Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 565–575

GENERATORS OF NEST ALGEBRAS

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1. Introduction. A collection of subspaces of a Hilbert space is called a nest if it is totally ordered by inclusion. The set of all bounded linear operators leaving invariant each member of a given nest forms a weakly-closed algebra, called a nest algebra. Nest algebras were introduced by J. R. Ringrose in [**9**]. The present paper is concerned with generating nest algebras as weakly-closed algebras, and in particular with the following question which was first raised by H. Radjavi and P. Rosenthal in [**8**], viz: Is every nest algebra on a separable Hilbert space generated, as a weakly-closed algebra, by two operators? That the answer to this question is affirmative is proved by first reducing the problem using the main result of [**8**] and then by using a characterization of nests due to J. A. Erdos [**2**]. For the special case of an ordered basis as defined by R. V. Kadison and I. M. Singer [**5**] the result is stated separately as a corollary to the main theorem. Finally an example is given to show that, even on separable space, a certain class of weakly-closed algebras, called the class of reflexive algebras, which contains the class of nest algebras, does not have this double, nor even finite, generation property.

This work formed part of a thesis submitted for a Ph.D. degree at the University of Toronto.

2. Notation and preliminaries. Throughout this paper the terms *Hilbert space*, *subspace* and *projection* will be used to mean *complex* Hilbert space, *closed* subspace and *orthogonal* projection respectively. The set of all bounded linear operators acting on the Hilbert space H and taking values in the Hilbert space K will be denoted by $B(H, K)$ and we write $B(H, H)$ as simply $B(H)$. For the inner-product in a Hilbert space we will use the notation $(\cdot|\cdot)$. The topology on $B(H)$ induced by the set of seminorms $A \to |(Ax|x)|$ $(x \in H)$ is called the *weak* operator topology. The orthogonal complement of a subspace N will be denoted by $H \ominus N$ and P_N will denote the projection with range N. It is easy to see that the operator $T \in B(H)$ leaves N invariant if and only if $(I - P_N)TP_N = 0$ where I is the identity operator on H. The symbol $\oplus$ will always denote *orthogonal* direct sum. The symbol $\subseteq$ will be used for set inclusion, while $\subset$ will be reserved for proper inclusion and $\setminus$ will be used for set theoretic difference. Our measure theoretic terminology follows [**4**] and most of the notation, definitions and results we use concerning von Neumann algebras are to be found in [**1**]. As in [**8**], if H is a Hilbert space and $\mathscr{S} \subseteq B(H)$

Received November 7, 1972.

is any subset we let Lat $\mathscr{S}$ denote the set of all subspaces left invariant by every member of $\mathscr{S}$ and, for any collection $\mathscr{F}$ of subspaces of H, let Alg $\mathscr{F}$ denote the set of all operators in $B(H)$ which leave every member of $\mathscr{F}$ invariant. Thus

$$\text{Lat}\,\mathscr{S} = \{N : N \text{ a subspace of } H,\, TN \subseteq N \ (T \in \mathscr{S})\}$$

$$\text{Alg}\,\mathscr{F} = \{T \in B(H) : TN \subseteq N \ (N \in \mathscr{F})\}.$$

Again as in [**8**], we use the abbreviation 'm.a.s.a.' for 'maximal abelian self-adjoint algebra'.

If $\{N_\alpha\}$ is any collection of subspaces of the Hilbert space H, $\vee N_\alpha$ denotes the smallest subspace of H containing each N_α and $\wedge N_\alpha$ denotes the largest subspace of H contained in each N_α. A family $\mathscr{N}$ of subspaces of H is called a *nest* if it is totally ordered by inclusion, i.e. if, for any pair M, N of elements of $\mathscr{N}$, at least one of the inclusions $M \subseteq N$, $N \subseteq M$ is valid. A nest $\mathscr{N}$ is said to be *complete* if

(i) $(0), H \in \mathscr{N}$;

(ii) for any non-empty subset $\mathscr{N}_0$ of $\mathscr{N}$, the subspaces

$$\bigwedge_{N \in \mathscr{N}_0} N \text{ and } \bigvee_{N \in \mathscr{N}_0} N$$

are both members of $\mathscr{N}$.

If $\mathscr{N}$ is a complete nest of subspaces of H and $(0) \subset N \in \mathscr{N}$, we define N_- to be the subspace $\vee \{M : M \in \mathscr{N}, M \subset N\}$ and let $(0)_- = (0)$. The completeness of $\mathscr{N}$ clearly implies that $N_- \in \mathscr{N}$ for every $N \in \mathscr{N}$. If $N_- \neq N$, N_- is called the *immediate predecessor* of N in $\mathscr{N}$. For a given nest $\mathscr{N}$, Alg $\mathscr{N}$ is called the *nest algebra associated with* or *determined by* $\mathscr{N}$. It is not difficult to show that Alg $\mathscr{N}$ is a weakly-closed algebra for any nest $\mathscr{N}$. By virtue of [**9**, Lemma 3.2], in the theory of nest algebras we may restrict our attention to complete nests. Let $\mathscr{N}$ be a complete nest of subspaces of H. If M and N are elements of $\mathscr{N}$ then the corresponding projections P_M and P_N commute. Consequently, if $\mathscr{E}$ is the set of projections onto the members of $\mathscr{N}$ then $\mathscr{E}$ is a self-adjoint abelian subset of $B(H)$ and the von Neumann algebra generated by $\mathscr{E}$ is abelian. This von Neumann algebra is called the *core* of $\mathscr{N}$ and will be denoted by $\mathscr{C}$. If $\mathscr{C}'$ denotes the commutant of $\mathscr{C}$, i.e. the set $\{T \in B(H) : TS = ST (S \in \mathscr{C})\}$, then obviously

