POLYTOPES WITH AN AXIS OF SYMMETRY

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1. Introduction. During the last few years, Branko Grünbaum, Micha Perles, and others have made extensive use of Gale transforms and Gale diagrams in investigating the properties of convex polytopes. Up to the present, this technique has been applied almost entirely in connection with combinatorial and enumeration problems. In this paper we begin by showing that Gale transforms are also useful in investigating properties of an essentially metrical nature, namely the symmetries of a convex polytope. Our main result here (Theorem (10)) is that, in a manner that will be made precise later, the symmetry group of a polytope can be represented faithfully by the symmetry group of a Gale transform of its vertices. If a d-polytope $P \subset E^d$ has an axis of symmetry A (that is, A is a linear subspace of E^d such that the reflection in A is a symmetry of P), then it is called *axi-symmetric*. Using Gale transforms we are able to determine, in a simple manner, the possible numbers and dimensions of axes of symmetry of axi-symmetric polytopes.

The last part of the paper (§§ 5–9) is concerned with enumeration problems. Apart from Perles' recent determination of the number of simplicial d-polytopes with $d + 3$ vertices, no major enumeration problems have been solved within the last fifty years (see [**2**, 6.6] for a short history of the subject). It was therefore surprising to discover that if we restrict our attention to axi-symmetric polytopes with no vertices on the axis, then in many cases the determination of the number of combinatorial types becomes tractable.

Following the notation of [**2**], $c(v, d)$ will be used for the number of combinatorial types of d-polytopes with v vertices, and $c_s(v, d)$ for the corresponding number of simplicial polytopes. We write $c^*(v, d, a)$ for the number of combinatorial types of d-polytopes with an a-dimensional axis of symmetry, the star signifying that none of the vertices lies on the axis, so that $v = 2n$ is an even integer. Again the suffix s will be used to signify that only simplicial polytopes are to be considered.

In §§ 5, 6, and 7, $c^*(v, d, a)$ will be determined for certain values of v, d, and a, the results being tabulated at the end of the paper. It is not unexpected that the case of simplicial polytopes is easier, and in § 8 (Theorem (29)) we shall give a general expression for $c_s^*(2n, d, a)$ for all d and a in the case

$$n = \max(d - a, a + 1),$$

that is, as we shall show in § 2, in the case of polytopes with a minimum number

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of vertices. This general expression is in terms of $c_s(v', a)$ $(v' \leqq d - a)$, and therefore the explicit numerical value can be determined only in a few cases.

In the course of this investigation, we have accidentally discovered a number of curious identities, the simplest being

$$c^*(2(d-1), d, 1) = c(d+1, d-1) \qquad (d \geqq 3).$$

Although this is easily proved (it is an immediate consequence of [**2**, 6.1] and (23)), we can find no adequate "geometrical" explanation for this identity.

We wish to thank Keith Lloyd for his assistance in connection with the applications of Pólya's Theorem to the enumeration problems, and also Günter Ewald, whose discussions with one of us at an early stage in the investigation proved very helpful.

2. Gale transforms. Let P be a d-polytope (that is, a d-dimensional convex polytope) in E^d. Then a subspace A of E^d is called an *axis of symmetry* of P if reflection in A is a symmetry of P. In other words, to every vertex w of P, which does not lie on A, corresponds another vertex w' of P which is the reflection of w in A. The vertices w and w' are said to form a *pair* with respect to the axis A, and the line segment joining w to w' is bisected orthogonally by A. We write $a = \dim A$, and if $0 \leqq a \leqq d - 1$, then P is called an *axi-symmetric* polytope. If $a = 0$, then it is also called *centrally symmetric.*

In this section we investigate the special properties of the Gale transforms of the sets of vertices of axi-symmetric polytopes. For the most part we shall follow the notation and terminology of [**2**, 5.4; **3**, Chapter 2].

