$B_r^p(F_n)$ HAS NO NONTRIVIAL IDEMPOTENTS

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Abstract

We show that there is no nontrivial idempotent in the reduced group ℓ^p-operator algebra $B_r^p(F_n)$ of the free group F_n on n generators for each positive integer n.

1. Introduction

For a unital Banach algebra A, an *idempotent* in A is an element a with $a^2 = a$. Obviously, both the zero element 0 and the unit element I are idempotents in A. An idempotent a in A is a *nontrivial idempotent* if a is neither 0 nor I. In 1949, Kadison and Kaplansky conjectured that if Γ is a torsion-free discrete group, then the reduced group C^*-algebra $C_r^*(\Gamma)$ has no nontrivial idempotents (or, equivalently, projections). Since then, there have been many achievements around the conjecture, but whether it is true is still unknown.

An important approach to the Kadison–Kaplansky conjecture is through the Baum–Connes conjecture, namely, if Γ is a torsion-free discrete group that satisfies the Baum–Connes conjecture (actually, surjectivity of the assembly map is sufficient), then $C_r^*(\Gamma)$ has no nontrivial idempotents (see [1]). This includes a large class of groups. For example, Higson and Kasparov [10] showed that the Baum–Connes conjecture is true for T-amenable groups, which include amenable groups and free groups, Lafforgue [13] and Mineyev and Yu [15] proved that the Baum–Connes conjecture holds for hyperbolic groups. Hence, if Γ is a torsion-free discrete T-amenable group or hyperbolic group, then $C_r^*(\Gamma)$ contains no nontrivial idempotents. For hyperbolic groups, there is another way to study the Kadison–Kaplansky conjecture, which is due to Puschnigg [20] by using local cyclic homology.

There are many more results when Γ is a free group F_n. It was first shown by Pimsner and Voiculescu [19] that $C_r^*(F_n)$ has no nontrivial idempotents (see also [2, 3]).

For a discrete group Γ, if we consider the left regular representation of Γ on ℓ^p-space $\ell^p(\Gamma)$, then we can define the reduced group ℓ^p-operator algebra $B_r^p(\Gamma)$ of Γ for $p \in [1, \infty]$ (see Definition 2.1). These ℓ^p-operator algebras have been studied intensively; for example, the K-theory of some reduced group ℓ^p-operator algebras [14, 16], the rigidity results of reduced group ℓ^p-operator algebras for $p \neq 2$ [7], and the simplicity and tracial state of some reduced group ℓ^p-operator algebras [7, 9, 17]. Phillips also posed a question concerning the existence of nontrivial idempotents in $B_r^p(F_n)$ [18].

QUESTION 1.1 [18, part of Problem 9.3]. *For $n \in \{2, 3, \ldots\}$ and $p \in [1, \infty)$, does $B_r^p(F_n)$ have nontrivial idempotents?*

We answer this question (see Example 3.10) by using property $(RD)_q$ of groups (see Definition 3.2) introduced by Liao and Yu [14]. The main result of this paper is the following theorem.

THEOREM 1.2 (Theorem 3.8). *Let $p \in [1, \infty]$, q be its dual number and Γ be a discrete group. Assume Γ has property $(RD)_q$. If $C_r^*(\Gamma)$ has no nontrivial idempotents, then both $B_r^p(\Gamma)$ and $B_r^q(\Gamma)$ also have no nontrivial idempotents.*

Since the groups with property (RD) have property $(RD)_q$ for any $q \in [1, 2]$, we have the following corollary.

COROLLARY 1.3 (Corollary 3.9). *Let Γ be a discrete group. Assume Γ has property (RD). If $C_r^*(\Gamma)$ has no nontrivial idempotents, then for any $p \in [1, \infty]$, $B_r^p(\Gamma)$ also has no nontrivial idempotents.*

Haagerup [8] proved that the free group F_n has property (RD). Then combining the result of Pimsner and Voiculescu in [19] stating that $C_r^*(F_n)$ has no nontrivial idempotents with the above corollary, shows that $B_r^p(F_n)$ has no nontrivial idempotents for any $p \in [1, \infty]$. This answers Question 1.1 of Phillips.