$$\mathscr{C} \subseteq \mathscr{C}' \subseteq \text{Alg}\,\mathscr{N}.$$

3. Statement and reduction of the problem. Let H be a Hilbert space. Following [**8**], a sub-algebra $\mathfrak{A}$ of $B(H)$ is called *reflexive* if $\mathfrak{A} = \text{Alg Lat}\,\mathfrak{A}$; i.e. if whenever Lat $\mathfrak{A} \subseteq$ Lat B, then $B \in \mathfrak{A}$. The main result (Theorem 2) of [**8**] is the following, which we restate here for reference.

THEOREM 3.1. *If $\mathfrak{A}$ is a weakly-closed sub-algebra of $B(H)$ which contains a m.a.s.a. and for which* Lat $\mathfrak{A}$ *is a nest, then $\mathfrak{A}$ is reflexive.*

Obviously if $\mathfrak{A}$ is a reflexive algebra and Lat $\mathfrak{A}$ is a nest then $\mathfrak{A}$ is a nest algebra. The converse statement also holds. Indeed, if $\mathfrak{A}$ is a nest algebra then $\mathfrak{A} = \text{Alg}\,\mathscr{N}$ for some complete nest $\mathscr{N}$ and clearly $\mathfrak{A}$ is reflexive. But Lat $\mathfrak{A} = \mathscr{N}$ by [**9**, Theorem 3.4]. This fact leads to an equivalent restatement of a question that was first raised in [**8**], viz:

> Is every nest algebra on a separable space generated, as a weakly-closed algebra, by two operators?

The main result of this paper (Theorem 5.1) establishes that the answer to the above question is affirmative. The proof of this is reduced to the construction of a single operator with certain properties by application of Theorem 3.1 and the following result which is due to J. von Neumann [**7**].

THEOREM 3.2. *Every m.a.s.a. on a separable space is generated, as a weakly-closed algebra, by a single operator.*

Let $\mathscr{N}$ be a complete nest of subspaces of the separable Hilbert space H and let $\text{Alg}\,\mathscr{N}$, $\mathscr{E}$, $\mathscr{C}$ be as in the preceding section. If $\mathscr{D}$ is a m.a.s.a. containing the core $\mathscr{C}$ of $\mathscr{N}$ then $\mathscr{C} \subseteq \mathscr{D} \subseteq \text{Alg}\,\mathscr{N}$. If $A \in \text{Alg}\,\mathscr{N}$ then $\mathscr{N} \subseteq \text{Lat}\,A \cap \text{Lat}\,\mathscr{D}$. If there were some $A \in \text{Alg}\,\mathscr{N}$ such that $\mathscr{N} \supseteq \text{Lat}\,A \cap \text{Lat}\,\mathscr{D}$ then, taking B to be an operator which generates $\mathscr{D}$ as a weakly-closed algebra, which exists by Theorem 3.2, it would follow that the weakly-closed algebra generated by A and B is $\text{Alg}\,\mathscr{N}$. For if we denote this weakly-closed algebra by $\mathfrak{A}$ we have $\mathfrak{A} \subseteq \text{Alg}\,\mathscr{N}$ and so $\mathscr{N} \subseteq \text{Lat}\,\mathfrak{A}$. But $\mathscr{D} \subseteq \mathfrak{A}$ and $\text{Lat}\,\mathfrak{A} \subseteq \mathscr{N}$ by the properties of A and so $\text{Lat}\,\mathfrak{A} = \mathscr{N}$. Theorem 3.1 then shows that $\mathfrak{A} = \text{Alg Lat}\,\mathfrak{A} = \text{Alg}\,\mathscr{N}$.

Since for any m.a.s.a. $\mathscr{D}$ it is true that the subspace M is invariant under every element of $\mathscr{D}$ if and only if $P_M \in \mathscr{D}$, the proof that the above question has an affirmative answer is reduced to exhibiting

(i) a m.a.s.a. $\mathscr{D}$ containing the core $\mathscr{C}$ of $\mathscr{N}$;

(ii) an operator $A \in \text{Alg}\,\mathscr{N}$ with the property that whenever $(I - E)AE = 0$ with E a projection belonging to $\mathscr{D}$ and I denoting the identity operator on H, then $E \in \mathscr{E}$.