Let P be an axi-symmetric d-polytope in E^d, and choose the coordinate system in E^d so that the origin $o \in A$, and each point can be written in the form (x, y), with $x \in A$ and $y \in A^\perp$, the $(d-a)$-dimensional orthogonal complement of A passing through o. Then the vertices of P may be written

$$(1) \qquad \begin{aligned} u_i &= (z_i, o), & i &= 1, \ldots, m, \\ w_j &= (x_j, y_j), & & \\ w_j' &= (x_j, -y_j), & & \left.\right\} y_j \neq o, j = 1, \ldots, n, \end{aligned}$$

so that P has $m + 2n$ vertices, namely m lying on A, and the remaining $2n$ forming n pairs with respect to A. The set of points

$$(2) \qquad V_A = \{z_1, \ldots, z_m, x_1, \ldots, x_n\} \subset A$$

will be called the *axis figure* of P with respect to A, and the centrally symmetric set of points

$$(3) \qquad V_C = \{\pm y_1, \ldots, \pm y_n\} \subset A^\perp$$

will be called the *coaxis figure.* It should be noted that neither the axis figure nor the coaxis figure is, in general, the set of vertices of a (convex) polytope.

Clearly the polytope P is completely determined by V_A, V_C, and the one-to-one correspondence $\pm y_j \leftrightarrow x_j$ between the pairs of V_C and a subset of V_A.

Since A is a-dimensional, V_A must contain at least $a + 1$ points, and so $m + n \geqq a + 1$. Since A is $(d - a)$-dimensional, V_C must contain at least $2(d - a)$ points, and so $n \geqq d - a$. These lead to the relation

$$(4) \qquad n \geqq \max(a + 1 - m, d - a),$$

an inequality to which we shall frequently refer throughout the paper.

Any Gale transform of $V = \text{vert } P$ (the set of vertices of P) is $(m + 2n - d - 1)$-dimensional, and the first theorem describes its special properties.

(5) THEOREM. *Let P be a d-polytope with an a-dimensional axis of symmetry A, and with $m + 2n$ vertices: $u_1, \ldots, u_m$ on A, and $w_1, w_1', \ldots, w_n, w_n'$ paired with respect to A. Then there is a Gale transform $\bar{V}$ of* vert P *which has an $(m + n - a - 1)$-dimensional axis of symmetry $\bar{A}$ containing the transforms $\bar{u}_1, \ldots, \bar{u}_m$ of the vertices $u_1, \ldots, u_m$. The set of points*

$$\{\bar{u}_1, \ldots, \bar{u}_m, \bar{w}_1 + \bar{w}_1', \ldots, \bar{w}_n + \bar{w}_n'\} \subset \bar{A}$$

is a Gale transform of the axis figure V_A of P, and the centrally symmetric set of points

$$\{\pm(\bar{w}_1 - \bar{w}_1'), \ldots, \pm(\bar{w}_n - \bar{w}_n')\} \subset \bar{A}^{\perp}$$

is a c.s. transform (see [3] for definition) *of the coaxis figure V_C of P. (Here $\bar{A}^{\perp}$ is the $(n + a - d)$-dimensional orthogonal complement of $\bar{A}$ in $E^{m+2n-d-1}$* $=$ lin $\bar{V}$, *the linear hull of $\bar{V}$.)*

Proof. Let

$$(\lambda_{1k}, \ldots, \lambda_{mk}, \mu_{1k}, \ldots, \mu_{nk}), \qquad k = 1, \ldots, m + n - a - 1,$$

be a basis of the set of affine dependences of V_A, as given in (2), and let

$$(\nu_{1h}, \ldots, \nu_{nh}), \qquad h = 1, \ldots, n + a - d,$$

be a basis for the set of linear dependences of $V_C^+ = \{y_1, \ldots, y_n\}$. Then

$$(6) \qquad \left.\begin{array}{ll} \left.\begin{array}{l} \sum_{i=1}^{m} \lambda_{ik} z_i + \sum_{j=1}^{n} \mu_{jk} x_j = o, \\ \sum_{i=1}^{m} \lambda_{ik} + \sum_{j=1}^{n} \mu_{jk} = 0, \end{array}\right\} & k = 1, \ldots, m + n - a - 1, \\ \sum_{j=1}^{n} \nu_{jh} y_j = o, & h = 1, \ldots, n + a - d, \end{array}\right\},$$

from which it follows that

$$\left.\begin{array}{l} \sum_{i=1}^{m} 2\lambda_{ik} u_i + \sum_{j=1}^{n} \mu_{jk}(w_j + w_j') = o, \\ \sum_{i=1}^{m} 2\lambda_{ik} + \sum_{j=1}^{n} (\mu_{jk} + \mu_{jk}) = 0, \end{array}\right\} \quad k = 1, \ldots, m + n - a - 1,$$