Apart from free groups, these results can also be applied to torsion-free hyperbolic groups (see Example 3.11), torsion-free groups with polynomial growth and torsion-free cocompact lattices of $\mathrm{SL}(3, \mathbb{R})$ (see Example 3.12).

2. Preliminaries

In this section, we will recall some relevant concepts. Let $p \in [1, \infty]$ and let Γ be a discrete group. The *group algebra* $\mathbb{C}\Gamma$ is the algebra of all finitely supported functions $f : \Gamma \to \mathbb{C}$ equipped with the multiplication

$$f * g := \sum_{\alpha, \gamma \in \Gamma} (f_\alpha g_\gamma)(\alpha\gamma),$$

for any two elements $f = \sum_{\alpha \in \Gamma} f_\alpha \alpha$ and $g = \sum_{\gamma \in \Gamma} g_\gamma \gamma$ in $\mathbb{C}\Gamma$. The *left regular representation* of Γ on $\ell^p(\Gamma)$, denoted by $\lambda : \Gamma \to \mathcal{B}(\ell^p(\Gamma))$, is defined by

$$(\lambda(\gamma)\xi)(\alpha) := \xi(\gamma^{-1}\alpha),$$

for any $\gamma, \alpha \in \Gamma$ and $\xi \in \ell^p(\Gamma)$. For any $f = \sum_{\alpha \in \Gamma} f_\alpha \alpha$ in $\mathbb{C}\Gamma$, the *reduced norm* of f, denoted by $\|f\|_{\mathcal{B}(\ell^p(\Gamma))}$, is defined to be

$$\|f\|_{\mathcal{B}(\ell^p(\Gamma))} := \left\| \sum_{\alpha \in \Gamma} f_\alpha \lambda(\alpha) \right\|.$$

DEFINITION 2.1. Let $p \in [1, \infty]$ and let Γ be a discrete group.

(1) The *reduced group ℓ^p-operator algebra* of Γ, denoted by $B_r^p(\Gamma)$, is the reduced norm closure of the group algebra $\mathbb{C}\Gamma$.
(2) For any $f = \sum_{\alpha \in \Gamma} f_\alpha \alpha$ in $\mathbb{C}\Gamma$, let $f^* = \sum_{\alpha \in \Gamma} \overline{f_\alpha} \alpha^{-1}$. The *reduced group involution ℓ^p-operator algebra* of Γ, denoted by $B_r^{p,*}(\Gamma)$, is the completion of $\mathbb{C}\Gamma$ with respect to the norm

$$\|f\|_{B_r^{p,*}(\Gamma)} := \max\{\|f\|_{\mathcal{B}(\ell^p(\Gamma))}, \|f^*\|_{\mathcal{B}(\ell^p(\Gamma))}\}.$$

REMARK 2.2. The reduced group ℓ^p-operator algebra $B_r^p(\Gamma)$ is a Banach algebra. Generally, these algebras are not the same for different p (see [6, 14]). The reduced group involution ℓ^p-operator algebra $B_r^{p,*}(\Gamma)$, defined by Liao and Yu in [14], is a Banach $*$-algebra and there exist some groups G such that $B_r^{p,*}(G) \neq B_r^p(G)$ for $p \neq 2$ (see [14]).

REMARK 2.3. For $p \in (1, \infty)$, let q be its dual number (that is, $1/p + 1/q = 1$). The dual space of $\ell^p(\Gamma)$ is $\ell^q(\Gamma)$. If f^* is a bounded operator on $\ell^p(\Gamma)$, then f is a bounded operator on $\ell^q(\Gamma)$ and $\|f^*\|_{\mathcal{B}(\ell^p(\Gamma))} = \|f\|_{\mathcal{B}(\ell^q(\Gamma))}$ for any $f \in \mathbb{C}\Gamma$. As a consequence, $B_r^{p,*}(\Gamma) = B_r^{q,*}(\Gamma)$. Obviously, $B_r^{1,*}(\Gamma) = B_r^1(\Gamma) = \ell^1(\Gamma)$ and $B_r^{\infty,*}(\Gamma) = B_r^\infty(\Gamma) = \ell^1(\Gamma)$. In addition, when $p = 2$, then $B_r^{2,*}(\Gamma) = B_r^2(\Gamma)$ is a C^*-algebra, called the *reduced group C^*-algebra* of Γ, which we shall denote by $C_r^*(\Gamma)$.

