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STRONG SKEW COMMUTATIVITY PRESERVING MAPS ON RINGS

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Abstract

Let $\mathcal{A}$ be a unital ring with involution. Assume that $\mathcal{A}$ contains a nontrivial symmetric idempotent and $\phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear surjective map. We prove that if ϕ preserves strong skew commutativity, then $\phi(A) = ZA + f(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}_s(\mathcal{A})$ satisfies $Z^2 = I$ and f is a map from $\mathcal{A}$ into $\mathcal{Z}_s(\mathcal{A})$. Related results concerning nonlinear strong skew commutativity preserving maps on von Neumann algebras are given.

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1. Introduction

Let $\mathcal{A}$ be a ring with involution. For $A, B \in \mathcal{A}$, we denote by $[A, B]_* = AB - BA^*$ the skew Lie product of A and B. The skew Lie product was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [12–15]) and in the problem of characterizing ideals (see, for example, [3, 10]). Recall that a map ϕ from $\mathcal{A}$ into itself is called a skew commutativity preserving map if $[\phi(A), \phi(B)]_* = 0$ whenever $[A, B]_* = 0$ for all $A, B \in \mathcal{A}$. There have been a number of papers on the study of skew commutativity preserving maps (see [4, 5] and references therein).

More especially, a map ϕ is called a strong skew commutativity preserving map on $\mathcal{A}$ if $[\phi(A), \phi(B)]_* = [A, B]_*$ for all $A, B \in \mathcal{A}$. Clearly, strong skew commutativity preserving maps must be skew commutativity preserving maps, but the inverse is not true generally. In [6], Cui and Park characterized nonlinear surjective strong skew commutativity preserving maps ϕ on factor von Neumann algebras $\mathcal{A}$, that is, ϕ has the form $\phi(A) = \psi(A) + h(A)I$ for all $A \in \mathcal{A}$, where $\psi : \mathcal{A} \to \mathcal{A}$ is a linear bijective

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map satisfying $\psi(A)\psi(B) - \psi(B)\varphi(A)^* = AB - BA^*$ for all $A, B \in \mathcal{A}$ and h is a real functional on $\mathcal{A}$ with $h(0) = 0$. In [11], Qi and Hou generalized Cui and Park's results to prime rings with involution and proved that every nonlinear surjective strong skew commutativity preserving map ϕ on a unital prime ring $\mathcal{A}$ with involution has the form $\phi(A) = \lambda A + f(A)$ for all $A \in \mathcal{A}$, where $\lambda \in \{-1, 1\}$ and f is a map from $\mathcal{A}$ into $\mathcal{Z}_s(\mathcal{A})$ (the symmetric center of $\mathcal{A}$). Qi and Hou in [11] also characterized the nonlinear surjective strong skew commutativity preserving maps of von Neumann algebras without central summands of type I_1. In this article, we continue this line of investigation and characterize nonlinear strong skew commutativity preserving maps on general rings with involution.

This article is organized as follows. In Section 2, we deal with the case that $\mathcal{A}$ is a ring with the unit I and involution having a nontrivial symmetric idempotent P, that is, $P^* = P$ and $P^2 = P$. We set $\mathcal{Z}_s(\mathcal{A})$, the symmetric central element of $\mathcal{A}$, that is, $\mathcal{Z}_s(\mathcal{A}) = \{Z \in \mathcal{A} : Z^* = Z \text{ and } ZA = AZ \text{ for all } A \in \mathcal{A}\}$. Assume that $\phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear surjective strong skew commutativity preserving map. We show that $\phi(A) = ZA + f(A)$ for all $A \in \mathcal{A}$, where $Z \in \mathcal{Z}_s(\mathcal{A})$ satisfies $Z^2 = I$ and f is a map from $\mathcal{A}$ into $\mathcal{Z}_s(\mathcal{M})$. In Section 3, we give several applications of the results in the above section for some operator algebras. Particularly, we characterize nonlinear surjective strong skew commutativity preserving maps on factor von Neumann algebras and von Neumann algebras without central summands of type I_1.

2. Characterization of strong skew commutativity preserving maps

In this section, we discuss the nonlinear surjective strong skew commutativity preserving maps on unital rings with involution.