The existence of such a m.a.s.a. and such an operator, given a complete nest of subspaces of a separable Hilbert space is established by using the characterization of such nests given in [**2**].

4. Characterization of complete nests. In [**2**] Erdos constructs, for a given complete nest $\mathscr{N}$ of subspaces of a separable Hilbert space H, a canonical nest that is unitarily equivalent to $\mathscr{N}$. The result is generalized in [**3**] to giving a complete set of unitary invariants for a certain wider class of nests. The following introductory remarks are taken from [**3**] and are only those necessary to understand the terminology of the result in [**2**] that we use. In the following $\mathscr{N}$ will denote a complete nest of subspaces of the separable Hilbert space H.

If $M, N \in \mathcal{N}$ and $M \subset N$ the set $\{K \in \mathcal{N} : M \subset K \subset N\}$ is denoted by (M, N) and is called the *open order interval determined by M and N*. The interpretation of the symbols $[M, N)$, $(M, N]$ and $[M, N]$ should be obvious. An abuse of notation is frequently used by writing $(0, N)$, $[0, N)$, $(0, N]$ and $[0, N]$ instead of $((0), N)$ etc. The collection of all open order intervals is a base for a topology on $\mathcal{N}$, called the *order topology*. With this topology $\mathcal{N}$ is compact and metrizable (compactness follows from [**6**, p. 162, Problem C]; metrizability is shown in [**3**, Theorem 2.2]). Hence, as shown in [**4**], the Borel sets of $\mathcal{N}$ are the elements of the σ-ring generated by all closed, or all open, subsets of $\mathcal{N}$ and all Borel measures are finite. Let R be the ring of subsets of $\mathcal{N}$ generated by all the open order intervals of $\mathcal{N}$. Then any member of R can be written as a finite disjoint union $\bigcup_{i=1}^{n} R_i$ where each R_i is either an open order interval or a singleton. Let $x \in H$ be arbitrary and define the set function μ_x on R as follows. For open order intervals and singletons let

$$\mu_x((N_1, N_2)) = ((E_{2-} - E_1)x|x)$$
$$\mu_x(\{N_1\}) = ((E_1 - E_{1-})x|x)$$

where E_i, E_{i-} are the projections with ranges N_i, N_{i-} respectively $(i = 1, 2)$. Extend the definition of μ_x to the general member of R in the obvious way. Then it can be shown that μ_x is a countably additive set function on R [**3**, Lemma 3.1] which extends to a measure on the Borel subsets of $\mathcal{N}$. We denote this Borel measure by μ_x and denote the set of Borel subsets of $\mathcal{N}$ by $\mathcal{B}$. In this way, every vector $x \in H$ gives rise to a Borel measure μ_x defined on $\mathcal{N}$.

The core $\mathcal{C}$ of $\mathcal{N}$ is an abelian von Neumann algebra and since the underlying space is separable, it is a well-known result (see e.g. [**1**, p. 19]) that $\mathcal{C}$ has a separating vector, i.e. there is a vector $x \in H$ such that $Ax = 0$ and $A \in \mathcal{C}$ implies $A = 0$. Separating vectors for $\mathcal{C}$ will be called separating for $\mathcal{N}$. In the characterization given in [**2**] a major role is played by the Borel measures of the form μ_x arising from separating vectors x for $\mathcal{N}$.

If μ is any Borel measure on $\mathcal{B}$ we will use the usual notation $L^2(\mathcal{N}, \mu)$ for the Hilbert space of (equivalence classes of) Borel measurable, μ-square summable functions defined on $\mathcal{N}$. For any Borel subset δ of $\mathcal{N}$, χ_δ will denote the characteristic function of δ and $\chi_\delta L^2(\mathcal{N}, \mu)$ will denote the following subspace of $L^2(\mathcal{N}, \mu)$

$$\chi_\delta L^2(\mathcal{N}, \mu) = \{ f \in L^2(\mathcal{N}, \mu) : f = 0 \text{ a.e.}[\mu] \text{ on } \mathcal{N} \backslash \delta\}.$$

Theorem 11 of [**2**] shows that, for any unit vector $x \in H$ separating $\mathcal{N}$, there is a family $\{\beta_i\}_{i=1}^{\kappa}$ of Borel subsets of $\mathcal{N}$ with $1 \leqq \kappa \leqq \infty$ and $\mathcal{N} = \beta_1$, $\beta_n \supseteq \beta_{n+1}$ and a unitary transformation of H onto

$$\hat{H} = \bigoplus_{i=1}^{\kappa} L^2(\mathcal{N}, \mu_i)$$

where the Borel measures $\{\mu_i\}_{i=1}^{\kappa}$ are given by

$$\mu_i(\delta) = \mu_x(\delta \cap \beta_i) \qquad (\delta \in \mathcal{B}, 1 \leqq i \leqq \kappa)$$

such that the image of an element N of $\mathscr{N}$ under this transformation is

$$\hat{N} = \bigoplus_{i=1}^{\kappa} \chi_{[0,N]} L^2(\mathscr{N}, \mu_i).$$

(In the following, if $\kappa = \infty$ the index set is to be taken as the set of positive integers.)