For any discrete group Γ, we have the following relation between $B_r^{p,*}(\Gamma)$ and $B_r^p(\Gamma)$.

LEMMA 2.4. *Let $p \in [1, \infty]$, then the identity map on $\mathbb{C}\Gamma$ extends to a contractive, injective homomorphism of Banach algebras,*

$$\iota_{*,p} : B_r^{p,*}(\Gamma) \to B_r^p(\Gamma),$$

so that $B_r^{p,*}(\Gamma)$ is contained in $B_r^p(\Gamma)$. Similarly, $B_r^{p,*}(\Gamma)$ is contained in $B_r^q(\Gamma)$, where q is the dual number of p.

PROOF. Obviously, $\iota_{*,p}$ is a contractive homomorphism and we prove that it is injective. Assume $\iota_{*,p}(T) = 0$ and choose a family of elements $\{f_i\}_{i \in I}$ in $\mathbb{C}\Gamma$ which converges to T in $B_r^{p,*}(\Gamma)$. Since $\max\{\|f_i\|_{\ell^p(\Gamma)}, \|f_i\|_{\ell^q(\Gamma)}\} \leq \|f_i\|_{B_r^{p,*}(\Gamma)}$, it follows that $\{f_i\}_{i \in I}$ converges to an element f in $\ell^p(\Gamma) \cap \ell^q(\Gamma)$ (where q is the dual number of p) and

$T = \lambda(f)$ in $B_r^{p,*}(\Gamma)$, which means

$$T = \sum_{\alpha \in \Gamma} f_\alpha \lambda(\alpha), \tag{2.1}$$

as operators on $\ell^p(\Gamma)$ and $\ell^q(\Gamma)$, where $f = \sum_{\alpha \in \Gamma} f_\alpha \alpha$ (which may be an infinite sum) and λ is the left regular representation of Γ on $\ell^p(\Gamma)$ and $\ell^q(\Gamma)$. By the assumption, $T = 0$ as an operator on $\ell^p(\Gamma)$. Thus, $f = 0$ as a vector in $\ell^p(\Gamma) \cap \ell^q(\Gamma)$ which implies $f_\alpha = 0$ for any $\alpha \in \Gamma$. Then by (2.1), $T = 0$ in $B_r^{p,*}(\Gamma)$, which implies that $\iota_{*,p}$ is an injective homomorphism.

By a similar argument, $B_r^{p,*}(\Gamma)$ is contained in $B_r^q(\Gamma)$. □

The following proposition is due to Liao and Yu. For the convenience of the reader, we include its proof.

PROPOSITION 2.5 [14, Proposition 2.4]. *Let $p \in [1, \infty]$ and let Γ be a discrete group. Then the identity map on $\mathbb{C}\Gamma$ extends to a contractive, injective homomorphism of Banach algebras,*

$$\iota_{p,2} : B_r^{p,*}(\Gamma) \to C_r^*(\Gamma).$$

PROOF. By Remark 2.3, any element $T \in B_r^{p,*}(\Gamma)$ is not only a bounded operator on $\ell^p(\Gamma)$ with norm less than $\|T\|_{B_r^{p,*}(\Gamma)}$, but also a bounded operator on $\ell^q(\Gamma)$ with norm less than $\|T\|_{B_r^{p,*}(\Gamma)}$, where q is the dual number of p. By the Riesz–Thorin interpolation theorem, T is a bounded operator on $\ell^2(\Gamma)$ with norm less than $\|T\|_{B_r^{p,*}(\Gamma)}$, which implies that $\iota_{p,2}$ is a contractive homomorphism.