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and

$$\left.\begin{aligned} \sum_{j=1}^{n} \nu_{jh}(w_j - w_j') &= o, \\ \sum_{j=1}^{n} (\nu_{jh} - \nu_{jh}) &= 0, \end{aligned}\right\} \quad h = 1, \ldots, n + a - d.$$

Since $(m + n - a - 1) + (n + a - d) = m + 2n - d - 1$, we conclude that the linearly independent vectors

$$\begin{aligned} &(2\lambda_{1k}, \ldots, 2\lambda_{mk}, \mu_{1k}, \mu_{1k}, \ldots, \mu_{nk}, \mu_{nk}), && k = 1, \ldots, m + n - a - 1, \\ &(0, \ldots, 0, \nu_{1h}, -\nu_{1h}, \ldots, \nu_{nh}, -\nu_{nh}), && h = 1, \ldots, n + a - d, \end{aligned}$$

form a basis for the set of affine dependences of

$$\text{vert } P = \{u_1, \ldots, u_m, w_1, w_1', \ldots, w_n, w_n'\}.$$

Thus a Gale transform of vert P is given by

$$\begin{aligned} \bar{u}_i &= (2\bar{z}_i, o), && i = 1, \ldots, m, \\ \left.\begin{aligned} \bar{w}_j &= (\bar{x}_j, \bar{y}_j), \\ \bar{w}_j' &= (\bar{x}_j, -\bar{y}_j), \end{aligned}\right\} & && j = 1, \ldots, n, \end{aligned}$$

where

$$\begin{aligned} \bar{z}_i &= (\lambda_{i1}, \ldots, \lambda_{i,m+n-a-1}), && i = 1, \ldots, m, \\ \left.\begin{aligned} \bar{x}_j &= (\mu_{j1}, \ldots, \mu_{j,m+n-a-1}), \\ \bar{y}_j &= (\nu_{j1}, \ldots, \nu_{j,n+a-d}), \end{aligned}\right\} & && j = 1, \ldots, n. \end{aligned}$$

Since, from (6), $\{\bar{z}_1, \ldots, \bar{z}_m, \bar{x}_1, \ldots, \bar{x}_n\}$ is a Gale transform of V_A, and $\{\pm\bar{y}_1, \ldots, \pm\bar{y}_n\}$ is a c.s. transform of V_C, the assertions of the theorem follow immediately.

In the theorem it should be noted that $\bar{w}_j$ may (exceptionally) lie on $\bar{A}$, and then $\bar{w}_j$ and $\bar{w}_j'$ coincide. In this case we may regard $\bar{w}_j$ and $\bar{w}_j'$ as being interchanged by the reflection in $\bar{A}$.

The relationship between a set of points and its Gale transform is symmetrical, as is clear from the geometrical interpretation [**3**, Chapter 4]. However, the Gale transform is only defined to within linear equivalence, and affinely equivalent sets of points have the same Gale transforms. This introduces complications in discussing properties of polytopes deduced from corresponding properties of their Gale transforms. The problem can be dealt with in one of two ways. One way is to generalize the concept of an axis of symmetry by allowing affine reflections, that is, involutory affine transformations which preserve the polytope, although, as we shall see in the next section, this approach introduces no significant generalization. The second way is to adopt the convention that, whenever we say that a polytope P has a certain property, then we mean that some polytope affinely equivalent to P has the stated property. Since it is slightly more simple, we shall follow the second approach here. Thus we are able to state the converse of Theorem (5) as follows.

(7) CONVERSE. *If a Gale transform $\bar{V}$ of the set of vertices of a d-polytope P with v vertices has a b-dimensional axis of symmetry $\bar{A}$ $(0 \leqq b \leqq v - d - 1)$ containing m points $\bar{u}_1, \ldots, \bar{u}_m$ of $\bar{V}$, then the polytope P has an axis of symmetry A of dimension $a = \frac{1}{2}(v + m) - b - 1$ containing exactly m vertices of P.*

Notice that $\{\bar{u}_1, \ldots, \bar{u}_m\}$ may be a proper subset of the set of points of $\bar{V}$ on $\bar{A}$, but when this happens, the remaining points of $\bar{V}$ on $\bar{A}$ must occur in coincident pairs. We regard the points of these pairs as being interchanged by the reflection in $\bar{A}$. Thus each axis of symmetry of a Gale transform may correspond to a number of distinct axes (of various dimensions) of the polytope P. This will be investigated in more detail in the next section.