5. Main theorem. Using the above characterization of nests we will now prove

THEOREM 5.1. *Every nest algebra on a separable space is generated, as a weakly-closed algebra, by two operators.*

Proof. As we showed in section 3, the theorem follows if we can exhibit a m.a.s.a. and an operator with certain properties. The results of the preceding section enable us to restrict our attention to the complete nest

$$\hat{\mathscr{N}} = \left\{ \bigoplus_{i=1}^{\kappa} \chi_{[0,N]} L^2(\mathscr{N}, \mu_i) : N \in \mathscr{N} \right\}$$

of subspaces of $\hat{H}$. In this case, a m.a.s.a. containing the core $\hat{\mathscr{C}}$ of the nest $\hat{\mathscr{N}}$ is evident. Indeed, the von Neumann algebra $\hat{\mathscr{D}}$ generated by the projections with ranges of the form

$$\bigoplus_{i=1}^{\kappa} \chi_{\delta_i} L^2(\mathscr{N}, \mu_i)$$

where $\{\delta_i\}_{i=1}^{\kappa}$ is a collection of Borel subsets of $\mathscr{N}$, is a m.a.s.a. on $\hat{H}$ (for this see [**1**, p. 118, p. 19]) and obviously contains $\hat{\mathscr{C}}$. Moreover, the range of any projection of $\hat{\mathscr{D}}$ is of the above form. The theorem will be proved if we can construct an operator $A \in \operatorname{Alg} \hat{\mathscr{N}}$ such that, whenever $(I - E)AE = 0$ and E is a projection belonging to $\hat{\mathscr{D}}$, then $E \in \hat{\mathscr{E}}$, the set of projections onto the members of $\hat{\mathscr{N}}$. Notice that $\mu_1 = \mu_x$ and that, for each i, $\mu_i \ll \mu_x$ with $d\mu_i/d\mu_x = \chi_{\beta_i}$ (Radon-Nikodym derivative). We let $H_i = L^2(\mathscr{N}, \mu_i)$ for $1 \leqq i \leqq \kappa$, so that

$$\hat{H} = \bigoplus_{i=1}^{\kappa} H_i.$$

The first step in the construction is to define an operator in $B(H_1)$ with certain properties. Before doing this however, two preliminary lemmas are needed. A proof of the first lemma appears in [**10**].

LEMMA 5.1. *If $\{M_\alpha\}$ is a totally ordered family of subspaces of a separable Hilbert space, and if $N = \bigcap_\alpha M_\alpha$, then there is a countable sub-family $\{M_{\alpha_i}\}$ such that $N = \bigcap_i M_{\alpha_i}$ and $M_{\alpha_{i+1}} \subseteq M_{\alpha_i}$ for each i.*

LEMMA 5.2. *For any fixed i with $1 \leqq i \leqq \kappa$, if δ is a Borel subset of $\mathscr{N}$ and the function $L \rightarrow \mu_i(\delta \cap (L, H])$ $(L \in \mathscr{N})$ vanishes a.e. $[\mu_i]$ on $\mathscr{N} \backslash \delta$ then there is a subspace $N \in \mathscr{N}$ such that $\chi_\delta H_i = \chi_{[0,N]} H_i$.*

Proof. If $\delta' = \delta \cup \gamma$ where γ is the subset of $\mathcal{N} \backslash \delta$ on which the function does not vanish then $\mu_i(\gamma) = 0$, $\chi_\delta H_i = \chi_{\delta'} H_i$ and $\mu_i(\delta' \cap (L, H]) = 0$ if $L \in \mathcal{N} \backslash \delta'$. If $\mathcal{N} \backslash \delta' = \emptyset$ then the result follows with $N = H$. Otherwise, let

$$M = \bigcap_{L \in \mathcal{N} \backslash \delta'} L.$$

Then $M \in \mathcal{N}$ as $\mathcal{N}$ is a complete nest and by Lemma 5.1 there is a non-increasing sequence $\{M_n\}_{n=1}^{\infty}$ of elements of $\mathcal{N} \backslash \delta'$ such that

$$M = \bigcap_{n=1}^{\infty} M_n.$$

Then

$$\delta' \cap (M, H] = \bigcup_{n=1}^{\infty} (\delta' \cap (M_n, H])$$

and so,

$$\mu_i(\delta' \cap (M, H]) = \lim_{n \to \infty} \mu_i(\delta' \cap (M_n, H]) = 0.$$

If we let $\delta'' = \delta' \backslash \gamma'$ where $\gamma' = \delta' \cap (M, H]$ then $\chi_{\delta''} H_i = \chi_{\delta'} H_i$ and $[0, M) \subseteq \delta'' \subseteq [0, M]$. Thus $\delta'' = [0, M]$ or $[0, M)$. If $M_- \subset M$ then $[0, M) = [0, M_-]$ and if $M_- = M$ then

$$\mu_i(\{M\}) = \mu_x(\beta_i \cap \{M\}) \leqq \mu_x(\{M\}) = 0$$

and so $\mu_i([0, M)) = \mu_i([0, M])$. This shows that $\chi_\delta H_i = \chi_{\delta'} H_i = \chi_{\delta''} H_i = \chi_{[0,M]} H_i$ or $\chi_{[0,M_-]} H_i$ and since $M_- \in \mathcal{N}$ the lemma is proved.