Now we show that $\iota_{p,2}$ is injective. Assume $\iota_{p,2}(T) = 0$. As in the proof of Lemma 2.4, there exists a vector $f \in \ell^p(\Gamma) \cap \ell^q(\Gamma)$ such that $T = \lambda(f)$ as operators on $\ell^p(\Gamma)$ and $\ell^q(\Gamma)$. By the Riesz–Thorin interpolation theorem again, $T = \lambda(f)$ as operators on $\ell^2(\Gamma)$. By the assumption, $T = 0$ in $C_r^*(\Gamma)$ which implies that $f = 0$, and thus $T = 0$ in $B_r^{p,*}(\Gamma)$ which implies that $\iota_{p,2}$ is an injective homomorphism. □

Recall that an idempotent in a unital Banach space is called nontrivial if it is neither the zero element 0 nor the unit element I.

COROLLARY 2.6. *Let $p \in [1, \infty]$ and Γ be a discrete group. If $C_r^*(\Gamma)$ has no nontrivial idempotents, then $B_r^{p,*}(\Gamma)$ also has no nontrivial idempotents.*

PROOF. If e is a nontrivial idempotent in $B_r^{p,*}(\Gamma)$, then $\iota_{p,2}(e)$ is a nontrivial idempotent in $C_r^*(\Gamma)$. Since $\iota_{p,2}$ is an injective homomorphism, we get a contradiction. □

3. Idempotents and property $(RD)_q$

In this section, we will explore nontrivial idempotents in $B_r^p(\Gamma)$ and in $B_r^q(\Gamma)$ from nontrivial idempotents in $C_r^*(\Gamma)$ by using property $(RD)_q$ of the group Γ, where $q \in [1, 2]$ and p is the dual number of q. First, we give the following key lemma.

LEMMA 3.1. *Let A be a Banach algebra. Then A has no nontrivial idempotents if and only if for some (any) dense subset $F \subseteq A$, the spectrum $sp(a)$ of a in A is connected for any $a \in F$.*

PROOF. First, if A has no nontrivial idempotents, we want to show that the spectrum of any element in A is connected. Assume it is not true, namely, there exists an element $a \in A$, such that $sp(a)$ is disconnected. Then there exist two disjoint open subsets $U, V \subseteq \mathbb{C}$ such that $sp(a) \subseteq U \cup V$, $sp(a) \cap U \neq \emptyset$ and $sp(a) \cap V \neq \emptyset$. Let f be a function on $U \cup V$ such that $f|_U = 0$ and $f|_V = 1$. Then f is a holomorphic function on the neighbourhood of $sp(a)$ and $f^2 = f$. Applying holomorphic functional calculus, we obtain an idempotent $f(a)$ in A and by the spectral mapping theorem, $sp(f(a)) = \{0, 1\}$, which implies that $f(a)$ is a nontrivial idempotent. Thus, we get a contradiction.

For the other direction, assume A has a nontrivial idempotent a. Then $1 - a$ is also a nontrivial idempotent and $sp(a) = sp(1 - a) = \{0, 1\}$. Since F is dense in A, there exists an element $b \in F$ such that $\|b - a\| < \min\{1/(4(2\|a\| + 1)), \|a\|\}$, which implies $\|b^2 - b\| < 1/4$. Thus, $sp(b) \subset \{x \in \mathbb{C} : \mathrm{Re}(x) \neq 1/2\}$. Let χ be a function such that $\chi(x) = 1$ for $\mathrm{Re}(x) > 1/2$ and $\chi(x) = 0$ for $\mathrm{Re}(x) < 1/2$. Since the holomorphic functional calculus by χ is norm continuous in the neighbourhood of a, there exists $\delta < \min\{1/(4(2\|a\| + 1)), \|a\|\}$ such that

$$\|\chi(b') - a\| = \|\chi(b') - \chi(a)\| < 1$$

for some $b' \in F$ with $\|b' - a\| < \delta$. Thus, $sp(b') \cap \{x \in \mathbb{C} : \mathrm{Re}(x) < 1/2\} \neq \emptyset$, as otherwise, $\chi(b') = 1$, which implies that a is invertible. By a similar argument for $1 - a$, we see that $sp(b') \cap \{x \in \mathbb{C} : \mathrm{Re}(x) > 1/2\} \neq \emptyset$. In conclusion, $sp(b')$ is disconnected, contradicting the assumption that the spectrum of any element in F is connected. □

