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A NOTE ON POWERFUL NUMBERS IN SHORT INTERVALS

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Abstract

We investigate uniform upper bounds for the number of powerful numbers in short intervals $(x, x+y]$. We obtain unconditional upper bounds $O(y/\log y)$ and $O(y^{11/12})$ for all powerful numbers and $y^{1/2}$-smooth powerful numbers, respectively. Conditional on the *abc*-conjecture, we prove the bound $O(y/\log^{1+\epsilon} y)$ for squarefull numbers and the bound $O(y^{(2+\epsilon)/k})$ for k-full numbers when $k \geq 3$. These bounds are related to Roth's theorem on arithmetic progressions and the conjecture on the nonexistence of three consecutive squarefull numbers.

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1. Introduction and main result

A number n is *squarefull* if its prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ satisfies $a_i \geq 2$ for all $1 \leq i \leq r$. Similarly, a number n is *k-full* if $a_i \geq k$ for $1 \leq i \leq r$. For example, $72 = 2^3 \cdot 3^2$ is squarefull and $243 = 3^5$ is 5-full. Let $Q_k(x)$ denote the number of k-full numbers which are less than or equal to x. It is known that

$$Q_k(x) = \prod_p \Big(1 + \sum_{m=k+1}^{2k-1} \frac{1}{p^{m/h}}\Big) x^{1/k} + O(x^{1/(k+1)}), \tag{1.1}$$

where the product is over all primes (see, for example, [1, 4]). There are also estimates for the number of k-full numbers in short intervals $(x, x+y]$ with $y = o(x)$. For moderate size y, there are some asymptotic results. For example, Trifonov [8] and Liu [6] respectively obtained

$$Q_2(x + x^{1/2+\theta}) - Q_2(x) \sim \frac{\zeta(3/2)}{2\zeta(3)} x^{\theta} \quad \text{for } \frac{19}{154} = 0.12337\ldots < \theta < \frac{1}{2},$$

and

$$Q_3(x + x^{2/3+\theta}) - Q_3(x) \sim \frac{\zeta(4/3)}{3\zeta(4)} x^{\theta} \quad \text{for } \frac{5}{42} = 0.11904\ldots < \theta < \frac{1}{3}.$$

What happens when y is very small, say $y \ll x^{1/2}$ or even $y \ll \log x$? For such short intervals, one can only expect suitable upper bounds rather than asymptotic formulae. Thus, in this note, we are interested in finding uniform upper bounds for $Q_k(x+y) - Q_k(x)$ with $1 \le y \le x$ that are independent of x. By comparing k-full numbers with perfect kth powers, we suspect the following conjecture to be true.

CONJECTURE 1.1. Given an integer $k \ge 2$ and a real number $x \ge 1$, there exists some constant $C_k \ge 1$ such that

$$Q_k(x+y) - Q_k(x) \le C_k y^{1/k}$$

uniformly over $1 \le y \le x$.

We are far from proving this at the moment. The current best upper bound,

$$Q_k(x+y) - Q_k(x) \ll \frac{y \log\log(y+2)}{\log(y+2)}, \tag{1.2}$$

was obtained by De Koninck *et al.* [3]. We improve (1.2) slightly.

THEOREM 1.2. *Given an integer $k \ge 2$ and a real number $x \ge 1$, we have*

$$Q_k(x+y) - Q_k(x) \ll \frac{y}{\log(y+1)} \tag{1.3}$$

uniformly over $1 \le y \le x$.

In fact, we shall prove the following more general result concerning squarefull numbers in arithmetic progression over short intervals which gives Theorem 1.2 immediately, as k-full numbers are included in squarefull numbers.

THEOREM 1.3. *Given real numbers $x \ge 1$ and $0 < \alpha < 1$ and integers $q > 0$ and r with $\gcd(r,q) = 1$, we have*

$$\sum_{\substack{x<n\le x+y \\ n \text{ squarefull} \\ n \equiv r \,(\mathrm{mod}\, q)}} 1 \ll_\alpha \frac{y}{\phi(q)\log(y+1)}$$

uniformly over $1 \le y \le x$ and $1 \le q \le y^{1-\alpha}$.