THEOREM 2.1. *Let $\mathcal{A}$ be a ring with the unit I and involution. Assume that $\mathcal{A}$ contains a nontrivial symmetric idempotent P such that $X\mathcal{A}P = 0$ implies $X = 0$ and $X\mathcal{A}(I - P) = 0$ implies $X = 0$ and $\phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear surjective strong skew commutativity preserving map, that is, ϕ satisfies*

$$[\phi(A), \phi(B)]_* = [A, B]_*$$

for all $A, B \in \mathcal{A}$. Then there exist an element $Z \in \mathcal{Z}_s(\mathcal{A})$ with $Z^2 = I$ and a map $f : \mathcal{A} \to \mathcal{Z}_s(\mathcal{A})$ such that $\phi(A) = ZA + f(A)$ for all $A \in \mathcal{A}$.

PROOF. We shall organize the proof of Theorem 2.1 in a series of claims.

Claim 1. Let $X \in \mathcal{A}$. Then $P\mathcal{A}X = 0$ implies $X = 0$ and $(I - P)\mathcal{A}X = 0$ implies $X = 0$.

If $P\mathcal{A}X = 0$, then we have $X^*\mathcal{A}^*P = 0$, that is, $X^*\mathcal{A}P = 0$. It follows from the property of P that $X^* = 0$. So, $X = 0$.

Similarly, one can prove that $(I - P)\mathcal{A}X = 0$ implies $X = 0$.

Claim 2. Let $A \in \mathcal{A}$. Then $PAPXP = PXPA^*P$ for all $X \in \mathcal{A}$ implies $PAP = PZ$, where $Z \in \mathcal{Z}_s(\mathcal{A})$.

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Taking $X = P$, we get $PAP = PA^*P$. It follows that $PAPXP = PXPAP$. Then we let

$$\mathcal{A}PAP\mathcal{A} = \Big\{\sum_{j=1}^{n} X_jPAPY_j : X_j, Y_j \in \mathcal{A}, n \in \mathrm{N}\Big\}$$

and

$$\mathcal{A}P\mathcal{A} = \Big\{\sum_{j=1}^{n} X_jPY_j : X_j, Y_j \in \mathcal{A}, n \in \mathrm{N}\Big\}.$$

It is easy to see that $\mathcal{A}PAP\mathcal{A}$ is an ideal of $\mathcal{A}P\mathcal{A}$. We can define an additive map $\psi : \mathcal{A}P\mathcal{A} \to \mathcal{A}PAP\mathcal{A}$ by $\psi(\sum_{j=1}^{n} X_jPY_j) = \sum_{j=1}^{n} X_jPAPY_j$. The map is well defined. Indeed, if $\sum_{j=1}^{n} X_jPY_j = 0$, we have $\sum_{j=1}^{n} X_jPY_jYPAP = 0$ for all $Y \in \mathcal{A}$. It follows from the fact that $PAPXP = PXPAP$ that $\sum_{j=1}^{n} X_jPAPY_jYP = 0$. From the property of P, we obtain $\sum_{j=1}^{n} X_jPAPY_j = 0$, which means that ψ is well defined. Moreover, the following equations hold

$$\begin{aligned}
\psi\Big(\sum_{i=1}^{n} X_iPY_i \sum_{j=1}^{m} X'_jPY'_j\Big) &= \psi\Big(\sum_{j=1}^{m}\Big(\sum_{i=1}^{n} X_iPY_i\Big)X'_jPY'_j\Big)\\
&= \sum_{j=1}^{m}\Big(\sum_{i=1}^{n} X_iPY_i\Big)X'_jPAPY'_j\\
&= \sum_{i=1}^{n} X_iPY_i \sum_{j=1}^{m} X'_jPAPY'_j\\
&= \sum_{i=1}^{n} X_iPY_i\psi\Big(\sum_{j=1}^{m} X'_jPY'_j\Big).
\end{aligned}$$

This implies that ψ is a left $\mathcal{A}P\mathcal{A}$-module homomorphism, that is, $\psi \in \mathrm{Hom}_{\mathcal{A}P\mathcal{A}}(\mathcal{A}P\mathcal{A}, \mathcal{A}PAP\mathcal{A})$. By [7, Ch. 4, Theorem 4.9], there exists an element $Z \in \mathcal{A}PAP\mathcal{A}$ such that $\psi(X) = XZ$ for all $X \in \mathcal{A}P\mathcal{A}$. In particular,

$$PAP = \psi(P) = PZ.$$

Next we shall show that $Z \in \mathcal{Z}_s(\mathcal{A})$. In fact, for any $Y \in \mathcal{A}$, $X = \sum_{i=1}^{n} X_iPY_i \in \mathcal{A}P\mathcal{A}$,

$$\psi(XY) = \psi\Big(\sum_{i=1}^{n} X_iPY_iY\Big) = \sum_{j=1}^{n} X_iPAPY_iY = \psi(X)Y.$$