For a discrete group Γ, a *length function* on Γ is a function $l : \Gamma \to [0, \infty)$ such that:

(1) $l(\gamma) = 0$ if and only if γ is the identity element;
(2) $l(\gamma^{-1}) = l(\gamma)$ for any $\gamma \in \Gamma$;
(3) $l(\gamma_1 + \gamma_2) \leq l(\gamma_1) + l(\gamma_2)$ for any $\gamma_1, \gamma_2 \in \Gamma$.

Let e be the identity element of Γ. For any $n \geq 0$, denote by $B_n(e)$ the set of all elements γ in Γ with $l(\gamma) \leq n$.

DEFINITION 3.2. Let $q \in [1, \infty]$ and let Γ be a discrete group. We say that Γ has *property* $(RD)_q$ (with respect to a length function l) if there exists a polynomial $\mathcal{P}$ such that for any function $f \in \mathbb{C}\Gamma$ with support in $B_n(e)$, we have

$$\|f\|_{\mathcal{B}(\ell^q(\Gamma))} \leq \mathcal{P}(n)\|f\|_{\ell^q(\Gamma)}.$$

REMARK 3.3. The property $(RD)_q$ (more generally, defined for locally compact groups) was introduced by Liao and Yu (see [14, Section 4]) to compute the K-theory of $B_r^q(\Gamma)$ and $B_r^{q,*}(\Gamma)$. It is obvious that every group has property $(RD)_1$. When $q \in (2, \infty]$, Liao and Yu proved that a countable discrete group has property $(RD)_q$ with respect to a length function l if and only if it has polynomial growth in l (see [14,

Section 4]). When $p = 2$, the property $(RD)_2$, called *property* (RD), was introduced by Jolissaint [11] and has important applications to the Novikov conjecture (see [4]) and the Baum–Connes conjecture (see [13]).

The following theorem is due to Lafforgue. Two different proofs given by Liao and Yu and Pisier can be found in [14, Theorem 4.4].

THEOREM 3.4 (V. Lafforgue). *If Γ is a discrete group with property $(RD)_q$ for some $q > 1$, then it has property $(RD)_{q'}$ for any $q' \in (1, q)$. In particular, property (RD) implies property $(RD)_q$ for any $q \in (1, 2)$.*

Let B be a unital Banach algebra and A a subalgebra of B containing the unit element of B. We say that A is *stable under the holomorphic functional calculus* in B if for every $a \in A$ and f holomorphic in a neighbourhood of $sp_B(a)$, the element $f(a)$ of B lies in A, where $sp_B(a)$ is the spectrum of a in B. We say that A is a *spectral invariant subalgebra* of B if $sp_A(a) = sp_B(a)$ for any element $a \in A$.

Schweitzer showed that these two notions are equivalent.

LEMMA 3.5 [21, Lemma 1.2]. *Let B be a unital Banach algebra and A a Fréchet subalgebra of B containing the unit element of B. Then A is stable under the holomorphic functional calculus in B if and only if A is spectral invariant in B.*

The significance of property $(RD)_q$ for groups is the following proposition proved by Liao and Yu.

