DENSE SUBGRAPHS AND CONNECTIVITY

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A proper subgraph of a connected linear graph is said to disconnect the graph if removing it leaves a disconnected graph. In this paper we characterize, in the following sense, the disconnecting subgraphs of a fixed connected graph. We define two distinct types of disconnecting subgraphs (isthmuses and articulators) which are minimal in the sense that no proper subgraph of either type can disconnect the graph. We then show that any disconnecting subgraph must contain either an isthmus or an articulator. We also define a set of subgraphs (called dense) which form a lattice. We show that the union of the minimal dense subgraphs contains all isthmuses and articulators. In terms of these subgraphs we investigate some of the consequences of assuming that a disconnecting subgraph must contain at least m points.

1. Definitions. A (linear, undirected) graph G is a finite set of elements $p_1, p_2, \ldots, p_n$ called *points*, and a set of ordered pairs of these elements defining a symmetric, non-reflexive binary relation. Two points occurring in an ordered pair are said to be *neighbours*. A *subgraph* of G is a subset of the points of G together with all the ordered pairs in G containing only elements of the subset. A subgraph is thus determined by its set of points when the binary relation of G is understood.

Two distinct points, p and q, in G are said to be *connected* by a *path* of *length* k if there exist $k + 1$ distinct points $p = p_1, p_2, \ldots, p_{k+1} = q$ such that the ordered pairs (p_i, p_{i+1}), for $i = 1, 2, \ldots, k$, are in G. The *distance* between two points is the length of the shortest path between them. The *diameter* of the graph is the greatest distance between pairs of points in the graph.

A graph having only one point or more than one point and every pair of points connected is also called *connected*. If every pair of points are neighbours the graph is called *completely connected*. A graph which is not connected is called *disconnected*. The null graph is disconnected.

The *union*, *intersection*, and *difference* of two subgraphs G_1 and G_2 is the subgraph whose set of points is the union, intersection, or difference of the sets of points of G_1 and G_2. We denote the union by $G_1 + G_2$ and the difference by $G_1 - G_2$.

If a graph is not connected it is the union of a set of disjoint subgraphs each one of which is connected and such that the union of any two is not connected. This set is unique and we refer to it as the *partition* of the graph.

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We say that a proper subgraph G' of a connected graph G *disconnects* G if $G - G'$ is disconnected. We shall be interested in ways of disconnecting a fixed connected graph G containing n points and to this end we introduce two definitions.

A *k-isthmus* of G is a completely connected subgraph which has k points, disconnects G, but does not properly contain a completely connected subgraph which disconnects G. A *k-articulator* of G is a subgraph G' which has k points, disconnects G, is not completely connected, and has the property that each subgraph in the partition of $G - G'$ has a neighbour of each point in G'. We shall use the generic terms *isthmus* and *articulator* when the number of points is irrelevant.

For example, if we denote G pictorially with lines representing the relation between points we can see the isthmuses and articulators in the following connected graphs:

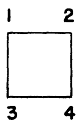

FIGURE 1

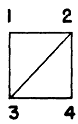

FIGURE 2

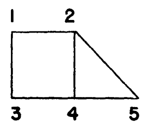

FIGURE 3

In Figure 1 the subgraphs with point sets $\{1, 4\}$ and $\{2, 3\}$ are articulators but there are no isthmuses. In Figure 2 $\{2, 3\}$ is an isthmus but there are no articulators. In Figure 3 $\{2, 4\}$ is an isthmus and $\{1, 4\}$ and $\{2, 3\}$ are articulators.

We now define a type of subgraph which we shall prove has a close connection with the isthmuses and articulators of G. A connected subgraph G' is called *dense* if $G' = G$ or if every point in $G - G'$ has a neighbour in G'. A dense proper subgraph which is contained in no other dense subgraph except G we call *D-maximal*; a dense subgraph containing no other dense subgraph we call *D-minimal*. We let S_D denote the collection of dense subgraphs of G ordered by inclusion together with the empty graph ϕ. We let Γ denote the union of all D-minimal subgraphs of G. Unless otherwise stated all dense subgraphs, isthmuses, and articulators are those of G.