LEMMA 5.3. *Define*

$$(Bf)(M) = f(M) + \int_{(M,H]} f(L) d\mu_x(L) \quad (M \in \mathcal{N}, f \in H_1).$$

Then $B \in B(H_1)$ *and*

(i) $\|B\| \leqq 2$;

(ii) $B(\chi_{[0,N]} H_1) \subseteq \chi_{[0,N]} H_1$ *for every* $N \in \mathcal{N}$;

(iii) *If* δ *is a Borel subset of* $\mathcal{N}$ *and* $B(\chi_\delta H_1) \subseteq \chi_\delta H_1$ *then there is a subspace* $N \in \mathcal{N}$ *such that* $\chi_\delta H_1 = \chi_{[0,N]} H_1$.

Proof. If f is Borel measurable and satisfies

$$\int_{\mathcal{N}} |f(L)|^2 d\mu_x(L) < \infty$$

then the function

$$M \to \int_{(M,H]} f(L) d\mu_x(L) \quad (M \in \mathcal{N})$$

is clearly Borel measurable and

$$\int_{\mathcal{N}} \left| \int_{(M,H]} f(L) d\mu_x(L) \right|^2 d\mu_x(M)$$

$$= \int_{\mathcal{N}} \left| \int_{\mathcal{N}} f(L) \chi_{(M,H]}(L) d\mu_x(L) \right|^2 d\mu_x(M)$$

$$\leqslant \int_{\mathcal{N}} |f(L)|^2 d\mu_x(L)$$

using the Cauchy-Schwarz Inequality and the fact that $\mu_x(\mathcal{N}) = ||x||^2 = 1$. It is clear that if g is Borel measurable and $g = f$ a.e. $[\mu_x]$ then the functions

$$M \to \int_{(M,H]} f(L) d\mu_x(L), \quad M \to \int_{(M,H]} g(L) d\mu_x(L)$$

are equal a.e. $[\mu_x]$. Since linearity is obvious it follows that

$$f(\cdot) \to \int_{(\cdot,H]} f(L) d\mu_x(L) \quad (f \in H_1)$$

defines an operator, call it $\hat{B} \in B(H_1)$, such that $||\hat{B}|| \leqq 1$. Hence, if I is the identity operator on H_1, $I + \hat{B} = B \in B(H_1)$ and (i) follows immediately. If $f \in H_1$ and $f = 0$ a.e. $[\mu_x]$ on $(N, H]$ where $N \in \mathcal{N}$ then if $L \in (N, H]$, $(L, H] \subseteq (N, H]$ and so $f = 0$ a.e. $[\mu_x]$ on $(L, H]$. Thus $(Bf)(L) = f(L)$ and this proves (ii). If $\delta \in \mathscr{B}$ and $B(\chi_\delta H_1) \subseteq \chi_\delta H_1$ then, in particular $B\chi_\delta \in \chi_\delta H_1$ and so the function $M \to \mu_x(\delta \cap (M, H])$ $(M \in \mathcal{N})$ vanishes a.e. $[\mu_x]$ on $\mathcal{N} \backslash \delta$. By Lemma 5.2 with $i = 1$, result (iii) follows.

For every i let $F_i \in B(H_1)$ be the projection with range $\chi_{\beta_i} H_1$. Then F_i is 'multiplication by χ_{β_i}' and since $\mathcal{N} = \beta_1 \supseteq \beta_2 \supseteq \beta_3 \ldots$ we have $I = F_1 \geqq F_2 \geqq F_3 \ldots$. For any fixed i with $1 \leqq i \leqq \kappa$ the spaces H_i and $F_i H_1$ may be identified. More precisely, there is a natural unitary transformation from H_i onto $F_i H_1$. In fact, the transformation $U_i : H_i \to F_i H_1$ defined by $U_i f = \chi_{\beta_i} f$ $(f \in H_i)$ is such a transformation with inverse given by $U_i^{-1} f = f$ $(f \in F_i H_1)$. This unitary transformation has the property that, for any Borel subset δ of $\mathcal{N}$, the image of $\chi_\delta H_i$ under U_i is $\chi_\delta H_1 \cap \chi_{\beta_i} H_1$.

For every i, j with $1 \leqq i,\ j \leqq \kappa$ let $B_{ij} \in B(H_j, H_i)$ be the operator $B_{ij} = U_i^{-1} F_i B F_j U_j$, where $B \in B(H_1)$ is the operator as defined in Lemma 5.3. Then

$$(B_{ij} f)(M) = \chi_{\beta_i \cap \beta_j}(M) f(M)$$

$$+ \chi_{\beta_i}(M) \int_{(M,H]} f(L) d\mu_j(L) \quad (f \in H_j, M \in \mathcal{N})$$

and $B_{11} = B$.