Using a similar technique, we can obtain some power savings over (1.3) for *smooth* k-full numbers in short intervals.

THEOREM 1.4. *Given an integer $k \ge 2$ and a real number $x \ge 1$, we have*

$$\sum_{\substack{x<n\le x+y \\ n\ k\text{-full} \\ p^+(n) \le y^{1/2}}} 1 \le \sum_{\substack{x<n\le x+y \\ n \text{ squarefull} \\ p^+(n) \le y^{1/2}}} 1 \ll y^{11/12} \tag{1.4}$$

uniformly over $1 \le y \le x$. Here $p^+(n)$ stands for the largest prime factor of n.

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One may increase the exponent $1/2$ up to 1 and obtain a similar power saving upper bound.

The bound (1.4) lends evidence towards Conjecture 1.1 and shows that the difficulty lies with *nonsmooth* k-full numbers. Another piece of evidence comes from the famous *abc*-conjecture. It was proved in [3] that, given any $\delta > 0$, the interval

$$(x, x + x^{1-(2+\delta)/k}] \tag{1.5}$$

contains at most one k-full number for sufficiently large x under the *abc*-conjecture. From this, one has the following result.

THEOREM 1.5. *Assume the abc-conjecture. Given an integer $k \geq 2$ and real numbers $\delta > 0$ and $x \geq 1$, we have*

$$Q_k(x + y) - Q_k(x) \ll_{\epsilon,k} y^{(2+\delta)/k} \tag{1.6}$$

uniformly over $1 \leq y \leq x$.

We shall modify the proof in [3] concerning (1.5) slightly to correct an inaccuracy (since the a, b, c in the application of the *abc*-conjecture might not be relatively prime). Then we apply it to obtain Theorem 1.5. Observe that (1.5) or (1.6) give us nothing nontrivial when $k = 2$. To remedy this, we shall prove the following conditional result which improves (1.3) slightly by a small power of a logarithm.

THEOREM 1.6. *The abc-conjecture implies that for some absolute constant $c > 0$,*

$$Q_2(x + y) - Q_2(x) \ll \frac{y}{\log^{1+c}(y + 1)}$$

uniformly over $1 \leq y \leq x$.

The proof relies on the following recent breakthrough result of Bloom and Sisask on the density of integer sequences without three-term arithmetic progressions.

THEOREM 1.7 (Bloom–Sisask, [2]). *Let $N \geq 2$ and $A \subset \{1, 2, \ldots, N\}$ be a set with no nontrivial three-term arithmetic progressions, that is, solutions to $x + y = 2z$ with $x \neq y$. Then*

$$|A| \ll \frac{N}{(\log N)^{1+c}},$$

where $c > 0$ is an absolute constant.

This paper is organised as follows. First, we will prove Theorems 1.3 and 1.4 using the Brun–Titchmarsh inequality and ideas from Shiu's generalisation [7]. Then we will prove Theorem 1.5 using the *abc*-conjecture. Finally, we will prove Theorem 1.6 by establishing the nonexistence of three-term arithmetic progressions for squarefull numbers in short intervals.

Notation. We use $|A|$ to denote the number of elements in a finite set A and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. We let $p_-(n)$ and $p^+(n)$ be the smallest and the largest prime factor of n, respectively. The symbols $f(x) = O(g(x))$

and $f(x) \ll g(x)$ are equivalent to $|f(x)| \le Cg(x)$ for some constant $C > 0$. Also, $f(x) = O_{\lambda_1,\ldots,\lambda_r}(g(x))$ and $f(x) \ll_{\lambda_1,\ldots,\lambda_r} g(x)$ mean that the implicit constant may depend on $\lambda_1, \ldots, \lambda_r$. Furthermore, $f(x) = o(g(x))$ means $\lim_{x\to\infty} f(x)/g(x) = 0$ and $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$. Finally, the summation symbol $\sum'$ signifies that a sum is over squarefull numbers only.

