambda(x)|^{p^*(s)}\,\mathrm{d}x + o(\varepsilon^s)\\ &= \frac{\varepsilon^s}{|\xi|^s}\int_{\mathbb{R}^N} |z_1^\lambda(x)|^{p^*(s)}\,\mathrm{d}x + o(\varepsilon^s).\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x = \frac{\varepsilon^s}{|\xi|^s}\int_{\mathbb{R}^N} |z_1^\lambda(x)|^{p^*(s)}\,\mathrm{d}x + o(\varepsilon^s).$$

(ii) If $N > ps$ and $\lambda = \lambda^{**}$, then $b(\lambda) = N/p^*(s)$. From (1.9) it follows that

$$\begin{aligned}\varepsilon^s \int_{|x|<|\xi|/2\varepsilon} |z_1^\lambda(x)|^{p^*(s)}\,\mathrm{d}x &= \varepsilon^s \int_0^{|\xi|/2\varepsilon} |z_1^\lambda(r)|^{p^*(s)} r^{N-1}\,\mathrm{d}r\\ &= \varepsilon^s \int_1^{|\xi|/2\varepsilon} |z_1^\lambda(r)|^{p^*(s)} r^{N-1}\,\mathrm{d}r + O(\varepsilon^s)\\ &\geqslant \theta(\lambda)^{p^*(s)}\varepsilon^s|\ln\varepsilon| + O(\varepsilon^s)\\ &= \theta(\lambda)^{p^*(s)}\varepsilon^s|\ln\varepsilon| + o(\varepsilon^s|\ln\varepsilon|).\end{aligned} \tag{3.4}$$

From (3.1)–(3.4) it follows that

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \geqslant \theta(\lambda)^{p^*(s)} \frac{\varepsilon^s|\ln\varepsilon|}{|\xi|^s} + o(\varepsilon^s|\ln\varepsilon|).$$

From (1.9) and by a similar argument we have that

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \leqslant \bar{\theta}(\lambda)^{p^*(s)} \frac{\varepsilon^s|\ln\varepsilon|}{|\xi|^s} + o(\varepsilon^s|\ln\varepsilon|).$$

(iii) If $N > \max\{ps, s+1\}$ and $\lambda > \lambda^{**}$, then $b(\lambda) < N/p^*(s)$. From (1.9) it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x\\ &\geqslant \theta(\lambda)^{p^*(s)} \int_{\mathbb{R}^N} \frac{\varepsilon^{s-N}\,\mathrm{d}x}{|x+\xi|^s(|x/\varepsilon|^{a(\lambda)/\delta} + |x/\varepsilon|^{b(\lambda)/\delta})^{N-s}}\\ &= \theta(\lambda)^{p^*(s)}\varepsilon^{p^*(s)\beta} \int_{\mathbb{R}^N} \frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{-(N-s)}}{|x|^s|x-\xi|^{p^*(s)a(\lambda)}}\,\mathrm{d}x\\ &= \theta(\lambda)^{p^*(s)}\varepsilon^{p^*(s)\beta} \int_{\mathbb{R}^N} \biggl(\frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)a(\lambda)}} - \frac{1}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}\biggr)\\ &\quad + \theta(\lambda)^{p^*(s)}\varepsilon^{p^*(s)\beta} \int_{\mathbb{R}^N} \frac{\mathrm{d}x}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}.\end{aligned} \tag{3.5}$$