We call G *m-connected* if $G - G'$ is connected for every subgraph G' containing fewer than m points.

The subgraph of neighbours of a point p in a subgraph G' we denote by $G'(p)$.

2. Dense Subgraphs. Suppose G_1 is a dense subgraph and G_2 is a subgraph containing G_1. Since every point in $G_2 - G_1$ has a neighbour in the connected graph G_1 it follows that G_2 is connected. Also every point in $G - G_2$ has a neighbour in G_2 (in fact in G_1). Therefore we have

LEMMA 2.1. *A subgraph which contains a dense subgraph is also dense.*

Let G_1 and G_2 be two dense subgraphs. By this lemma their union is also dense. Their intersection need not be dense but if it is not it cannot contain a dense subgraph, again by this lemma. Therefore we have

THEOREM 2.2. *S_D is a lattice in which the* l.u.b. *is the graph union, and the* g.l.b. *is the graph intersection when the intersection is dense and otherwise is ϕ.*

Applying Lemma 2.1 to the definition of D-maximal we get

LEMMA 2.3. *A subgraph is D-maximal if and only if it is connected and contains $n - 1$ points.*

Suppose $n \geqslant 2$ and let d denote the diameter of G. Let p_1 and p_2 be points such that the distance between them is d. If $G - p_1$ is not connected let $\{G_i\}$ denote its partition. Suppose p_2 is in G_1 and let p_3 be a point in G_2 which is a neighbour of p_1. Any path between p_2 and p_3 must pass through p_1 since removing p_1 disconnects an otherwise connected graph. Thus the distance between p_2 and p_3 is $d + 1$ which is a contradiction. It follows that $G - p_1$, and $G - p_2$ by symmetry, is connected and so a D-maximal subgraph. We have proved

THEOREM 2.4. *If $n \geqslant 2$ then G contains at least two D-maximal subgraphs. Thus every point is contained in a proper dense subgraph.*

We shall show that if $G = \Gamma$ and the D-minimal subgraphs are mutually disjoint then G is completely connected. First we need

LEMMA 2.5. *Let $\{\Gamma_i\}$ be a collection of mutually disjoint dense subgraphs whose union is G. If at least one of the Γ_i contains more than one point then there is a dense subgraph containing none of the Γ_i.*

We shall prove this by constructing the desired dense subgraph G'.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be those subgraphs among the Γ_i containing more than one point.

If $s = 1$ let G' denote an arbitrary D-maximal subgraph of Γ_1. Every point in Γ_1 has a neighbour in G' and every other point in G is dense and so is a neighbour to every point in G'. Thus G' is a dense subgraph of G properly contained in Γ_1 and disjoint from the other Γ_i, as desired.

Now suppose $s \geqslant 2$. Choose an arbitrary point q in Γ_1. Let $q_1 = q$ and q_i be a neighbour of q in Γ_i for $i = 2, 3, \ldots, s$. Since each Γ_i $(i = 1, 2, \ldots, s)$ contains more than one point, it contains a D-maximal subgraph G_i which contains q_i. Let G' be the union of the G_i.

Each of the G_i is connected and each has a neighbour of q or contains q so G' is connected. Let p be an arbitrary point in G distinct from q. Let Γ_i be that subgraph containing p. If Γ_i contains no other points p is a neighbour of every point in G'. If Γ_i has more than one point then either p has a neighbour in G_i (if p is not in G_i or G_i contains more than one point) or is a neighbour of q (if p is the only point in G_i). Thus G' is dense and we can complete our argument as before.