As a special example, consider the octahedron in E^3. This, along with a Gale transform of its set of vertices, is illustrated in Figure 1. It is easily verified

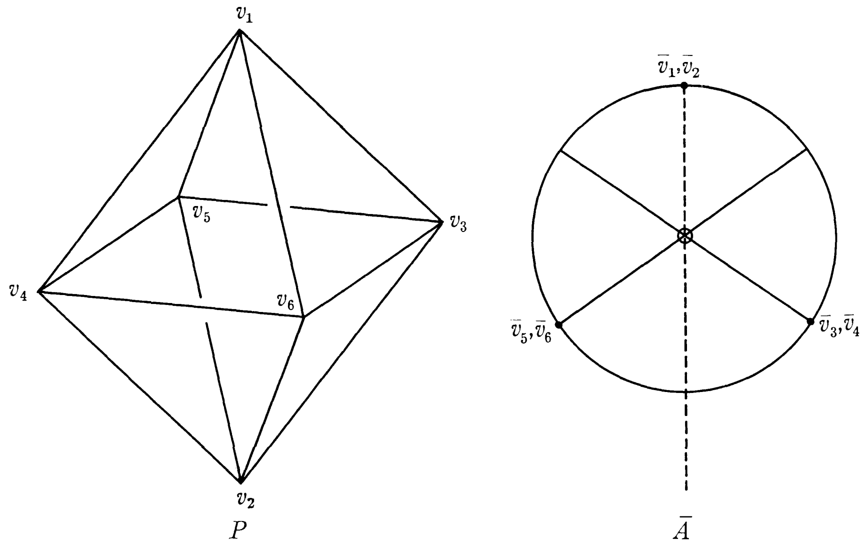

FIGURE 1

that the assertions of Theorem (5) hold. For the converse, let us consider the axis of symmetry $\bar{A}$ of $\bar{V}$ marked in the figure. If we take $m = 0$, so that the points $\bar{v}_1$ and $\bar{v}_2$ are regarded as being interchanged by the reflection in $\bar{A}$, then $\bar{A}$ corresponds to an axis of symmetry A of P, of dimension $\frac{1}{2}(6 + 0) - 1 - 1 = 1$, namely the line joining the mid-points of the edges $[v_3, v_5]$ and $[v_4, v_6]$ of P (or equally, the line joining the mid-points of the edges $[v_3, v_6]$ and $[v_4, v_5]$). If we take $m = 2$, so that $\bar{v}_1$ and $\bar{v}_2$ are each fixed under the reflection in $\bar{A}$, then $\bar{A}$ will correspond to an axis of dimension $\frac{1}{2}(6 + 2) - 1 - 1 = 2$ of P containing the vertices v_1 and v_2, namely either of the two planes spanned by v_1, v_2, and one of the lines of symmetry just mentioned.

Two extreme cases of the theorem and its converse are of interest.

First suppose that $a \geqq \frac{1}{2}(m + d - 1)$ (in which case we shall say that the dimension of the axis is *large*). Then, by (4), $n \geqq a + 1 - m$. If n takes its minimal value $n = a + 1 - m$ (a case with which we shall be particularly concerned later), then the axis figure V_A is the set of vertices of a simplex, and its Gale transform consists of $a + 1$ points coincident at the origin. Hence we have the following.

(8) COROLLARY. *If* $a \geqq \frac{1}{2}(m + d - 1)$ *and* $n = a + 1 - m$, *then every Gale transform of the set of* $v = m + 2n$ *vertices of* P *is centrally symmetric. Conversely, if a Gale transform of* vert P *is centrally symmetric about the origin* o, *and* m *points lie at* o, *then* P *has an axis of symmetry* A *of dimension* $\frac{1}{2}(v + m) - 1$, *and the axis figure* V_A *is the set of vertices of a simplex.*

As before, in applying the corollary, a point of the Gale transform $\bar{V}$ of multiplicity r may be counted as s single points and t double points, where s and t are any positive integers satisfying $s + 2t = r$. Thus, for example, the m points mentioned in Corollary (8) need not be the only points of $\bar{V}$ at the origin.

In the second case, suppose that P has an axis of symmetry of dimension $a \leqq \frac{1}{2}(m + d - 1)$ (when we shall say that the dimension of the axis is *small*). From (4) we deduce that $n \geqq d - a$. If n takes its minimal value $n = d - a$, then the coaxis figure V_C consists of the set of vertices of a cross-polytope, and its c.s. transform consists of $2(d - a)$ points at the origin. We thus deduce the following.