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For $a, b \geqslant 0$ and $\tau > 1$, the following elementary inequality is well known:

$$0 \leqslant (a+b)^{\tau} - a^{\tau} \leqslant C(a^{\tau-1}b + b^{\tau}),$$

where $C = C(\tau) > 0$ is some constant. From $N > s+1$ it follows that

$$\begin{aligned}
&\left|\frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)a(\lambda)}} - \frac{1}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}\right| \\
&\quad = \frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}} \\
&\qquad \times ((\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{N-s} - (|x-\xi|^{(b(\lambda)-a(\lambda))/\delta})^{N-s}) \\
&\quad \leqslant C\frac{(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}} \\
&\qquad \times (|x-\xi|^{(b(\lambda)-a(\lambda))(N-s-1)/\delta}\varepsilon^{b(\lambda)-a(\lambda)/\delta} + \varepsilon^{p^*(s)(b(\lambda)-a(\lambda))}\Big) \\
&\quad = C\biggl(\frac{\varepsilon^{(b(\lambda)-a(\lambda))/\delta}(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)-((b(\lambda)-a(\lambda))(N-s-1))/\delta}} \\
&\qquad\qquad + \frac{\varepsilon^{p^*(s)(b(\lambda)-a(\lambda))}(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}\biggr). \qquad (3.6)
\end{aligned}$$

If $\varepsilon \to 0$, then it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\varepsilon^{(b(\lambda)-a(\lambda))/\delta}(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)-(b(\lambda)-a(\lambda))(N-s-1)/\delta}} \\
&\quad \leqslant C\varepsilon^{(b(\lambda)-a(\lambda))/\delta}\int_{|x-\xi|>|\xi|/2,\,|x|<2|\xi|} \frac{\mathrm{d}x}{|x|^s} \\
&\qquad + C\varepsilon^{(b(\lambda)-a(\lambda))/\delta}\int_{|x-\xi|>|\xi|/2,\,|x|>2|\xi|} \frac{\mathrm{d}x}{|x|^{s+p^*(s)b(\lambda)+p(b(\lambda)-a(\lambda))/(N-p)}} \\
&\qquad + C\varepsilon^{N-p^*(s)b(\lambda)}\int_0^{|\xi|/2\varepsilon} \frac{r^{N-1}\,\mathrm{d}r}{r^{p^*(s)b(\lambda)-(b(\lambda)-a(\lambda))(N-s-1)/\delta}(1+r^{b(\lambda)-a(\lambda)/\delta})^{N-s}} \\
&\quad = O(\varepsilon^{(b(\lambda)-a(\lambda))/\delta}) \\
&\quad = o(1) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

By a similar argument we also have

$$\int_{\mathbb{R}^N} \frac{\varepsilon^{p^*(s)(b(\lambda)-a(\lambda))}(\varepsilon^{b(\lambda)-a(\lambda)/\delta} + |x-\xi|^{b(\lambda)-a(\lambda)/\delta})^{s-N}}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}\,\mathrm{d}x = o(1) \quad \text{as } \varepsilon \to 0.$$

From (3.5) and (3.6) we have that

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \geqslant \theta(\lambda)^{p^*(s)}\varepsilon^{p^*(s)\beta}\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}} + o(\varepsilon^{p^*(s)\beta}).$$

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From (1.9) and by a similar argument we deduce that

$$\int_{\mathbb{R}^N} \frac{|z_\varepsilon^\lambda|^{p^*(s)}}{|x+\xi|^s}\,\mathrm{d}x \leqslant \bar{\theta}(\lambda)^{p^*(s)}\varepsilon^{p^*(s)\beta}\int_{\mathbb{R}^N}\frac{\mathrm{d}x}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}} + o(\varepsilon^{p^*(s)\beta}).$$

On the other hand, the following function $\varphi(\xi)$ is invariant by rotation:

$$\varphi(\xi) := \int_{\mathbb{R}^N}\frac{\mathrm{d}x}{|x|^s|x-\xi|^{p^*(s)b(\lambda)}}.$$

Moreover,

$$\varphi(\eta\xi) = \eta^{p^*(s)(\delta-b(\lambda))}\varphi(\xi) = \eta^{-p^*(s)\beta}\varphi(\xi) \quad \forall\eta>0,$$
$$\varphi(\xi) = |\xi|^{-p^*(s)\beta}\varphi(\xi/|\xi|) = |\xi|^{-p^*(s)\beta}\varphi(e_1) = |\xi|^{-p^*(s)\beta}\delta_{\lambda,s}.$$

Thus, the proof of this lemma is complete. □

4. Proof of the main results

Now we prove Theorems 1.1–1.4. To proceed, some preliminary lemmas are needed.