LEMMA 5.4. *The collection* $\{B_{ij}\}_{1 \leqq i,j \leqq \kappa}$ *of operators satisfies*

https://doi.org/10.4153/CJM-1974-052-8 Published online by Cambridge University Press

(i) $||B_{ij}|| \leqq 2$ *for all* i *and* j;

(ii) $B_{ij}(\chi_{[0,N]}H_j) \subseteq \chi_{[0,N]}H_i$ *for every* $N \in \mathscr{N}$ *and every* i, j;

(iii) *for any fixed* i, *if* δ *is a Borel subset of* $\mathscr{N}$ *and* $B_{ii}(\chi_\delta H_i) \subseteq \chi_\delta H_i$ *then there is a subspace* $N \in \mathscr{N}$ *such that* $\chi_\delta H_i = \chi_{[0,N]}H_i$;

(iv) *if* i *and* j *are fixed with* $i \neq j$ *and if* $M, N \in \mathscr{N}$ *with* $M \subseteq N$ *and* $B_{ij}(\chi_{[0,N]}H_j) \subseteq \chi_{[0,M]}H_i$ *then* $\chi_{[0,N]}H_k = \chi_{[0,M]}H_k$ *where* $k = \max(i, j)$.

Proof. (i) follows from the definition and Lemma 5.3 (i). Now let i, j and $N \in \mathscr{N}$ be fixed. That $B_{ij}(\chi_{[0,N]}H_j) \subseteq \chi_{[0,N]}H_i$ is equivalent to showing that

$$F_iBF_j(\chi_{[0,N]}H_1 \cap \chi_{\beta_j}H_1) \subseteq \chi_{[0,N]}H_1 \cap \chi_{\beta_i}H_1$$

and this follows from Lemma 5.3 (ii) and the fact that $F_k(\chi_\delta H_1) \subseteq \chi_\delta H_1$ for every k and every $\delta \in \mathscr{B}$. This proves (ii). Let i be fixed and let $\delta \in \mathscr{B}$ such that $B_{ii}(\chi_\delta H_i) \subseteq \chi_\delta H_i$. Then since

$$(B_{ii}\chi_\delta)(M) = \chi_{\beta_i}(M)\chi_\delta(M) + \chi_{\beta_i}(M)\mu_i(\delta \cap (M, H]) \qquad (M \in \mathscr{N})$$

it follows that

$$\begin{aligned} 0 &= \mu_i(\{M \in \mathscr{N} \backslash \delta : \chi_{\beta_i}(M)\mu_i(\delta \cap (M, H]) \neq 0\}) \\ &= \mu_i(\beta_i \cap \{M \in \mathscr{N} \backslash \delta : \mu_i(\delta \cap (M, H]) \neq 0\}) \\ &= \mu_i(\{M \in \mathscr{N} \backslash \delta : \mu_i(\delta \cap (M, H]) \neq 0\}). \end{aligned}$$

In other words, the function $M \rightarrow \mu_i(\delta \cap (M, H])$ $(M \in \mathscr{N})$ vanishes a.e. $[\mu_i]$ on $\mathscr{N} \backslash \delta$. By Lemma 5.2 the result (iii) follows. Now let i and j be fixed with $i \neq j$ and suppose that $M, N \in \mathscr{N}$ with $M \subseteq N$ and

$$B_{ij}(\chi_{[0,N]}H_j) \subseteq \chi_{[0,M]}H_i.$$

Then since

$$\begin{aligned} (B_{ij}\chi_{[0,N]})(L) = \chi_{\beta_i \cap \beta_j}(L)\chi_{[0,N]}(L) \\ + \chi_{\beta_i}(L)\mu_j([0, N] \cap (L, H]) \quad (L \in \mathscr{N}) \end{aligned}$$

it follows that

$$\begin{aligned} 0 &= \mu_i(\{L \in (M, H] : \chi_{\beta_i \cap \beta_j}(L)\chi_{[0,N]}(L) \\ &\qquad + \chi_{\beta_i}(L)\mu_j([0, N] \cap (L, H]) \neq 0\}) \\ &= \mu_i(\beta_i \cap \{L \in (M, H] : \chi_{\beta_j}(L)\chi_{[0,N]}(L) + \mu_j([0, N] \cap (L, H]) \neq 0\}) \\ &= \mu_i(\{L \in (M, H] : \chi_{\beta_j}(L)\chi_{[0,N]}(L) + \mu_j([0, N] \cap (L, H]) \neq 0\}). \end{aligned}$$

In other words, the function

$$L \rightarrow \chi_{\beta_j}(L)\chi_{[0,N]}(L) + \mu_j([0, N] \cap (L, H]) \qquad (L \in \mathscr{N})$$

vanishes a.e. $[\mu_i]$ on $(M, H]$. But on $\beta_j \cap [0, N] \cap (M, H] \subseteq (M, H]$ this function is certainly not zero, so if $k = \max(i, j)$,

$$\begin{aligned} 0 &= \mu_i(\beta_j \cap [0, N] \cap (M, H]) = \mu_i(\beta_j \cap (M, N]) \\ &= \mu_x(\beta_i \cap \beta_j \cap (M, N]) = \mu_x(\beta_k \cap (M, N]) = \mu_k((M, N]). \end{aligned}$$

Thus $\chi_{[0,N]}H_k = \chi_{[0,M]}H_k$ and the proof of the lemma is complete.

