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PROOF OF A CONJECTURE OF GOULDEN AND JACKSON

ANDREI OKOUNKOV

ABSTRACT. We prove an integration formula involving Jack polynomials conjectured by I. P. Goulden and D. M. Jackson in connection with enumeration of maps in surfaces.

In this note we prove an integration formula involving Jack symmetric polynomials which was conjectured by I. P. Goulden and D. M. Jackson in [GJ], Conjecture 3.4, in connection with enumeration of maps in surfaces. We refer the reader to [GJ] and [GHJ] for a discussion of the role of that formula.

Our proof is based on an integration formula of C. F. Dunkl [D] and some results of [OO]. Instead of Dunkl's formula one could probably use related formulas of K. Kadell [K] or I. Cherednik [C] to obtain another proof.

Let $P_\lambda(x;\alpha)$ denote the Jack symmetric polynomial with parameter $\alpha > 0$ (see [M2], section VI.10). The normalization of the polynomial $P_\lambda(x;\alpha)$ is chosen so that the coefficient of the monomial $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ equals 1. We shall also use the integral form $J_\lambda(x;\alpha)$ of the Jack polynomial which is normalized so that the coefficient of the monomial $x_1x_2\ldots x_{|\lambda|}$ equals $|\lambda|!$. If $|\lambda|$ is even then let

$$\left[|y|^{|\lambda|}\right] J_\lambda(y;\alpha)$$

denote the coefficient of the polynomial

$$|x|^{|\lambda|} := \left(\sum x_i^2\right)^{|\lambda|/2}$$

in the power-sum expansion of the polynomial $J_\lambda(x;\alpha)$. If $|\lambda|$ is odd, we set by definition $[|y|^{|\lambda|}]\, J_\lambda(y;\alpha) = 0$.

Set

$$c(n,\alpha) := \int_{\mathbb{R}^n} |V(x)|^{2/\alpha}\, e^{-|x|^2/2}\, dx\,, \quad n = 1,2,\ldots\,,$$

where

$$V(x) := \prod_{i<j}(x_i - x_j)$$

is the Vandermonde determinant. The exact value of the constant $c(n,\alpha)$ was given in [M1], formula 4.1, but we shall not need it.

The following formula, conjectured by I. P. Goulden and D. M. Jackson, is our main result:

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THEOREM.

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} J_\lambda(x;\alpha)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2\alpha}\,dx = J_\lambda(1,\dots,1;\alpha)\Big(\big[|y|^{|\lambda|}\big]J_\lambda(y;\alpha)\Big)\,.$$

PROOF. The following integral is a specialization of the integration formula due to Dunkl (see [D], Proposition 2.1)

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} K(x,y)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2}\,dx = e^{|y|^2/2}\,. \tag{1}$$

Here the integral kernel $K(x,y)$ is the solution of the difference-differential equations

$$D_i K(x,y) = y_i K(x,y)\,, \quad i = 1,\dots,n \tag{2}$$

normalized by

$$K(0,y) = 1\,.$$

The operators D_i in (2) are the commuting Dunkl operators

$$D_i f := \frac{\partial}{\partial x_i} f + \frac{1}{\alpha}\sum_{j\neq i}\frac{f(x) - f(s_{ij}\cdot x)}{x_i - x_j}\,,$$

where s_{ij} is the permutation of i-th and j-th coordinate. The function $K(x,y)$ has the following symmetry properties

$$K(x,y) = K(y,x),$$
$$K(s\cdot x, s\cdot y) = K(x,y), \quad s \in S(n).$$

Now consider the Bessel function

$$F(x,y) := \frac{1}{n!}\sum_{s\in S(n)} K(s\cdot x, y)\,.$$

From (1) we clearly have

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} F(x,y)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2}\,dx = e^{|y|^2/2}\,. \tag{3}$$

The function $F(x,y)$ is symmetric in both x and y and satisfies the differential equations (the difference part disappears because of the symmetry of $F(x,y)$)

$$f(D_1,\dots,D_n)\,F(x,y) = f(y_1,\dots,y_n)F(x,y)$$

for any *symmetric* polynomial f. For example, taking the polynomial

$$f(D_1,\dots,D_n) = D_1^2 + \cdots + D_n^2$$

one obtains

$$\Big(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha}\sum_{i\neq j}\frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}\Big)F(x,y) = |y|^2 F(x,y)\,.$$

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It is easy to see from differential equations for Jack polynomials that the function $F(x,y)$ can be obtained by the following limit transition from Jack polynomials with parameter α. By symmetry, we can assume that

$$y_1 \geq \cdots \geq y_n\,,$$

then

$$F(x,y) = \lim_{M\to\infty} \frac{P_{[My]}\big(1+x_1/M,\ldots,1+x_n/M;\alpha\big)}{P_{[My]}(1,\ldots,1;\alpha)}\,, \tag{4}$$

where $[My]$ stands for the integer part of the vector My.