PROPOSITION 3.6 [14, Proposition 4.6]. *Let $p \in [1, \infty]$ and q be its dual number. Let Γ be a discrete group with property $(RD)_q$ with respect to a length function l. Then for sufficiently large $t > 0$, the space $S_q^t(\Gamma)$ of elements $f \in \ell^q(\Gamma)$ such that*

$$\|f\|_{S_q^t} := \|\gamma \to (1 + l(\gamma))^t f(\gamma)\|_{\ell^q(\Gamma)} < \infty$$

is a Banach algebra for the norm $\| \cdot \|_{S_q^t}$. It is contained in $B_r^{p,*}(\Gamma)$, $B_r^p(\Gamma)$ and $B_r^q(\Gamma)$, and stable under holomorphic functional calculus in each of these three algebras.

Combining this proposition with Lemma 3.5 gives the following corollary.

COROLLARY 3.7. *Let p, q be as above and let Γ be a discrete group with property $(RD)_q$. Then for any $f \in \mathbb{C}\Gamma$,*

$$sp_{B_r^{p,*}(\Gamma)}(f) = sp_{B_r^p(\Gamma)}(f) = sp_{B_r^q(\Gamma)}(f) = sp_{S_q^t(\Gamma)}(f).$$

Now, we are ready to state and prove our main theorem.

THEOREM 3.8. *Let $p \in [1, \infty]$, q be its dual number and Γ be a discrete group. Assume Γ has property $(RD)_q$. If $C_r^*(\Gamma)$ has no nontrivial idempotents, then both $B_r^p(\Gamma)$ and $B_r^q(\Gamma)$ also have no nontrivial idempotents.*

PROOF. By Corollary 2.6, $B_r^{p,*}(\Gamma)$ has no nontrivial idempotents. Thus by Lemma 3.1, the spectrum $sp_{B_r^{p,*}(\Gamma)}(f)$ is connected for any $f \in \mathbb{C}\Gamma$. If Γ has property $(RD)_q$, then by

Corollary 3.7, $sp_{B_r^p(\Gamma)}(f) = sp_{B_r^q(\Gamma)}(f) = sp_{B_r^{p,*}(\Gamma)}(f)$ are all connected for any $f \in \mathbb{C}\Gamma$. Thus, by Lemma 3.1 again, both $B_r^p(\Gamma)$ and $B_r^q(\Gamma)$ have no nontrivial idempotents. □

Combining this theorem with Theorem 3.4 and the fact that every group has property $(RD)_1$ gives the following corollary.

COROLLARY 3.9. *Let Γ be a discrete group with property (RD). If $C_r^*(\Gamma)$ has no nontrivial idempotents, then for any $p \in [1, \infty]$, $B_r^p(\Gamma)$ also has no nontrivial idempotents.*

EXAMPLE 3.10. We apply the main theorem to free groups F_n for any positive integer n. Haagerup [8] proved that the free group F_n has property (RD) and Pimsner and Voiculescu [19] proved that the reduced group C^*-algebra $C_r^*(F_n)$ has no nontrivial idempotents. Thus by Corollary 3.9, $B_r^p(F_n)$ has no nontrivial idempotents for any $p \in [1, \infty]$. This answers Question 1.1 raised by Phillips.

EXAMPLE 3.11. In this example, we consider a torsion-free hyperbolic group Γ. Jolissaint [11] and de la Harpe [5] proved that hyperbolic groups have property (RD). There are at least two different ways to prove that $C_r^*(\Gamma)$ has no nontrivial idempotents. One is as a corollary of the Baum–Connes conjecture for hyperbolic groups (see Lafforgue [13] and Mineyev and Yu [15]). Another, due to Puschnigg, uses local cyclic homology (see [20]). Thus by Corollary 3.9, $B_r^p(\Gamma)$ has no nontrivial idempotents for any $p \in [1, \infty]$.

EXAMPLE 3.12. For a torsion-free discrete group Γ, if Γ satisfies the Baum–Connes conjecture (actually, surjectivity of the assembly map is sufficient), then the reduced group C^*-algebra $C_r^*(\Gamma)$ has no nontrivial idempotents (see [1]). Thus by Corollary 3.9, for any $p \in [1, \infty]$, $B_r^p(\Gamma)$ has no nontrivial idempotents for every torsion-free discrete group Γ which has property (RD) and satisfies the Baum–Connes conjecture. Apart from hyperbolic groups, such groups Γ can also be finitely generated, torsion-free groups with polynomial growth (see [10, 11]) and torsion-free cocompact lattices of $SL(3, \mathbb{R})$ (see [12, 13]).