It follows that $XYZ = XZY$, that is, $\mathcal{A}P\mathcal{A}(YZ - ZY) = 0$. Thus, by Claim 1, we get $YZ = ZY$ for all $Y \in \mathcal{A}$. Furthermore, since $PAP = PA^*P$, we get $PZ = PZ^*$ and so $P\mathcal{A}(Z - Z^*) = 0$. It follows from Claim 1 that $Z = Z^*$. Hence, $Z \in \mathcal{Z}_s(\mathcal{A})$, as desired.

In what follows, let $P_1 = P$ and $P_2 = I - P_1$. We denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{M}_{22}$ and each element $A \in \mathcal{A}$ can be written as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{A}_{ij}$, $i, j = 1, 2$.

Claim 3. $\phi(P_1) = ZP_1 + Z_{P_1}$, where $Z, Z_{P_1} \in \mathcal{Z}_s(\mathcal{A})$.

For any $X \in \mathcal{A}$, since $[P_1, [P_1, [P_1, X]_*]_*]_* = [P_1, X]_*$,

$$[P_1, [P_1, [\phi(P_1), \phi(X)]_*]_*]_* = [\phi(P_1), \phi(X)]_*.$$

It follows from the surjectivity of ϕ that

$$[P_1, [P_1, [\phi(P_1), X]_*]_*]_* = [\phi(P_1), X]_*. \tag{2.1}$$

Taking $X = A_{12} \in \mathcal{A}_{12}$ in (2.1), we get $A_{12}\phi(P_1)^* P_1 = P_2\phi(P_1)A_{12}$, which implies that $P_2\phi(P_1)A_{12} = 0$ and $A_{12}\phi(P_1)^* P_1 = 0$. By Claim 1 and the property of P, $P_2\phi(P_1)P_1 = P_2\phi(P_1)^* P_1 = 0$, that is, $P_2\phi(P_1)P_1 = P_1\phi(P_1)P_2 = 0$.

Moreover, taking $X = A_{11} \in \mathcal{A}_{11}$ in (2.1), we obtain $P_1\phi(P_1)A_{11} = A_{11}\phi(P_1)^* P_1$. By Claim 2, there exists an element $Z_1 \in \mathcal{Z}_s(\mathcal{A})$ such that $P_1\phi(A_{11})P_1 = Z_1 P_1$. With a similar argument, we have $P_2\phi(P_1)P_2 = Z_2 P_2$, where $Z_2 \in \mathcal{Z}_s(\mathcal{A})$. Hence,

$$\phi(P_1) = Z_1 P_1 + Z_2 P_2 = (Z_1 - Z_2)P_1 + Z_2.$$

We denote $Z = Z_1 - Z_2$ and $Z_{P_1} = Z_2$. Then $\phi(P_1) = ZP_1 + Z_{P_1}$.

Finally, we need to prove that $Z \neq 0$. On the contrary, if $Z = 0$, then $\phi(P_1) = Z_2 \in \mathcal{Z}_s(\mathcal{A})$. It follows that $[P_1, X]_* = [\phi(P_1), \phi(X)]_* = 0$ for all $X \in \mathcal{A}$. This leads to $P_1 \in \mathcal{Z}_s(\mathcal{A})$. So, $(I - P_1)\mathcal{A}P_1 = 0$. From the property of P_1, we get $I - P_1 = 0$, which is impossible as P_1 is nontrivial.

In the sequel, Z is the symmetric central element in Claim 3.

Claim 4. Let $A_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$. Then:

(a) $\phi(X) = P_1$ implies $X_{ij} = 0$;
(b) $\phi(X) = A_{ij}$ implies $X_{ij} = ZA_{ij}$ and $X_{ji} = 0$.

It follows from Claim 3 that

$$[P_1, X]_* = [\phi(P_1), \phi(X)]_* = [ZP_1, P_1]_* = 0.$$

Then $X_{12} = X_{21} = 0$. So, (a) is true.

Similarly, by Claim 3,

$$[P_1, X]_* = [\phi(P_1), \phi(X)]_* = [ZP_1, A_{12}]_* = ZA_{12},$$

which implies that $X_{12} = ZA_{12}$ and $X_{21} = 0$. If $\phi(X) = A_{21}$, we can obtain $X_{21} = ZA_{21}$ and $X_{12} = 0$ in the same way. So, (b) is true.