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Recall that $\hat{H} = \bigoplus_{i=1}^{\kappa} H_i$ where $H_i = L^2(\mathscr{N}, \mu_i)$ and $1 \leqq \kappa \leqq \infty$. Since for every i and j the operator $B_{ij} \in B(H_j, H_i)$ satisfied $\|B_{ij}\| \leqq 2$, there exist positive scalars $\{\tau_{ij}\}_{1 \leqq i, j \leqq \kappa}$ (e.g., $\tau_{ij} = 2^{-(i+j)/2}$ for every i, j) such that the $\kappa \times \kappa$ matrix with operator entries which has as i-jth element the operator $\tau_{ij}B_{ij}$, represents an operator $A \in B(\hat{H})$. It will be shown that

(i) $A \in \operatorname{Alg} \hat{\mathscr{N}}$;

(ii) if $(I - E)AE = 0$ where I is the identity operator on $\hat{H}$ and E is a projection belonging to $\hat{\mathscr{D}}$ then $E \in \hat{\mathscr{E}}$.

Firstly, let $N \in \mathscr{N}$ be arbitrary. Then

$$\hat{N} = \bigoplus_{i=1}^{\kappa} \chi_{[0,N]} H_i$$

is the corresponding element of $\hat{\mathscr{N}}$. To show that $A\hat{N} \subseteq \hat{N}$ we have to show that $(I - P_{\hat{N}})AP_{\hat{N}} = 0$. Using the representations of A and $P_{\hat{N}}$ as $\kappa \times \kappa$ matrices with operator entries this reduces to showing that $(I_i - E_i)B_{ij}E_j = 0$ for every i and j where I_i is the identity operator on H_i and $E_k \in B(H_k)$ $(1 \leqq k \leqq \kappa)$ is the projection with range $\chi_{[0,N]}H_k$. That is, we have to show that $B_{ij}(\chi_{[0,N]}H_j) \subseteq \chi_{[0,N]}H_i$ for every i and j. But this is so, by Lemma 5.4 (ii). This proves that $A \in \operatorname{Alg} \hat{\mathscr{N}}$.

Now let E be a projection belonging to $\hat{\mathscr{D}}$ satisfying $(I - E)AE = 0$. Then the range of E is of the form

$$\bigoplus_{i=1}^{\kappa} \chi_{\delta_i} H_i$$

where $\{\delta_i\}_{i=1}^{\kappa}$ is a collection of Borel subsets of $\mathscr{N}$. For every i let $P_i \in B(H_i)$ be the projection with range $\chi_{\delta_i}H_i$. Then $(I - E)AE = 0$ implies that $(I_i - P_i)B_{ij}P_j = 0$ for every i and j, and in particular that

$$B_{ii}(\chi_{\delta_i}H_i) \subseteq \chi_{\delta_i}H_i$$

for every i. By Lemma 5.4 (iii), for every i there is a subspace $N_i \in \mathscr{N}$ such that $\chi_{\delta_i}H_i = \chi_{[0,N_i]}H_i$. Also, for any $i > 1$ we have

$$B_{1i}(\chi_{[0,N_i]}H_i) \subseteq \chi_{[0,N_1]}H_1$$
$$B_{i1}(\chi_{[0,N_1]}H_1) \subseteq \chi_{[0,N_i]}H_i$$

and since either $N_1 \subseteq N_i$ or $N_i \subseteq N_1$, Lemma 5.4 (iv) shows that $\chi_{[0,N_i]}H_i = \chi_{[0,N_1]}H_i$ for every i. Thus the range of E is the subspace $\hat{N}_1 = \bigoplus_{i=1}^{\kappa} \chi_{[0,N_1]}H_i$ and so $E \in \hat{\mathscr{E}}$. From our previous remarks this completes the proof of Theorem 5.1.

Remark. From any collection $\{B_{ij}\}_{1 \leqq i, j \leqq \kappa}$ of operators satisfying $B_{ij} \in B(H_j, H_i)$ and the four conditions of Lemma 5.4 we can construct an operator $A \in B(\hat{H})$ with the 'right' properties with respect to the nest $\hat{\mathscr{N}}$. However, inasmuch as the collection $\{B_{ij}\}_{1 \leqq i, j \leqq \kappa}$ actually used is derived, using the U_i's and F_j's, from a single operator B on H_1, it is in this sense, canonically related to the nest $\hat{\mathscr{N}}$.