We can now prove

THEOREM 2.6. *If $G = \Gamma$ and the D-minimal subgraphs are mutually disjoint then G is completely connected.*

Given the hypothesis, if any D-minimal subgraph contains more than one point we can apply Lemma 2.5 to obtain a dense subgraph not containing any D-minimal subgraph. Since this is absurd every D-minimal subgraph contains exactly one point. Therefore every point of G is dense and so G is completely connected.

Now we prove

THEOREM 2.7. *If a point p is not D-minimal then $\Gamma(p)$ disconnects G.*

Suppose $G - \Gamma(p)$ is connected. It contains p and every point in $\Gamma(p)$ is a neighbour of p so that $G - \Gamma(p)$ is dense. Thus $G - \Gamma(p)$ contains a D-minimal subgraph G' having no neighbours of p. This is possible only if $G' = p$.

We have incidentally proved

LEMMA 2.8. *If $G - (G(p) - H)$ is connected it is dense.*

LEMMA 2.9. *$G - G(p)$ is connected if and only if p is D-minimal.*

3. Connectivity. We begin by finding a necessary and sufficient condition that a subgraph be an articulator or an isthmus.

If G' is an articulator or an isthmus then $G - G'$ is not connected. Let p be any point in G' and consider $G - G' + p$. If G' is an articulator every subgraph in the partition of $G - G'$ contains a neighbour of p so $G - G' + p$ is connected. Likewise every point in $G' - p$ has a neighbour in $G - G'$ and so in $G - G' + p = G - (G' - p)$. Therefore $G - G' + p$ is dense as is $G - G''$ for every proper subgraph G'' of G'. If G' is an isthmus then $G' - p$ is completely connected so $G - G' + p$ is connected. Every point in $G' - p$ is a neighbour of p so $G - G' + p$ is dense as is $G - G''$ for every proper subgraph G'' of G'.

Now suppose G' is a subgraph which disconnects G but $G - G' + p$ is connected for every point p in G'. Then such a point must have a neighbour in every subgraph of the partition of $G - G'$ so G' is an articulator if it is not completely connected and an isthmus if it is. Thus we have

THEOREM 3.1. *A subgraph G' is either an articulator or an isthmus if and only if it disconnects G and $G - G''$ is connected (and so dense) for every proper subgraph G'' of G'.*

COROLLARY. *An articulator does not properly contain an articulator. An articulator does not contain an isthmus and conversely.*

Let G' be an articulator or isthmus and let p be any point in G'. Then, by Theorem 3.1, $G - G' + p$ is dense and so contains a D-minimal subgraph which must contain p since $G - G'$ is not dense. It follows that every point in G' is in Γ. That is

THEOREM 3.2. *All articulators and all isthmuses are contained in* Γ.

Let G' be any subgraph which disconnects G. It is either an articulator or an isthmus or, by Theorem 3.1, contains a proper subgraph G'' which disconnects G. By repeating the argument on G'' we are eventually led to the case when $G - p$ is not connected for a point p. Since such a point p is an isthmus we have

THEOREM 3.3. *A subgraph which disconnects G contains an articulator or an isthmus.*

We now turn to some of the consequences of m-connectivity and obtain

THEOREM 3.4. *If G is m-connected then*

1. *$G - G'$ is dense for every subgraph G' containing less than m points, and conversely.*

2. *G contains no k-isthmus for $k = 1, 2, \ldots, m - 1$ and no k-articulator for $k = 2, 3, \ldots, m - 1$, and conversely.*

3. *Γ contains at least m points as does $\Gamma(p)$ for every point p in G which is not D-minimal.*

4. *The intersection of any $m - 1$ D-maximal subgraphs is dense.*

5. *An m-articulator of Γ is an m-articulator of G.*

6. *Either G is completely connected and $n = m$, or it is not and $n \geqslant m + 2$.*

7. *If p is a point in $G - \Gamma$ and q is a point in $G(p)$ then $G(q)$ has more than m points.*

Let G be m-connected and G' be a subgraph containing less than m points. Suppose there is a point p in G' without a neighbour in $G - G'$. Then $G - G' + p = G - (G' - p)$ is not connected contrary to the definition of m-connected. It follows that every point p in G' has a neighbour in the connected subgraph $G - G'$ which is thus dense. The converse is clear and so part 1 is proved.