Claim 5. $\phi(A_{ii}) = ZA_{ii}$ for all $A_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$.

For any $A_{11} \in \mathcal{A}_{11}$ and $A_{12} \in \mathcal{A}_{12}$, by the surjectivity of ϕ, there exists an element $X \in \mathcal{A}$ such that $\phi(X) = A_{12}$. Then $[X, A_{11}]_* = [\phi(X), \phi(A_{11})]_* = [A_{12}, \phi(A_{11})]_*$, that is,

$$XA_{11} - A_{11}X^* = A_{12}\phi(A_{11}) - \phi(A_{11})A_{12}^*. \tag{2.2}$$

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Multiplying (2.2) by P_1 from the left-hand side and P_2 from the right-hand side, we arrive at $-A_{11}X^*P_2 = A_{12}\phi(A_{11})P_2$. By Claim 4(b), this is equivalent to

$$A_{12}\phi(A_{11})P_2 = 0.$$

It follows from Claim 1 that $P_2\phi(A_{11})P_2 = 0$. Similarly, we have $[A_{11}, X]_* = [\phi(A_{11}), \phi(X)]_* = [\phi(A_{11}), A_{12}]_*$, that is,

$$A_{11}X - XA_{11}^* = \phi(A_{11})A_{12} - A_{12}\phi(A_{11})^*. \tag{2.3}$$

Multiplying (2.3) by P_2 from both sides, we obtain $P_2\phi(A_{11})A_{12} = 0$, which implies $P_2\phi(A_{11})P_1 = 0$. At the same time, multiplying (2.3) by P_1 from the left-hand side and P_2 from the right-hand side, and combining the fact that $P_2\phi(A_{11})P_2 = 0$, we arrive at $A_{11}XP_2 = P_1\phi(A_{11})A_{12}$. Using Claim 4(b) again, we have $ZA_{11}A_{12} = P_1\phi(A_{11})A_{12}$. Therefore, $P_1\phi(A_{11})P_1 = ZA_{11}$ holds for all $A_{11} \in \mathcal{A}_{11}$.

For any $A_{21} \in \mathcal{A}_{21}$, by the surjectivity of ϕ, there exists an element $X \in \mathcal{A}$ such that $\phi(X) = A_{21}$. Then $[X, A_{11}]_* = [\phi(X), \phi(A_{11})]_* = [A_{21}, \phi(A_{11})]_*$, that is,

$$XA_{11} - A_{11}X^* = A_{21}\phi(A_{11}) - \phi(A_{11})A_{21}^*. \tag{2.4}$$

Multiplying (2.4) by P_2 from both sides, we get $A_{21}\phi(A_{11})P_2 = 0$. It follows from Claim 1 that $P_1\phi(A_{11})P_2 = 0$. Hence, $\phi(A_{11}) = ZA_{11}$.

Similarly, $\phi(A_{22}) = ZA_{22}$ can be obtained.

Claim 6. $Z^2 = I$ and $\phi(A_{ij}) = ZA_{ij}$ for all $A_{ij} \in \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$.

For any $B_{12} \in \mathcal{A}_{12}$ and $B_{21} \in \mathcal{A}_{21}$, by the surjectivity of ϕ, there exist $U, V, W \in \mathcal{A}$ such that $\phi(U) = P_1$, $\phi(V) = B_{12}$ and $\phi(W) = B_{21}$. Then we have the following three equations.

$$[U, A_{12}]_* = [\phi(U), \phi(A_{12})]_* = [P_1, \phi(A_{12})]_*,$$
$$[V, A_{12}]_* = [\phi(V), \phi(A_{12})]_* = [B_{12}, \phi(A_{12})]_*$$

and

$$[W, A_{12}]_* = [\phi(W), \phi(A_{12})]_* = [B_{21}, \phi(A_{12})]_*,$$

which are equivalent to

$$UA_{12} - A_{12}U^* = P_1\phi(A_{12}) - \phi(A_{12})P_1, \tag{2.5}$$
$$VA_{12} - A_{12}V^* = B_{12}\phi(A_{12}) - \phi(A_{12})B_{12}^* \tag{2.6}$$