2. Some preparations

Lemma 2.1. *For any $X \ge 1$,*

$$\sum_{X<n\le X^2}{}' \frac{1}{n} \ll X^{-1/2}.$$

Proof. From (1.1), $Q_2(X) \ll X^{1/2}$. By partial summation, the above sum is

$$\int_X^{X^2} \frac{1}{u}\, dQ(u) = \frac{Q(X^2)}{X^2} - \frac{Q(X)}{X} + \int_X^{X^2} \frac{Q(u)}{u^2}\, du \ll \frac{1}{X^{1/2}} + \int_X^{X^2} \frac{1}{u^{3/2}}\, du \ll \frac{1}{X^{1/2}}. \qquad \square$$

Lemma 2.2 (Brun–Titchmarsh inequality). *Let $q \ge 1$ and r be integers satisfying* $\gcd(r, q) = 1$. *Suppose $q < y \le x$ and $z \ge 2$. Then,*

$$\sum_{\substack{x<n\le x+y \\ n\equiv r \pmod q \\ p_-(n)>z}} 1 \ll \frac{y}{\phi(q)\log z} + z^2.$$

The above bound is still true when $y \le q$ or $y < 1$ since there is at most one term in the sum. The estimate follows from the Selberg upper bound sieve method (see, for example, [5, page 104]).

Finally, let us recall the *abc*-conjecture. For any nonzero integer m, the kernel of m is

$$\kappa(m) := \prod_{p|m} p.$$

Conjecture 2.3 (*abc*-conjecture). For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for any integers a, b, c with $a + b = c$ and $\gcd(a, b) = 1$, we have

$$\max\{|a|, |b|, |c|\} \le C_\epsilon \kappa(abc)^{1+\epsilon}.$$

3. Proof of Theorem 1.3

Our proof is inspired by Shiu [7] on the Brun–Titchmarsh theorem for multiplicative functions. We may assume that $y \ge 2^{2/\alpha}$ for the theorem is clearly true when $1 \le y < 2^{2/\alpha}$ by choosing a large enough implicit constant. Recall that $1 \le q \le y^{1-\alpha}$ for some $\alpha > 0$. Let $z = y^{\alpha/2} \ge 2$. Any squarefull number n in $[x, x + y]$ can be factored as

$$n = \underbrace{p_1^{a_1} \cdots p_j^{a_j}}_{b_n} \underbrace{p_{j+1}^{a_{j+1}} \cdots p_s^{a_s}}_{d_n} \quad \text{with } p_1 < p_2 < \cdots < p_s,$$

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where j is the greatest index such that $p_1^{a_1} \cdots p_j^{a_j} \le z$. Hence, $b_n \le z < b_n p_{j+1}^{a_{j+1}}$. Note that j may be 0 (the product is an empty product) if $p_1^{a_1} > z$. In this case, $b_n = 1$ and $d_n = n$. Also, since $n \equiv r \pmod q$ with $\gcd(r, q) = 1$, we must have $\gcd(b_n, q) = 1 = \gcd(d_n, q)$.

Case 1: $b_n > z^{1/2}$. As $q \le y^{1-\alpha}$ and $z = y^{\alpha/2}$, the number of such squarefull numbers is bounded by

$$\sideset{}{'}\sum_{\substack{z^{1/2} < b \le z \\ \gcd(b,q)=1}} \sum_{\substack{x < n \le x+y \\ b \mid n \\ n \equiv r \ (\mathrm{mod}\ q)}} 1 \le \sideset{}{'}\sum_{z^{1/2} < b \le z} \Big(\frac{y/b}{q} + 1 \Big) \ll \frac{y}{qz^{1/4}} + z^{1/2} \ll_\alpha \frac{y}{\phi(q) \log y} \tag{3.1}$$

by (1.1) and Lemma 2.1.

