A CONVERSE TO THE LOG-LOG LAW FOR MARTINGALES

Dedicated to the memory of Hanna Neumann

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ABSTRACT

For sums of independent and identically distributed random variables x_n, the Hartman-Wintner law of the iterated logarithm is equivalent to $x_n \in L_2$. We show that this is also true when the x_n form a stationary, ergodic martingale difference sequence. This is accomplished by extending a theorem of Volker Strassen to the present context.

1. Introduction

In this paper we consider a sequence $\{x_j\}$ of identically distributed random variables on a probability space $(\Omega, \mathscr{A}, p)$ and the following two conditions:

(A) $$x_n \in L_2 \text{ and } E(x_n) = 0,\ E(x_n^2) = K^2 \text{ for some } K,$$

(B) $$p\{\limsup_{n\to\infty}(x_1 + \cdots + x_n)/(2n \log\log n)^{\frac{1}{2}} = T^2\} = 1 \text{ for some } T.$$

As usual, E denotes Lebesgue integration with respect to p of $\mathscr{A}$-measurable functions on Ω. For fixed $t > 0$, L_t is the set of functions f on Ω for which $E(|f|^t) < \infty$, while for $A \in \mathscr{A}$, "A a.e." means $p(A) = 1$.

If the x_n are independent, (A) implies (B), with $T^2 = K^2$, by the celebrated Hartman-Winter law of the iterated logarithm [2]. In fact (B) implies (A), with $K^2 = T^2$, as shown by Strassen [6], who thus elucidated the true nature of the log-log law for sums of independent, identically distributed random variables. Like the central limit theorem ($(x_1 + \cdots + x_n)/(nK^2)^{\frac{1}{2}}$ converges in law to the unit normal distribution if and only if (A)), it is a second order result.

Even without independence such a characterization of the law of the iterated logarithm may obtain. Specifically, suppose that the x_n form a *martingale difference sequence* with respect to an increasing sequence $(\phi, \Omega) = \mathscr{F}_0 \subseteq \cdots \subseteq F_n \subset \cdots$ of sigma sub-algebras of A; i.e., $x_n \in L_1$ is $\mathscr{F}_n$ measurable and

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$E(x_{n+1} | \mathscr{F}_n) = 0$ a.e., all n, where $E(\cdot | \mathscr{F}_j)$ denotes a version of the conditional expectation operator on the $\mathscr{F}_j$ measurable functions. Then if the x_n are also stationary and ergodic the Hartman-Wintner theorem extends and (A) still implies (B), as was shown by Stout [5]. Of course, $E(x_n) = 0$ is redundant in (A).

Interestingly, Strassen's theorem extends too, and (B) still implies (A), as will be shown in the next section. The law of the iterated logarithm remains a second order result in the martinglale case.

2. The result

We prove the following statement.

THEOREM. *Let $\{x_j\}$ be a stationary and ergodic sequence of random variables on a probability space $(\Omega, \mathscr{A}, p)$. If $\{x_j, \mathscr{F}_j\}$ is a martingale difference sequence and if condition* (B) *holds, then also condition* (A) *holds with $K^2 = T^2$.*

PROOF. Without loss of generality we may take Ω to be the set of all extended real valued sequences $\{\cdots, u_{-1}, u_0, u_1, \cdots\}$, the x_j to be the coordinate random variables on Ω, $\mathscr{A}$ to be the σ-algebra generated by the x_j, p to be the probability generated by finite dimensional distributions, and $\mathscr{F}_j$ to be the σ-field generated $\{x_k, -\infty < k \leqq j\}$. A standard construction assures that there is such a representation that preserves the original finite dimensional distributions, so that the given stochastic structure is not changed.

It suffices to prove $x_n \in L_2$ since $E(x_n^2) = T^2$ then follows directly from Stout's theorem. Also, the distribution of x_i may be taken as continuous since the distribution of $y_n = x_n + u_n$ is (the u_n are independent uniform random variables on $[-1, 1]$ and are independent of the x_n) and because the y_n form a stationary, ergodic martingale difference sequence and satsfy (i) $y_n \in L_2$ if and only if $x_n \in L_2$; (ii) lim sup $(y_1 + \cdots + y_n)/(2n \log \log n)^{\frac{1}{2}}$ is finite a.e. if and only if lim sup $(x_1 + \cdots + x_n)/(2n \log \log n)^{\frac{1}{2}}$ is.