and

$$WA_{12} - A_{12}W^* = B_{21}\phi(A_{12}) - \phi(A_{12})B_{21}^* \tag{2.7}$$

for all $A_{12} \in \mathcal{A}_{12}$. Multiplying (2.5)–(2.7) by P_2 from the left-hand side and P_1 from the right-hand side, respectively, we obtain $P_2\phi(A_{12})P_1 = 0$, $P_2\phi(A_{12})B_{12}^* = 0$ and $B_{21}\phi(A_{12})P_1 = 0$. It follows from Claim 1 and the property of P that $P_1\phi(A_{12})P_1 = 0$ and $P_2\phi(A_{12})P_2 = 0$. So, $\phi(A_{12}) = P_1\phi(A_{12})P_2$. Moreover, by Claim 3,

$$[P_1, A_{12}]_* = [\phi(P_1), \phi(A_{12})]_* = [ZP_1, \phi(A_{12})]_*.$$

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This leads to $A_{12} = Z\phi(A_{12})$. Similarly, we obtain $A_{21} = Z\phi(A_{21})$.

Next we shall show that $Z^2 = I$. In fact, for any $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$, we get $[A_{12}, A_{21}]_* = [\phi(A_{12}), \phi(A_{21})]_*$, that is, $A_{12}A_{21} = \phi(A_{12})\phi(A_{21})$. It follows that

$$Z^2 A_{12}A_{21} = Z\phi(A_{12})Z\phi(A_{21}) = A_{12}A_{21},$$

that is, $(Z^2 - I)A_{12}A_{21} = 0$. Fixing A_{12}, from the property of P, we get $(Z^2 - I)A_{12} = 0$. Then, using the property of P again, we have $(Z^2 - I)P = 0$ and so $(Z^2 - I)\mathcal{A}P = 0$. Hence, $Z^2 = I$, $\phi(A_{12}) = ZA_{12}$ and $\phi(A_{21}) = ZA_{21}$.

Claim 7. ϕ is almost additive, that is, for any $A, B \in \mathcal{A}$, $\phi(A + B) - \phi(A) - \phi(B) = Z_{A,B} \in \mathcal{Z}_s(\mathcal{A})$.

For any $X \in \mathcal{A}$, it is easy to check that

$$\begin{aligned} [\phi(A + B) - \phi(A) - \phi(B), \phi(X)]_* &= [\phi(A + B), \phi(X)]_* - [\phi(A), \phi(X)]_* - [\phi(B), \phi(X)]_* \\ &= [A + B, X]_* - [A, X]_* - [B, X]_* = 0. \end{aligned}$$

Since ϕ is surjective, we obtain a symmetric central element $Z_{A,B} \in \mathcal{Z}_s(\mathcal{A})$ such that $\phi(A + B) - \phi(A) - \phi(B) = Z_{A,B}$.

Now, by Claims 5–7,

$$\begin{aligned} \phi(A) &= \phi(A_{11} + A_{12} + A_{21} + A_{22}) \\ &= \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22}) + Z_A \\ &= Z(A_{11} + A_{12} + A_{21} + A_{22}) + Z_A \\ &= ZA + Z_A \end{aligned}$$

for all $A \in \mathcal{A}$, where $Z_A \in \mathcal{Z}_s(\mathcal{A})$. Define a map $f : \mathcal{A} \to \mathcal{Z}_s(\mathcal{A})$ by $f(A) = Z_A$ for all $A \in \mathcal{A}$. Then we have $\phi(A) = ZA + f(A)$, completing the proof. □

Because a unital prime ring with involution satisfies the hypotheses of Theorem 2.1 if it contains a nontrivial idempotent, the following result is immediate from Theorem 2.1, which was obtained in [11, Theorem 2.1].

Corollary 2.2. *Let $\mathcal{A}$ be a prime ring with the unit I and involution. Assume that $\mathcal{A}$ contains a nontrivial symmetric idempotent and $\phi : \mathcal{A} \to \mathcal{A}$ is a surjective map. If ϕ is a nonlinear surjective strong skew commutativity preserving map, then there exists a map $f : \mathcal{A} \to \mathcal{Z}_s(\mathcal{A})$ such that $\phi(A) = A + f(A)$ or $\phi(A) = -A + f(A)$ for all $A \in \mathcal{A}$.*

Proof. By Theorem 2.1, there exist an element $Z \in \mathcal{Z}_s(\mathcal{A})$ with $Z^2 = I$ and a map $f : \mathcal{A} \to \mathcal{Z}_s(\mathcal{A})$ such that $\phi(A) = ZA + f(A)$ for all $A \in \mathcal{A}$. We only need to prove that $Z = I$ or $Z = -I$. Indeed, since $\mathcal{A}$ is a prime ring with involution, we see that $\mathcal{Z}_s(\mathcal{A}) \subseteq C$, where C is the center of $Q_{ml}(\mathcal{A})$ and $Q_{ml}(\mathcal{A})$ is the maximal left ring of quotients of $\mathcal{A}$ (see [1, 2]). Noticing that C is a field [2, Theorem A.6], we have $Z = I$ or $Z = -I$ by $Z^2 = I$. □

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3. Applications

In this section, we will give several applications of the results in the above section.