Case 2: $b_n \le z^{1/2}$ and $p_-(d_n) \le z^{1/2}$. Then $p_{j+1} \le z^{1/2}$ and $p_{j+1}^{a_{j+1}} > z^{1/2}$ which implies $p_{j+1}^{-a_{j+1}} \le \min(z^{-1/2}, p_{j+1}^{-2})$ as $a_{j+1} \ge 2$. Hence, the sum

$$\sum_{p_{j+1} \le z^{1/2}} \frac{1}{p_{j+1}^{a_{j+1}}} \le \sum_{p_{j+1} \le z^{1/4}} z^{-1/2} + \sum_{z^{1/4} < p_{j+1} \le z^{1/2}} \frac{1}{p_{j+1}^2} \ll \frac{1}{z^{1/4}}.$$

Therefore, by replacing $p_{j+1}^{a_{j+1}}$ with a generic p^a, the number of squarefull numbers in this case is bounded by

$$\sum_{\substack{p \le z^{1/2} \\ \gcd(p,q)=1}} \sum_{\substack{x < n \le x+y \\ p^a \mid n \\ n \equiv r \ (\mathrm{mod}\ q)}} 1 \le \sum_{p \le z^{1/2}} \Big(\frac{y/p^a}{q} + 1 \Big) \ll \frac{y}{qz^{1/4}} + z^{1/2} \ll_\alpha \frac{y}{\phi(q) \log y}, \tag{3.2}$$

since $q \le y^{1-\alpha}$ and $z = y^{\alpha/2}$.

Case 3: $b_n \le z^{1/2}$ and $p_-(d_n) > z^{1/2}$. As $q \le y^{1-\alpha}$ and $z = y^{\alpha/2}$, the number of such squarefull numbers is bounded by

$$\sideset{}{'}\sum_{\substack{b \le z^{1/2} \\ \gcd(b,q)=1}} \sum_{\substack{x/b < n/b \le (x+y)/b \\ p_-(n/b) > z^{1/2} \\ (n/b) \equiv r\bar{b} \ (\mathrm{mod}\ q)}} 1 \ll \sideset{}{'}\sum_{b \le z} \Big(\frac{y/b}{\phi(q) \log z} + z \Big) \ll \frac{y}{\phi(q) \log z} + z^{3/2} \ll_\alpha \frac{y}{\phi(q) \log y} \tag{3.3}$$

by (1.1), Lemma 2.2 and the convergence of the sum of reciprocals of squarefull numbers (which follows from Lemma 2.1 for instance). Here $\bar{b}$ denotes the multiplicative inverse of b (mod$q$), that is, $b\bar{b} \equiv 1 \pmod q$.

Combining (3.1), (3.2) and (3.3), we have Theorem 1.3.

4. Proof of Theorem 1.4

This is very similar to the proof of Theorem 1.3, so we just highlight the necessary adjustments. We set $q = 1$ and $z = y^{1/3}$. The arguments for Case 1 and Case 2 are

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exactly the same as (3.1) and (3.2), and we get the bound

$$\frac{y}{z^{1/4}} + z^{1/2} \ll y^{11/12}.$$

It remains to deal with Case 3, where $b_n \le z^{1/2}$ and $z^{1/2} < p_-(d_n) \le y^{1/2}$ as the squarefull numbers are assumed to be $y^{1/2}$-smooth. Thus, with $p := p_-(d_n)$ and $d_n := p^2 d$, the number of squarefull numbers in this case is bounded by