Lemma 4.1. *Suppose that $N > \max\{p^2, p+1\}$, $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold and the constants λ_* and Λ_m are defined as in (1.13)–(1.18). Assume one of the following conditions holds:*

(i) $k_0 = 0$, $k = m = 1$, $l \geqslant 1$;

(ii) $k \geqslant 2$, $m \geqslant k_0+1$, $m+l \geqslant 2$, $0 < \lambda_m \leqslant \lambda_*$;

(iii) $k \geqslant 2$, $m \geqslant k_0+1$, $l \geqslant 0$, $\lambda_* < \lambda_m < \bar{\lambda}$, $\Lambda_m > 0$.

Then $A < \gamma_m^{-p/p^*(s)} S(\lambda_m)$.

Proof. (i) If $k_0 = 0, k = m = 1$ and $\lambda_1 > 0$, we deduce that

$$\int_{\mathbb{R}^N}\left(|\nabla z_\varepsilon^{\lambda_1}(x-a_1)|^p - \lambda_1\frac{|z_\varepsilon^{\lambda_1}(x-a_1)|^p}{|x-a_1|^p}\right) = \int_{\mathbb{R}^N}\left(|\nabla z_\varepsilon^{\lambda_1}(x)|^p - \lambda_1\frac{|z_\varepsilon^{\lambda_1}(x)|^p}{|x|^p}\right) = S(\lambda_1),$$
$$\int_{\mathbb{R}^N}\frac{\gamma_1|z_\varepsilon^{\lambda_1}(x-a_1)|^{p^*(s)}}{|x-a_1|^s} = \gamma_1, \qquad \sum_{j=1}^{l}\int_{\mathbb{R}^N}\frac{\mu_j|z_\varepsilon^{\lambda_1}(x-a_1)|^{p^*(s)}}{|x-b_j|^s} > 0.$$

Then the desired result follows.

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(ii) Suppose that $k \geqslant 2$ and $m \geqslant k_0 + 1 \geqslant 1$. If $\varepsilon \to 0$, from Lemma 3.1 it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \bigg(|\nabla z_\varepsilon^{\lambda_m}(x-a_m)|^p - \sum_{i=1}^{k} \lambda_i \frac{|z_\varepsilon^{\lambda_m}(x-a_m)|^p}{|x-a_i|^p} \bigg) \\
&\quad = \int_{\mathbb{R}^N} \bigg(|\nabla z_\varepsilon^{\lambda_m}(x-a_m)|^p - \lambda_m \frac{|z_\varepsilon^{\lambda_m}(x-a_m)|^p}{|x-a_m|^p} \bigg) - \sum_{i\neq m,\, i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{|z_\varepsilon^{\lambda_m}(x-a_m)|^p}{|x-a_i|^p} \\
&\quad = \int_{\mathbb{R}^N} \bigg(|\nabla z_\varepsilon^{\lambda_m}(x)|^p - \lambda_m \frac{|z_\varepsilon^{\lambda_m}(x)|^p}{|x|^p} \bigg) - \sum_{i\neq m,\, i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{|z_\varepsilon^{\lambda_m}(x)|^p}{|x+a_m-a_i|^p} \\
&\quad = S(\lambda_m) - \sum_{i\neq m, i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{|z_\varepsilon^{\lambda_m}(x)|^p}{|x+a_m-a_i|^p} \\
&\quad \leqslant S(\lambda_m) - \begin{cases} \varepsilon^p(\bar{\Lambda}_m + o(1)) \displaystyle\int_{\mathbb{R}^N} |z_1^{\lambda_m}(x)|^p, & 0 < \lambda_m < \lambda^*, \\ \varepsilon^p |\ln\varepsilon| (\Lambda_m + o(1)), & \lambda_m = \lambda^*, \\ (\delta_{\lambda,p}) \varepsilon^{p\beta_{\lambda_m}} (\Lambda_m + o(1)), & \lambda^* < \lambda_m < \bar{\lambda}. \end{cases}
\end{aligned}$$