The necessity of part 2 follows from Theorem 3.3 and the sufficiency is clear.

Since the second half of part 3 follows from Theorem 2.7 we must show that Γ contains at least m points. If G is completely connected then $G = \Gamma$ so Γ is m-connected and thus contains at least m points. If G is not completely connected it contains at least one point which is not D-minimal. That point has at least m neighbours in Γ so Γ contains at least m points.

Part 4 follows from part 1 and Lemma 2.3.

Part 5 follows from

LEMMA 3.5. *An articulator of* Γ *disconnects* G.

In proving this we do not assume that G is m-connected.

Let Γ' be an articulator of Γ and suppose $G - \Gamma'$ is connected. Every point in Γ' has a neighbour in $\Gamma - \Gamma'$ and so in $G - \Gamma'$. Therefore $G - \Gamma'$ is dense and so contains a D-minimal subgraph G'. But $G - \Gamma' - (G - \Gamma) = \Gamma - \Gamma'$ is not dense. This implies that $G - \Gamma$ contains some point of G' contrary to the definition of Γ. It follows that Γ' disconnects G.

Now suppose G is m-connected and Γ' contains m points. By this lemma and Theorem 3.3, Γ' contains either an articulator or an isthmus. But it cannot contain either properly by part 2. Since Γ' is not completely connected it is an articulator (of G).

If G is completely connected then $n = m$. Otherwise there are points p and q in G which are not connected so $G - p - q$ disconnects G. It follows that $m \leqslant n - 2$ and part 6 is proved.

If p is a point in $G - \Gamma$ and q is a point in $G(p)$ then if q is not D-minimal it has at least m neighbours in Γ as well as at least one (that is p) in $G - \Gamma$. If q is D-minimal then it has $n - 1$ neighbours. Since $G - \Gamma$ has a point G cannot be completely connected so $n - 1 > m$ and part 7 is proved.

As a partial converse of part 4 we prove

THEOREM 3.6. *If the intersection of any* $m \geqslant 2$ *D-maximal subgraphs is connected then there are no k-isthmuses or k-articulators for* $m \geqslant k \geqslant 2$.

Let G' be a k-isthmus or k-articulator with $m \geqslant k \geqslant 2$, and let p be an arbitrary point in G'. By Theorem 3.1 we know that $G - p$ is dense and so D-maximal. Thus $G - G'$ is the intersection of $k \leqslant m$ D-maximal subgraphs but is not connected contrary to the hypothesis of the theorem.

We complete this section with a few isolated results.

THEOREM 3.7. *If* G *is not completely connected but* $G(p)$ *is for some point* p *then* p *is in* $G - \Gamma$.

Suppose Γ' is a D-minimal subgraph containing p. If $\Gamma' = p$ then $G(p) = G - p$ so G is completely connected contrary to hypothesis. Therefore Γ' contains a neighbour q of p. Since every point which is a neighbour of p is a neighbour of q we see that $\Gamma' - p$ is dense, again contrary to hypothesis. Thus p is not in any D-minimal subgraph.

THEOREM 3.8. *The intersection of all dense subgraphs is exactly the subgraph of* 1*-isthmuses.*

Suppose p is a point contained in all dense subgraphs. Then $G - p$ does not contain a dense subgraph and so is not dense. Since p has a neighbour in $G - p$ the latter is not connected so p is a 1-isthmus. Conversely, if p is a 1-isthmus $G - p$ is not dense but G is so that every dense subgraph contains p.

Added in proof: In order to complete the statements of Theorems 3.1 and 3.3 we should have proved that *if G is not completely connected it contains a disconnecting subgraph.* This is trivial. For suppose G is not completely connected. Then it contains at least three points, at least two of which are not neighbours. Then $G - p - q$ disconnects G.

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