$$\begin{aligned}\sum_{b\le z^{1/2}}{}' \sum_{z^{1/2}<p\le y^{1/2}} \sum_{\substack{x/b<p^2 d\le (x+y)/b \\ p_-(d)>z^{1/2}}} 1 &= \sum_{b\le z^{1/2}}{}' \sum_{z^{1/2}<p\le y^{1/2}} \sum_{\substack{x/bp^2<d\le (x+y)/bp^2 \\ p_-(d)>z^{1/2}}} 1 \\ &\ll \sum_{b\le z^{1/2}}{}' \sum_{z^{1/2}<p\le y^{1/2}} \Big(\frac{y/(bp^2)}{\log z} + z\Big) \ll \frac{y}{z^{1/2}\log z} + \frac{z^{5/4}y^{1/2}}{\log y} \ll y^{11/12}\end{aligned}$$

by (1.1), Lemma 2.2 and the convergence of the sum of reciprocals of squarefull numbers. The above bounds together yield Theorem 1.4.

5. Proof of (1.5) and Theorem 1.5

Given an integer $k \ge 2$ and a small real number $\delta > 0$, we claim that the interval from (1.5), namely

$$(x, x + x^{1-(2+\delta)/k}],$$

contains at most one k-full number for all sufficiently large $x > C$ (in terms of δ and k) under the *abc*-conjecture.

Following De Koninck *et al.* [3], we suppose that the interval $(x, x + x^{1-(2+\delta)/k}]$ contains two k-full numbers, $b < c$. Then $c = a + b$ for some integer a with $0 < a \le x^{1-(2+\delta)/k}$. With $d = \gcd(a, b)$, the integers a/d, b/d and c/d are pairwise relatively prime. Note that $\kappa(n) \le n^{1/k}$ for any k-full number. Applying the *abc*-conjecture with $\epsilon = \delta/k$ to the equation $a/d + b/d = c/d$, we get

$$\begin{aligned}\frac{x}{d} < \frac{c}{d} &\le C_{\delta/k}\Big(\kappa\Big(\frac{a}{d}\Big)\kappa\Big(\frac{b}{d}\Big)\kappa\Big(\frac{c}{d}\Big)\Big)^{1+\delta/k} \le C_{\delta/k}\Big(\frac{a}{d}\cdot\kappa(b)\kappa(c)\Big)^{1+\delta/k} \\ &\le C_{\delta/k}\Big(\frac{x^{1-(2+\delta)/k}}{d}(2x)^{2/k}\Big)^{1+\delta/k} = 2^{(2/k)(1+\delta/k)}C_{\delta/k}\frac{x^{1-\delta^2/k^2}}{d^{1+\delta/k}} \\ &\le 2^{(2/k)(1+\delta/k)}C_{\delta/k}\frac{x^{1-\delta^2/k^2}}{d}.\end{aligned}$$

This implies

$$x^{\delta^2/k^2} \le 2^{(2/k)(1+\delta/k)}C_{\delta/k} \quad \text{or} \quad x \le (2^{(2/k)(1+\delta/k)}C_{\delta/k})^{k^2/\delta^2} =: C$$

and the claim follows.

Clearly, Theorem 1.5 is true for $1 \le y \le C$ by picking the implicit constant to be C. Now, for $C < y \le x$, the above claim implies that the interval

$$(x, x + y^{1-(2+\delta)/k}]$$

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contains at most one k-full number. By dividing the interval $(x, x+y]$ into subintervals of length $y^{1-(2+\delta)/k}$, we obtain

$$Q_k(x+y) - Q_k(x) \ll \frac{y}{y^{1-(2+\delta)/k}} \cdot 1 = y^{(2+\delta)/k},$$

which gives Theorem 1.5.