If $m \geqslant 2$, then we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{\gamma_i |z_\varepsilon^{\lambda_m}(x-a_m)|^{p^*(s)}}{|x-a_i|^s} \\
&= \int_{\mathbb{R}^N} \frac{\gamma_m |z_\varepsilon^{\lambda_m}(x-a_m)|^{p^*(s)}}{|x-a_m|^s} + \sum_{i=1}^{m-1} \int_{\mathbb{R}^N} \frac{\gamma_i |z_\varepsilon^{\lambda_m}(x-a_m)|^{p^*(s)}}{|x-a_i|^s} \\
&\geqslant \gamma_m + \begin{cases} \varepsilon^s \bigg(\displaystyle\sum_{i=1}^{m-1} \frac{\gamma_i}{|a_m-a_i|^s} + o(1) \bigg) \int_{\mathbb{R}^N} |z_1^{\lambda_m}(x)|^{p^*(s)}, & \lambda_m < \lambda^{**}, \\ (\theta(\lambda_m))^{p^*(s)} \varepsilon^s |\ln\varepsilon| \bigg(\displaystyle\sum_{i=1}^{m-1} \frac{\gamma_i}{|a_m-a_i|^s} + o(1) \bigg), & \lambda_m = \lambda^{**}, \\ (\theta(\lambda_m))^{p^*(s)} (\delta_{\lambda_m,s}) \varepsilon^{p^*(s)\beta_{\lambda_m}} \bigg(\displaystyle\sum_{i=1}^{m-1} \frac{\gamma_i}{|a_m-a_i|^{p^*(s)\beta_{\lambda_m}}} + o(1) \bigg), & \lambda_m > \lambda^{**}. \end{cases}
\end{aligned}$$

If $l \geqslant 1$, then it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{j=1}^{l} \frac{\mu_j |z_\varepsilon^{\lambda_m}(x-a_m)|^{p^*(s)}}{|x-b_j|^s} \\
&\quad = \sum_{j=1}^{l} \int_{\mathbb{R}^N} \frac{\mu_j |z_\varepsilon^{\lambda_m}(x)|^{p^*(s)}}{|x+a_m-b_j|^s}
\end{aligned}$$

$$\geqslant \begin{cases} \varepsilon^s \displaystyle\int_{\mathbb{R}^N} |z_1^{\lambda_m}(x)|^{p^*(s)} \biggl(\sum_{j=1}^{l} \frac{\mu_j}{|a_m - b_j|^s} + o(1)\biggr), & \lambda_m < \lambda^{**}, \\ (\theta(\lambda_m))^{p^*(s)} \varepsilon^s |\ln\varepsilon| \biggl(\displaystyle\sum_{j=1}^{l} \frac{\mu_j}{|a_m - b_j|^s} + o(1)\biggr), & \lambda_m = \lambda^{**}, \\ (\theta(\lambda_m))^{p^*(s)} (\delta_{\lambda_m,s}) \varepsilon^{p^*(s)\beta_{\lambda_m}} \biggl(\displaystyle\sum_{j=1}^{l} \frac{\mu_j}{|a_m - b_j|^{p^*(s)\beta_{\lambda_m}}} + o(1)\biggr), & \lambda_m > \lambda^{**}. \end{cases}$$

As in Lemma 3.1, for $N > p + (p-1)s$ we have

$$\lambda_m < \lambda_* \iff b(\lambda_m) > \frac{1}{p}(N - p + s) \iff p\beta_{\lambda_m} > s,$$

$$\lambda_m = \lambda_* \iff b(\lambda_m) = \frac{1}{p}(N - p + s) \iff p\beta_{\lambda_m} = s,$$

$$\lambda_m > \lambda_* \iff b(\lambda_m) < \frac{1}{p}(N - p + s) \iff p\beta_{\lambda_m} < s.$$