Recall that a von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a complex Hilbert space H containing the identity I. The center of $\mathcal{M}$ is denoted by $\mathcal{Z}(\mathcal{M}) = \{Z \in \mathcal{M} : ZM = MZ \text{ for all } M \in \mathcal{M}\}$. The algebra $\mathcal{M}$ is called a factor if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$. We denote that $\overline{A}$ is the central carrier of $A \in \mathcal{M}$. It is well known that the central carrier of A is the projection whose range is the closed linear span of $\{\mathcal{M}A(h) : h \in H\}$. For each self-adjoint operator $R \in \mathcal{M}$, the core of R, denoted by $\underline{R}$, is $\sup\{A \in \mathcal{Z}(\mathcal{M}) : A = A^*, A \leq R\}$. If $P \in \mathcal{M}$ is a projection and $\underline{P} = 0$, we call P a core-free projection. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$. We refer reader to [8] for the theory of von Neumann algebras.

THEOREM 3.1. *Let $\mathcal{M}$ be a factor von Neumann algebra and $\phi : \mathcal{M} \to \mathcal{M}$ a surjective strong skew commutativity preserving map. Then $\phi(A) = A$ or $\phi(A) = -A$ for all $A \in \mathcal{M}$.*

PROOF. It is well known that a factor von Neumann algebra is prime. Since $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$, we see that $\mathcal{Z}_s(\mathcal{M}) = \mathbb{R}I$. By Corollary 2.2, there exists a map $f : \mathcal{A} \to \mathbb{R}I$ such that $\phi(A) = A + f(A)$ or $\phi(A) = -A + f(A)$ for all $A \in \mathcal{M}$. We only need to prove that $f \equiv 0$. In fact, for any $A, B \in \mathcal{M}$, we have $[A, B]_* = [\phi(A), \phi(B)]_* = [A + f(A), B + f(B)]_*$, which implies $[A, f(B)]_* = 0$. Taking $A = iI$, we get $f(B) = 0$ for all $B \in \mathcal{M}$. Then $f \equiv 0$, completing the proof. □

THEOREM 3.2. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Assume that $\phi : \mathcal{M} \to \mathcal{M}$ is a surjective strong skew commutativity preserving map. Then there exists an operator $Z \in \mathcal{Z}_s(\mathcal{M})$ with $Z^2 = I$ such that $\phi(A) = ZA$ for all $A \in \mathcal{M}$.*

PROOF. From [9, Lemma 4], we know that each nonzero central projection of $\mathcal{M}$ is the central carrier of a core-free projection of $\mathcal{M}$. Since I is a nonzero central projection of $\mathcal{M}$, we can obtain a core-free projection $P \in \mathcal{M}$ with central carrier I. It follows from $\overline{P} = I$ that $\text{span}\{\mathcal{M}P(h) : h \in H\}$ is dense in H. For $A \in \mathcal{M}$, $A\mathcal{M}P = 0$ implies $A = 0$. Similarly, since $\overline{I - P} = I$, we see that $A\mathcal{M}(I - P) = 0$ implies $A = 0$. Hence, by Theorem 2.1, there exist an operator $Z \in \mathcal{Z}_s(\mathcal{M})$ with $Z^2 = I$ and a map $f : \mathcal{M} \to \mathcal{Z}_s(\mathcal{M})$ such that $\phi(A) = ZA + f(A)$ for all $A \in \mathcal{M}$.

Furthermore, since ϕ is a strong skew commutativity preserving map, for any $A, B \in \mathcal{M}$, we have $[A, B]_* = [\phi(A), \phi(B)]_* = [ZA + f(A), ZB + f(B)]_*$, which implies $[ZA, f(B)]_* = 0$. Taking $A = iI$, we get $Zf(B) = 0$ for all $B \in \mathcal{M}$. Multiplying the above equation by Z from the left-hand side, we get $f(B) = 0$ for all $B \in \mathcal{M}$. Hence, $\phi(A) = ZA$ for all $A \in \mathcal{M}$, completing the proof. □

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