The following formula for the limit (4) was obtained in [OO], formula 4.2.

$$F(x,y) = \sum_{\lambda} \frac{P_\lambda(x;\alpha)Q_\lambda(y;\alpha)}{(n/\alpha)_\lambda}\,, \tag{5}$$

where the sum is over all partitions with $\leq n$ parts,

$$Q_\lambda(y;\alpha) := \frac{1}{(P_\lambda, P_\lambda)} P_\lambda(y;\alpha), \tag{6}$$

and the generalized shifted factorial $(\,\cdot\,)_\lambda$ is defined by

$$(u)_\lambda := \prod_{(i,j)\in\lambda} \big(u + (j-1) - (i-1)/\alpha\big)\,.$$

The scalar product used in (6) is defined on the power-sum symmetric functions by

$$(p_\lambda, p_\mu) := \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda\mu}\,. \tag{7}$$

Now from (3) and (5) we have for

$$d = 0, 1, 2, \ldots$$

the following relation

$$\frac{1}{c(n,\alpha)} \sum_{|\lambda|=2d} \frac{Q_\lambda(y;\alpha)}{(n/\alpha)_\lambda} \int_{\mathbb{R}^n} P_\lambda(x;\alpha)\,|V(x)|^{2/\alpha}\, e^{-|x|^2/2}\, dx = \frac{|y|^{2d}}{2^d d!}\,.$$

Equivalently,

$$\frac{1}{c(n,\alpha)} \int_{\mathbb{R}^n} P_\lambda(x;\alpha)\,|V(x)|^{2/\alpha}\, e^{-|x|^2/2}\, dx = (n/\alpha)_\lambda \frac{\big(P_\lambda(y;\alpha), |y|^{|\lambda|}\big)}{2^{|\lambda|/2}(|\lambda|/2)!}\,. \tag{8}$$

By (7) we have

$$(|y|^{|\lambda|}, |y|^{|\lambda|}) = (2\alpha)^{|\lambda|/2}(|\lambda|/2)!\,. \tag{9}$$

Recall that J_λ is a scalar multiple of P_λ and (see, for example, [M2], formula 10.25)

$$J_\lambda(1,\ldots,1;\alpha) = a^{|\lambda|}(n/\alpha)_\lambda\,. \tag{10}$$

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Using (9) and (10) the formula (8) can be rewritten as follows

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} J_\lambda(x;\alpha)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2}\,dx = \alpha^{-|\lambda|/2} J_\lambda(1,\dots,1;\alpha)\Big(\big[|y|^{|\lambda|}\big]\,J_\lambda(y;\alpha)\Big). \tag{11}$$

provided $|\lambda|$ is even. (The integral clearly vanishes if $|\lambda|$ is odd.) In the above formula the factor

$$\Big(\big[|y|^{|\lambda|}\big]\,J_\lambda(y;\alpha)\Big)$$

stands for the coefficient of $|y|^{|\lambda|}$ in the power-sum expansion of the polynomial $J_\lambda(y;\alpha)$.

Now use the following change of variables

$$x = \frac{x'}{\sqrt{a}}$$

to obtain the statement of the theorem. ∎

REMARK. The following formula of C. Dunkl (see [D], Theorem 3.2) is a generalization of the formula (1)

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} K(x,y)\,K(x,z)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2}\,dx = e^{|y|^2/2+|z|^2/2}K(y,z)\,.$$

Using it and the very same argument based on the expansion (5) one can generalize the above theorem to give a formula for the integral

$$\frac{1}{c(n,\alpha)}\int_{\mathbb{R}^n} J_\lambda(x;a)\,J_\mu(x;a)\,|V(x)|^{2/\alpha}\,e^{-|x|^2/2}\,dx\,,$$

where λ and μ are two arbitrary partitions.

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Department of Mathematics
University of Chicago
5734 University Avenue
Chicago, IL
USA