For $0 < s < p$ and $N > p^2$, since $b(\lambda^*) = N/p$, $b(\lambda_*) = (N - p + s)/p$ and $b(\lambda^{**}) = N/p^*(s)$, from the facts that $b(\lambda)$ is strictly decreasing on the interval $(\delta, +\infty)$ and $b(\lambda^*) > b(\lambda_*) > b(\lambda^{**})$, we see that $0 < \lambda^* < \lambda_* < \lambda^{**} < \bar{\lambda}$.

Suppose that $m \geqslant k_0 + 1$, $m + l \geqslant 2$, $0 < \lambda_m < \lambda^*$ and $0 < s < p$. Then

$$\begin{aligned} A &\leqslant \int_{\mathbb{R}^N} \biggl(|\nabla z_\varepsilon^{\lambda_m}(x - a_m)|^p - \sum_{i=1}^{k} \lambda_i \frac{|z_\varepsilon^{\lambda_m}(x - a_m)|^p}{|x - a_i|^p}\biggr) \\ &\quad \times \biggl(\int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{\gamma_i |z_\varepsilon^{\lambda_m}(x - a_m)|^{p^*(s)}}{|x - a_i|^s} + \int_{\mathbb{R}^N} \sum_{j=1}^{l} \frac{\mu_j |z_\varepsilon^{\lambda_m}(x - a_m)|^{p^*(s)}}{|x - b_j|^s}\biggr)^{-p/p^*(s)} \\ &\leqslant (S(\lambda_m) + C\varepsilon^p)(\gamma_m + C\varepsilon^s)^{-p/p^*(s)} \\ &\leqslant (S(\lambda_m) + C\varepsilon^p)(\gamma_m^{p/p^*(s)} + C\varepsilon^s)^{-1} \\ &< \gamma_m^{-p/p^*(s)} S(\lambda_m). \end{aligned}$$

If $m \geqslant k_0 + 1$, $m + l \geqslant 2$ and $\lambda_m = \lambda^*$, for $\varepsilon > 0$ small we have

$$A \leqslant (S(\lambda_m) + C\varepsilon^p |\ln\varepsilon|)(\gamma_m^{p/p^*(s)} + C\varepsilon^s)^{-1} < \gamma_m^{-p/p^*(s)} S(\lambda_m).$$

If $m + l \geqslant 2$ and $\lambda^* < \lambda_m < \lambda_*$, then $p\beta_{\lambda_m} > s$ and $\lambda_m < \lambda^{**}$. As $\varepsilon \to 0$ we deduce

$$A \leqslant (S(\lambda_m) + C\varepsilon^{p\beta_{\lambda_m}})(\gamma_m^{p/p^*(s)} + C\varepsilon^s)^{-1} < \gamma_m^{-p/p^*(s)} S(\lambda_m).$$

(iii) If $\lambda_* \leqslant \lambda_m < \bar{\lambda}$ and $\Lambda_m > 0$, for $\varepsilon > 0$ small we have

$$A < \gamma_m^{-p/p^*(s)} S(\lambda_m).$$

The proof of this lemma is complete. □

https://doi.org/10.1017/S0013091508000187 Published online by Cambridge University Press

Lemma 4.2. *Suppose that $N > \max\{p^2, p+1\}$, $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold and the constant Θ_1 is defined as in (1.17). Assume that one of the following conditions is satisfied:*

(i) $k \geqslant 0$, $m \geqslant 0$, $l \geqslant 1$, $m + l \geqslant 2$;

(ii) $k \geqslant 1$, $m = 0$, $l = 1$ *and* $\Theta_1 > 0$.

Then $A < \mu_l^{-p/p^*(s)} S(0)$.