References

[1] P. Baum, A. Connes and N. Higson, 'Classifying space for proper actions and K-theory of group C^*-algebras', in: C^*-*Algebras: 1943–1993*, Contemporary Mathematics, 167 (ed. R. S. Doran) (American Mathematical Society, Providence, RI, 1994), 240–291.

[2] J. M. Cohen and A. Figà-Talamanca, 'Idempotents in the reduced C^*-algebra of a free group', *Proc. Amer. Math. Soc.* **103**(3) (1988), 779–782.

[3] A. Connes, 'Noncommutative differential geometry', *Publ. Math. Inst. Hautes Études Sci.* **62** (1985), 41–144.

[4] A. Connes and H. Moscovici, 'Cyclic cohomology, the Novikov conjecture and hyperbolic groups', *Topology* **29**(3) (1990), 345–388.

[5] P. de la Harpe, 'Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint', *C. R. Math. Acad. Sci. Paris* **307**(14) (1988), 771–774.

[6] E. Gardella and H. Thiel, 'Representations of p-convolution algebras on L_q-spaces', *Trans. Amer. Math. Soc.* **371**(3) (2019), 2207–2236.

[7]   E. Gardella and H. Thiel, 'Isomorphisms of algebras of convolution operators', *Ann. Sci. Éc. Norm. Supér. (4)*, to appear.

[8]   U. Haagerup, 'An example of a nonnuclear C^*-algebra, which has the metric approximation property', *Invent. Math.* **50**(3) (1978), 279–293.

[9]   S. Hejazian and S. Pooya, 'Simple reduced L_p-operator crossed products with unique trace', *J. Operator Theory* **74**(1) (2015), 133–147.

[10]   N. Higson and G. Kasparov. 'E-theory and KK-theory for groups which act properly and isometrically on Hilbert space', *Invent. Math.* **144**(1) (2001), 23–74.

[11]   P. Jolissaint, 'Rapidly decreasing functions in reduced C^*-algebras of groups', *Trans. Amer. Math. Soc.* **317**(1) (1990), 167–196.

[12]   V. Lafforgue, 'A proof of property (RD) for cocompact lattices of $SL(3, \mathbb{R})$ and $SL(3, \mathbb{C})$', *J. Lie Theory* **10**(2) (2000), 255–267.

[13]   V. Lafforgue, 'K-théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes', *Invent. Math.* **149**(1) (2002), 1–95.

[14]   B. Liao and G. Yu, 'K-theory of group Banach algebras and Banach property RD', Preprint, 2017, arXiv:1708.01982.

[15]   I. Mineyev and G. Yu, 'The Baum–Connes conjecture for hyperbolic groups', *Invent. Math.* **149**(1) (2002), 97–122.

[16]   N. C. Phillips, 'Crossed products of L_p operator algebras and the K-theory of Cuntz algebras on L_p spaces', Preprint, 2013, arXiv:1309.6406.

[17]   N. C. Phillips, 'Simplicity of reduced group Banach algebras', Preprint, 2019, arXiv:1909.11278.

[18]   N. C. Phillips, 'Open problems related to operator algebras on L_p spaces'. https://tinyurl.com/phillips-pdf.

[19]   M. Pimsner and D. Voiculescu, 'K-groups of reduced crossed products by free groups', *J. Operator Theory* **8**(1) (1982), 131–156.

[20]   M. Puschnigg, 'The Kadison–Kaplansky conjecture for word-hyperbolic groups', *Invent. Math.* **149**(1) (2002), 153–194.

[21]   L. B. Schweitzer, 'A short proof that $M_n(A)$ is local if A is local and Fréchet', *Internat. J. Math.* **3**(4) (1992), 581–589.

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