(9) COROLLARY. *If* $a \leqq \frac{1}{2}(m + d - 1)$ *and* $n = d - a$, *then every Gale transform of* vert P *consists of* m *single points and* n *double points* (*pairs of coincident points*), *with further coincidences between these points being allowed. Conversely, if a Gale transform of* vert P *consists of* m *single points and* n *double points* (*possibly with further coincidences*), *then* P *has an axis of symmetry of dimension* $d - n$, *and the coaxis figure* V_C *is the set of vertices of a cross-polytope.*

We remark that in both the cases covered by the corollaries the one-to-one correspondence between the pairs of points of the coaxis figure V_C and the subset of the axis figure V_A is irrelevant, in the sense that every such correspondence leads to the same combinatorial type of polytope. In fact, it is obvious that the various polytopes obtained are affinely equivalent, for the Gale transforms of their sets of vertices are identical.

3. Symmetry groups of polytopes. In § 2 we showed how an axi-symmetric polytope P could be represented by a Gale transform with an axis of symmetry, and conversely. Here we show that the whole group of symmetries of P can be represented simultaneously by symmetries of a Gale transform.

As indicated in the last section, it is convenient to set everything initially in a slightly more general context; we consider affine symmetries of the polytope P, that is, (non-singular) affine transformations which preserve P. If

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γ is an affine symmetry of P, then $P' = P\gamma$ coincides with (and so is affinely equivalent to) P. Thus a Gale transform $\bar{V}$ of vert P is also a Gale transform of vert P'. But γ induces a permutation of vert P. Writing $\bar{\gamma}$ for the corresponding permutation of the points of $\bar{V}$, we see that $\bar{V}\bar{\gamma}$ is also a Gale transform of vert P (and of vert P'), which shows that $\bar{\gamma}$ is induced by a (non-singular) linear transformation of lin $\bar{V}$. Conversely, if a permutation $\bar{\gamma}$ of $\bar{V}$ is induced by a linear transformation of lin $\bar{V}$, then, because of the symmetry between a set of points and its Gale transform, the corresponding permutation γ of vert P induces an affine symmetry of P.

From these facts we deduce that there is a one-to-one correspondence between the affine symmetries of P and the permutations of $\bar{V}$ which induce linear transformations of lin $\bar{V}$. Now each affine symmetry of P leaves fixed the centroid of the vertices of P, and so, after a suitable translation, the affine symmetry group of P is equivalent to a finite linear group. As is well-known [**7**], such a finite group of linear transformations is linearly equivalent to an orthogonal group, and we conclude that there is a polytope P_1 affinely equivalent to P, such that every affine symmetry of P_1 is a congruent transformation, that is, a symmetry of P_1. In an exactly similar way, there is a (new) Gale transform $\bar{V}_1$ of vert P (and hence of vert P_1 also) such that every linear transformation induced by a permutation of $\bar{V}_1$ is orthogonal. We have, in a sense, maximized the symmetry groups of both the polytope P and the Gale transform $\bar{V}$. Thus we have proved the following.

(10) THEOREM. *Let P be any convex polytope. Then there exists a Gale transform $\bar{V}$ of $V =$ vert P such that every symmetry γ of P corresponds to a symmetry $\bar{\gamma}$ of $\bar{V}$, in the sense that γ and $\bar{\gamma}$ induce the same permutations of V and $\bar{V}$, respectively.*

From now on we shall adopt the convention that whenever a Gale transform of vert P has the property stated in the theorem, we shall refer to it as *the* Gale transform of vert P (even though it is not, in general, unique). Notice that several elements of $S(P)$, the symmetry group of P, may correspond to the same element of $S(\bar{V})$, the symmetry group of $\bar{V}$. For example, in the case of the octahedron illustrated in Figure 1, both the identity and reflection in the plane containing the vertices v_3, v_4, v_5, and v_6 (corresponding to the transposition $\bar{v}_1 \leftrightarrow \bar{v}_2$ in $\bar{V}$) induce the identity transformation of lin $\bar{V}$. This motivates the following definition.

(11) *Definition.* If a permutation of the points of $\bar{V}$ induced by a symmetry of P corresponds to the identity transformation of lin $\bar{V}$, then this permutation is called an *intrinsic* symmetry of $\bar{V}$.