Proof. Consider the function

$$z_\varepsilon^0(x) = \varepsilon^{-\delta} U_{p,0}\biggl(\frac{|x|}{\varepsilon}\biggr)\biggl(\int_{\mathbb{R}^N} \frac{|U_{p,0}(|x|)|^{p^*(s)}}{|x|^s}\biggr)^{-1/p^*(s)},$$

where $U_{p,0}$ is the function defined in (1.10). Suppose $k \geqslant 1$. Arguing as in Lemmas 3.1 and 4.1, as $\varepsilon \to 0$ it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} \biggl(|\nabla z_\varepsilon^0(x-b_l)|^p - \sum_{i=1}^k \lambda_i \frac{|z_\varepsilon^0(x-b_l)|^p}{|x-a_i|^p}\biggr) &= \int_{\mathbb{R}^N} \biggl(|\nabla z_\varepsilon^0(x)|^p - \sum_{i=1}^k \lambda_i \frac{|z_\varepsilon^0(x)|^p}{|x+b_l-a_i|^p}\biggr) \\ &= S(0) - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|z_\varepsilon^0(x)|^p}{|x+b_l-a_i|^p} \\ &= S(0) - \varepsilon^p(\Theta_l + o(1)) \int_{\mathbb{R}^N} |z_1^0(x)|^p,\end{aligned}$$

where the integral

$$\int_{\mathbb{R}^N} |z_1^0(x)|^p$$

converges under the assumption $N > p^2$.

On the other hand, if $l \geqslant 2$ and $m \geqslant 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \sum_{j=1}^l \frac{\mu_j |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-b_j|^s} &= \int_{\mathbb{R}^N} \frac{\mu_l |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-b_l|^s} + \sum_{j=1}^{l-1} \int_{\mathbb{R}^N} \frac{\mu_j |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-b_j|^s} \\ &= \int_{\mathbb{R}^N} \frac{\mu_l |z_\varepsilon^0(x)|^{p^*(s)}}{|x|^s} + \sum_{j=1}^{l-1} \int_{\mathbb{R}^N} \frac{\mu_j |z_\varepsilon^0(x)|^{p^*(s)}}{|x+b_l-b_j|^s} \\ &= \mu_l + \varepsilon^s \biggl(\sum_{j=1}^{l-1} \frac{\mu_j}{|b_l-b_j|^s} + o(1)\biggr) \int_{\mathbb{R}^N} |z_1^0(x)|^{p^*(s)},\end{aligned}$$

$$\int_{\mathbb{R}^N} \sum_{i=1}^m \frac{\gamma_i |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-a_i|^s} = \varepsilon^s \biggl(\sum_{i=1}^m \frac{\gamma_i}{|b_l-a_i|^s} + o(1)\biggr) \int_{\mathbb{R}^N} |z_1^0(x)|^{p^*(s)},$$

where the integral

$$\int_{\mathbb{R}^N} |z_1^0(x)|^{p^*(s)}$$

converges under the assumption $N > p^2$.

(i) For $\varepsilon > 0$ small we have

$$
\begin{aligned}
A &\leqslant \int_{\mathbb{R}^N} \biggl(|\nabla z_\varepsilon^0(x-b_l)|^p - \sum_{i=1}^{k} \lambda_i \frac{|z_\varepsilon^0(x-b_l)|^p}{|x-a_i|^p} \biggr) \\
&\quad \times \biggl(\int_{\mathbb{R}^N} \sum_{i=1}^{m} \frac{\gamma_i |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-a_i|^s} + \int_{\mathbb{R}^N} \sum_{j=1}^{l} \frac{\mu_j |z_\varepsilon^0(x-b_l)|^{p^*(s)}}{|x-b_j|^s} \biggr)^{-p/p^*(s)} \\
&\leqslant (S(0) + C\varepsilon^p)(\mu_l + C\varepsilon^s)^{-p/p^*(s)} \\
&\leqslant (S(0) + C\varepsilon^p)(\mu_l^{p/p^*(s)} + C\varepsilon^s)^{-1} \\
&< \mu_l^{-p/p^*(s)} S(0).
\end{aligned}
$$