6. Proof of Theorem 1.6

First, we suppose $y \le x^{0.2}$. We claim that there is no nontrivial three-term arithmetic progression among the squarefull numbers in the interval $(x, x+y]$ under the abc-conjecture. Suppose the contrary. Then we have three squarefull numbers $x < a_1^2 b_1^3 < a_2^2 b_2^3 < a_3^2 b_3^3 \le x+y$ such that

$$a_1^2 b_1^3 = a_2^2 b_2^3 - d \quad \text{and} \quad a_3^2 b_3^3 = a_2^2 b_2^3 + d$$

for some positive integer d with $2d \le y$. Multiplying the above two equations, we get

$$a_1^2 a_3^2 b_1^3 b_3^3 = a_2^4 b_2^6 - d^2 \quad \text{or} \quad a_1^2 a_3^2 b_1^3 b_3^3 + d^2 = a_2^4 b_2^6.$$

Say $D^2 = \gcd(a_2^4 b_2^6, d^2)$ as the numbers are perfect squares. Then, the three integers

$$\frac{a_1^2 a_3^2 b_1^3 b_3^3}{D^2}, \quad \frac{d^2}{D^2}, \quad \frac{a_2^4 b_2^6}{D^2}$$

are pairwise relatively prime and we have the equation

$$\frac{a_1^2 a_3^2 b_1^3 b_3^3}{D^2} + \frac{d^2}{D^2} = \frac{a_2^4 b_2^6}{D^2}.$$

Now, by the abc-conjecture,

$$\begin{aligned}
\frac{x^2}{D^2} \le \frac{a_2^4 b_2^6}{D^2} &\ll_\epsilon \kappa\Big(\frac{a_1^2 a_3^2 b_1^3 b_3^3}{D^2} \frac{d^2}{D^2} \frac{a_2^4 b_2^6}{D^2}\Big)^{1+\epsilon} \\
&\ll_\epsilon \kappa(a_1^2 a_3^2 b_1^3 b_3^3)^{1+\epsilon} \kappa\Big(\frac{d^2}{D^2}\Big)^{1+\epsilon} \kappa(a_2^4 b_2^6)^{1+\epsilon} \\
&\ll_\epsilon (a_1 b_1 a_2 b_2 a_3 b_3)^{1+\epsilon} \Big(\frac{d}{D}\Big)^{1+\epsilon} \ll_\epsilon x^{3/2+3\epsilon/2} \frac{y^{1+\epsilon}}{D^{1+\epsilon}}.
\end{aligned}$$

Since $1 \le D \le d \le y$, this implies $x^{1/2-3\epsilon/2} \ll_\epsilon D^{1-\epsilon} y^{1+\epsilon} \ll y^2 \le x^{0.4}$, which is a contradiction for small enough ϵ, say $\epsilon = 0.01$, and sufficiently large $x > C$ (in terms of the implicit constant).

Clearly, the theorem is true for $1 \le y \le C$ by picking an appropriate implicit constant. So, we may assume $y > C$. Since arithmetic progressions are invariant under translation, we may shift the interval $(x, x+y]$ to $(0, y]$. Therefore, by Theorem 1.7, we have

$$Q_2(x+y) - Q_2(x) \ll \frac{y}{\log^{1+c} y},$$

which gives the theorem.

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Now, if $y > x^{0.2}$, one can simply divide the interval $(x, x + y]$ into subintervals of length $x^{0.2}$:

$$(x, x + x^{0.2}] \cup (x + x^{0.2}, x + 2x^{0.2}] \cup \cdots \cup \left(x + \left\lfloor \frac{y}{x^{0.2}} \right\rfloor x^{0.2}, x + \left(\left\lfloor \frac{y}{x^{0.2}} \right\rfloor + 1\right) x^{0.2}\right].$$

Then, over each interval $(x + ix^{0.2}, x + (i + 1)x^{0.2}]$, we have the bound

$$Q_2(x + (i + 1)x^{0.2}) - Q_2(x + ix^{0.2}) \ll \frac{x^{0.2}}{\log^{1+c} x}.$$

Summing over $\lfloor y/x^{0.2} \rfloor + 1$ of these intervals, we have

$$Q_2(x + y) - Q_2(x) \ll \frac{y}{x^{0.2}} \cdot \frac{x^{0.2}}{\log^{1+c} x} \ll \frac{y}{\log^{1+c} y},$$

which gives the theorem as well.

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