By way of contradiction, suppose that (B) holds while $x_n \notin L_2$. Using ideas from [3], fix $T > 0$ and choose numbers $C > 0$, $D < 0$ so that, writing $J = [D, C]$, the following conditions hold:

$$E(x_n | x_n \in J) = 0 \tag{1}$$

$$E(x_n^2 | x_n \in J) > 8T^2 \tag{2}$$

$$y = p\{x_n \in J\} > \tfrac{1}{2} \tag{3}$$

Define the random variables $U_n = I(\{x_n \in J\})$ and $V_n = I(\{x_n \notin J\})$ where, for $A \in \mathscr{A}$, $I(A)$ is the function from Ω to R taking values 1 when $\omega \in A$ and 0 when $\omega \notin A$. As both are measurable functions of x_n, the ergodic theorem applies to show that

$$(U_{-j} + \cdots + U_0 + \cdots + U_k)/(j + k + 1) \to y$$

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a.e. and

$$(V_{-j} + \cdots + V_0 + \cdots + V_k)/(j + k + 1) \to 1 - y$$

a.e., both as $j + k \to \infty$. $E(x_n^2) = \infty$ means $y < 1$ so that by (3), both $\sum_{l=-j}^{k} U_l$ and $\sum_{l=-j}^{k} V_l \to \infty$ as $j + k \to \infty$; the import of these statements is that $p\{x_n \in J$ for infinitely many $n \in M\} = p\{x_n \notin J$ for infinitely many $n \in M\} = 1$, where M is any infinite set of consecutive integers. Define (random) subsequences $\{\lambda(j)\}$ and $\{\mu(j)\}$ by.

$$\lambda(1) = \min\,(n > 0 : x_n \in J) \qquad \mu(1) = \min\,(n > 0 : x_n \notin J)$$

$$\lambda(j+1) = \min\,(n > \lambda(j) : x_n \in J) \qquad \mu(j+1) = \min\,(n > \mu(j) : x_n \notin J), j > 0$$

$$\lambda(j) = \max\,(n < \lambda(j+1) : x_n \in J) \qquad \mu(j) = \max\,(n < \mu(j+1) : x_n \notin J), j \leqq 0.$$

By the above remarks, the sequences are well-defined, except possibly on a null set; similarly with the sequences $\{Y_j\}$ and $\{Z_j\}$ defined by

$$Y_n = x_{\lambda(n)} \text{ and } Z_n = x_{\mu(n)} \tag{4}$$

Let G_n be the σ-subalgebra of A generated by $\{Y_i, -\infty < i \leqq n\}$, define $t_n^2 = E(Y_n^2 | G_{n-1})$ and, for $m \geqq 1$, $u_m^2 = t_1^2 + \cdots + t_m^2$. Observe that the recurrence times $\{\delta_j\}$, $\delta_n = \lambda(n) - \lambda(n-1)$ are stationary since the x_j are, and accordingly, $\{Y_j\}$ is also stationary, a fact which easily gives the stationarity of $\{t_j^2\}$.

Now Y_n is G_n-measurable and $E(Y_n | G_{n-1}) = 0$, a.e. Because $|Y_n| \leqq C - D$ a.e. and $p\{\sum_{n=1}^{\infty} t^2 = \infty\} > 0$, the conditions for Stout's [4] martingale analogue of the Kolmogoroff log-log law are satisfied. Accordingly,

$$p\{\limsup_{n \to \infty} (Y_1 + \cdots + Y_n)/(2u_n^2 \log\log u_n^2)^{\frac{1}{2}} = 1\} > 0. \tag{5}$$

Using the stationarity of $\{t_j^2\}$, the ergodic theorem and (2) imply that for any $\delta > 0$, $p\{u_n^2 \geqq 8nT^2\} > 1 - \delta$ for all sufficiently large n. Again by the ergodic theorem, $p\{Z_1 + \cdots + Z_{\lambda(n)-n} \geqq 0$ for infinitely many $n > 0\} > 0$. Finally $x_1 + \cdots + x_{\lambda(n)} = (Y_1 + \cdots + Y_n) + (Z_1 + \cdots + Z_{\lambda(n)-n})$ for $n > 0$, by (4). These facts combine with (5) to establish

$$p\{\limsup_{n \to \infty} (x_1 + \cdots + x_{\lambda(n)})/(8nT^2 \log\log n)^{\frac{1}{2}} \geqq 1\} > 0. \tag{6}$$

Finally, by (3) and the ergodic theorem, $p\{\lambda(n) \geqq 2n$ for only finitely many $n > 0\} = 1$ so that from (6)

$$p\{\limsup_{n \to \infty} (x_1 + \cdots + x_{\lambda(n)})/(3\lambda(n)T^2 \log\log \lambda(n))^{\frac{1}{2}} \geqq 1\} > 0, \tag{7}$$

a contradiction to (B) that completes the proof.

REMARK. We conclude by pointing out a related problem that remains open. If the x_n are independent, (B) is equivalent to $(x_1 + \cdots + x_n)/(nT^2)^{\frac{1}{2}}$ converging in law to the unit normal distribution, i.e., the central limit theorem. With the result of the preceding section, the martingle central limit theorem (see [1], e.g.) would be equivalent to the log-log law if the former result entailed $x_n \in L_2$.

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