(ii) In this case, arguing as above we deduce that

$$
\begin{aligned}
A &\leqslant \int_{\mathbb{R}^N} \biggl(|\nabla z_\varepsilon^0(x-b_1)|^p - \sum_{i=1}^{k} \lambda_i \frac{|z_\varepsilon^0(x-b_1)|^p}{|x-a_i|^p} \biggr) \biggl(\int_{\mathbb{R}^N} \frac{\mu_1 |z_\varepsilon^0(x-b_1)|^{p^*(s)}}{|x-b_1|^s} \biggr)^{-p/p^*(s)} \\
&\leqslant \mu_1^{-p/p^*(s)} (S(0) - C\varepsilon^p(\Theta_1 + o(1))) \\
&< \mu_1^{-p/p^*(s)} S(0).
\end{aligned}
$$

Thus, the proof of this lemma is complete. □

Proof of Theorem 1.1. Let $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ be a minimizing sequence for the best constant A. By the homogeneity of the quotient (1.12) we can assume that

$$
\sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\gamma_i |u_n|^{p^*(s)}}{|x-a_i|^s} + \sum_{j=1}^{l} \int_{\mathbb{R}^N} \frac{\mu_j |u_n|^{p^*(s)}}{|x-b_j|^s} = 1.
$$

From Ekeland's variational principle we can assume that the sequence has the Palais–Smale property:

$$
\begin{aligned}
o(\|v\|) = \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla v - \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{\lambda_i |u_n|^{p-2} u_n v}{|x-a_i|^p} \\
- A \biggl(\sum_{i=1}^{m} \int_{\mathbb{R}^N} \frac{\gamma_i |u_n|^{p^*(s)-2} u_n v}{|x-a_i|^s} + \sum_{j=1}^{l} \int_{\mathbb{R}^N} \frac{\mu_j |u_n|^{p^*(s)-2} u_n v}{|x-b_j|^s} \biggr)
\end{aligned}
$$

holds for all $v \in D^{1,p}(\mathbb{R}^N)$. Consequently,

$$
J'(u_n) \to 0, \qquad J(u_n) \to \biggl(\frac{1}{p} - \frac{1}{p^*(s)} \biggr) A.
$$

From the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_5)$ and by Lemmas 4.1 and 4.2 we have

$$
A < B^* = \min \biggl\{ \gamma_m^{-p/p^*(s)} S(\lambda_m), \mu_l^{-p/p^*(s)} S(0), \biggl(\sum_{i=1}^{m} \gamma_i + \sum_{j=1}^{l} \mu_j \biggr)^{-p/p^*(s)} S\biggl(\sum_{i=1}^{k} \lambda_i \biggr) \biggr\}.
$$

Consequently,

$$A < A^{-p/(p^*(s)-p)}(B^*)^{p^*(s)/(p^*(s)-p)}.$$

By Lemma 2.2 we conclude that $\{u_n\}$ has a subsequence converging strongly to some $u_0 \in H$. Furthermore,

$$J(u_0) = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)A.$$

Thus, u_0 achieves the infimum in (1.12). From the fact that $J(u_0) = J(|u_0|)$, it follows that $|u_0|$ is also a minimizer in (1.12) and therefore $v_0 = A^{1/(p^*(s)-p)}|u_0|$ is a non-negative solution of problem (1.1). By the maximum principle [**20**], we have that $v_0 > 0$ in $\mathbb{R}^N \setminus \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l\}$.

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. According to Lemmas 2.2, 2.6 and 4.1, the proof is similar to that of Theorem 1.1. We omit the details. □

Proof of Theorems 1.3 and 1.4. From Lemmas 2.2, 2.7 and 4.2 and arguing as in the proof of Theorem 1.1, the desired result follows. □

Acknowledgements. This work was supported partly by the National Natural Science Foundation of China (Grant no. 10771219) and the Science Foundation of SEAC of China (Grant no. 07ZN03).

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