6. A corollary. In the theory of triangular operator algebras introduced by R. V. Kadison and I. M. Singer [**5**] an important role is played by a certain class of such algebras called the ordered bases. Since, by [**5**, Theorem 3.1.1] every ordered basis is a nest algebra we have the following corollary of our main theorem.

COROLLARY 5.1. *Every ordered basis, on a separable space, is generated, as a weakly-closed algebra, by two operators.*

A particular instance of the above result was proved by H. Radjavi and P. Rosenthal in [**8**] and led to the formulation of the more general question regarding nest algebras.

7. An example. Having proved the theorem it is of interest to know to what extent it can be generalized. It is not difficult to see that the theorem is false without the separability condition. Indeed if H is a Hilbert space then $B(H)$, which is the nest algebra associated with the complete nest $\{(0), H\}$, is finitely generated as a weakly-closed algebra if and only if H is separable. Another generalization of this result would be to keep the separability condition and enlarge the class of weakly-closed algebras to which it applies. For any collection $\mathscr{F}$ of subspaces of the Hilbert space H, Alg$\mathscr{F}$ is a reflexive algebra and every reflexive algebra on H is of this form. Every reflexive algebra is weakly-closed and the class of reflexive algebras contain the class of nest algebras and the class of von Neumann algebras. The question of whether every von Neumann algebra is finitely generated is a well-known unsolved problem. Such considerations lead us to the following question. Is every reflexive algebra, on a separable space, finitely generated as a weakly-closed algebra? The answer is negative as the following example shows. It is due to Professor P. Rosenthal who has kindly permitted its inclusion in this paper.

Let H be a separable, infinite dimensional Hilbert space and let M be a subspace of H whose dimension and co-dimension are both infinite. Let $\mathscr{F}$ be the following collection of subspaces of H

$$\mathscr{F} = \{N : N \subseteq M \text{ or } N \supseteq M\}.$$

Then Alg$\mathscr{F}$ is a reflexive algebra and it will be shown that Alg$\mathscr{F}$ is not finitely generated as a weakly-closed algebra. Notice first that $\mathscr{F} \subseteq$ Lat Alg$\mathscr{F}$. If N is a subspace of H which is invariant under every operator belonging to Alg$\mathscr{F}$ then since $P_M \in$ Alg$\mathscr{F}$, $P_M N \subseteq N$. Hence $P_M P_N = P_N P_M$ and N has the decomposition $N = (N \cap M) \oplus (N \cap (H \ominus M))$. If

$$N \cap (H \ominus M) = (0)$$

then $N \subseteq M$ and so $N \in \mathscr{F}$. If $N \cap (H \ominus M)$ is not the zero subspace it will be shown that $M \subseteq N$ and hence that $N \in \mathscr{F}$. Let $y \in N \cap (H \ominus M)$ be any fixed unit vector and let $x \in M$ be arbitrary. Let F_x be the finite rank operator

https://doi.org/10.4153/CJM-1974-052-8 Published online by Cambridge University Press

defined by $F_x z = (z|y)x (z \in H)$. Then $F_x \in \text{Alg}\,\mathscr{F}$. For if $L \subseteq M$ then $F_x L = (0) \subseteq L$ and if $L \supseteq M$, $F_x L \subseteq M \subseteq L$. Thus $F_x N \subseteq N$ and since $y \in N$, $F_x y = x \in N$. Thus $M \subseteq N$. This shows that $\mathscr{F} = \text{Lat Alg}\,\mathscr{F}$.

Now let $A_1, A_2, \ldots, A_n$ be any finite collection of elements of $\text{Alg}\,\mathscr{F}$ and let $\mathfrak{A}$ be the weakly-closed algebra they generate. Then since $\text{Alg}\,\mathscr{F}$ is weakly-closed, $\mathfrak{A} \subseteq \text{Alg}\,\mathscr{F}$ and so $\mathscr{F} \subseteq \text{Lat}\,\mathfrak{A}$. It must be shown that $\mathfrak{A} \subset \text{Alg}\,\mathscr{F}$. To show this it is sufficient to show that $\mathscr{F} \subset \text{Lat}\,\mathfrak{A}$ by the preceding remark. Let $x \in H$ be any vector not belonging to M. Let $[x]$ denote the 1-dimensional subspace spanned by x. Then $[x] \vee M \in \mathscr{F}$ and so is invariant under each A_i for each i with $1 \leqq i \leqq n$. Hence for each i, there is a complex scalar α_i and a vector $m_i \in M$ such that $A_i x = \alpha_i x + m_i$. Since each 1-dimensional subspace of M belongs to $\mathscr{F}$, it follows that if $m_i \neq 0$ then m_i is an eigenvector of A_j. Let $N = \vee \{x, m_1, m_2, \ldots, m_n\}$. Then N is invariant under every element of $\mathfrak{A}$ and $N \notin \mathscr{F}$. Thus $\mathscr{F} \subset \text{Lat}\,\mathfrak{A}$ and $\mathfrak{A} \subset \text{Alg}\,\mathscr{F}$. This shows that $\text{Alg}\,\mathscr{F}$ is not finitely generated as a weakly-closed algebra.

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