The set of intrinsic symmetries of $\bar{V}$ forms a group isomorphic to a normal subgroup $I(P)$ of $S(P)$, and

$$S(P)/I(P) \cong S(\bar{V}).$$

In words, if $S(\bar{V})$ is the group of symmetries of the set of points of $\bar{V}$ (each point being counted with appropriate multiplicity), then every symmetry of P is represented by an element of $S(\bar{V})$ modulo an intrinsic symmetry of $\bar{V}$.

The symmetries of $I(P)$ are easily characterized using the Converse (7) of Theorem (5). For suppose that $\bar{V}$ contains v points in $v - d - 1$ dimensions, and that two of these points $\bar{w}$ and $\bar{w}'$ coincide. Then the intrinsic symmetry of $\bar{V}$ which transposes $\bar{w}$ and $\bar{w}'$ corresponds to an axis of symmetry $\bar{A}$ of dimension $v - d - 1$ containing $v - 2$ points of $\bar{V}$. This, in turn, corresponds to an axis of symmetry A of P of dimension

$$\tfrac{1}{2}(v + v - 2) - (v - d - 1) - 1 = d - 1,$$

and so P has a hyperplane of symmetry containing all but two of the vertices of P. It is clear also that to the reflection in such a hyperplane of symmetry of P corresponds an intrinsic symmetry of $\bar{V}$, and therefore $I(P)$ is generated by the reflections in such hyperplanes.

Referring to the example of the octahedron, $I(P)$ is generated by the reflections in three planes, each of which contains four vertices. In this case, $I(P)$ is an abelian group of order 8, $S(P)$ is a group of order 48, and $S(\bar{V})$ is the dihedral group of order 6. We also remark that the axis of symmetry $\bar{A}$ of $\bar{V}$ marked in Figure 1 corresponds to four axes of symmetry of P. The reflections in these axes differ by elements of $I(P)$; this situation occurs generally, and it is not a particular feature of the example.

4. The number of axes of symmetry of a polytope. In enumerating axi-symmetric polytopes, it is necessary to determine the number of axes of symmetry a polytope can have. We shall investigate this problem here.

Let A_1 and A_2 be two axes of symmetry of a d-polytope P. If there exists no symmetry γ of P such that $A_1\gamma = A_2$, then A_1 and A_2 will be called *essentially distinct*. More generally, the word *essentially* will be used in this context to mean "to within a symmetry of P". For example, an axis of symmetry A of P of dimension a is called *essentially unique* if any other a-dimensional axis of symmetry of P is equivalent to A under a symmetry of P. The remarks of the last section show that to each axis of symmetry $\bar{A}$ of $\bar{V}$, containing points of $\bar{V}$ only in coincident pairs, corresponds essentially one axis of symmetry of P containing no vertices of P (for such axes differ by, at most, symmetries of $I(P)$).

We begin with some easy results that can be obtained directly, without the use of Gale transforms. Let P be a d-polytope. It is obvious that P can have at most one centre of symmetry. If P has hyperplanes of symmetry, then the finite group generated by the corresponding reflections must, in fact, be generated by at most d of them [**1**]. Thus we have proved the following.

(12) THEOREM. *A d-polytope has at most d essentially distinct hyperplanes of symmetry.*

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If P has lines of symmetry, then an analogous result holds. The finite group G generated by the reflections in these lines may or may not contain the central reflection -1. If it does, put $G^* = G$, and if it does not, then write G^* for the direct product of G with the cyclic group of order 2 generated by -1. Now G^* contains a subgroup H generated by reflections in hyperplanes, namely, the products of the reflections in the lines of symmetry of P with -1. These reflections are in hyperplanes perpendicular to the lines of symmetry. As before, H is generated by at most d of the reflections, and so there are at most d lines of symmetry which are essentially distinct with respect to the group G^*. As these will also be distinct with respect to G, we have proved the following.

(13) THEOREM. *A d-polytope P has at most d essentially distinct lines of symmetry.*

In our enumeration of axi-symmetric polytopes, we shall be particularly interested in the cases where no vertices of the polytope lie on the axes. Here we can obtain rather stronger results. The first of these refines Theorem (12). Suppose that the d-polytope P has hyperplanes of symmetry containing no vertices, and that the finite group generated by the reflections in these hyperplanes is generated by r of them. If a point w lies on none of these hyperplanes, and so is not fixed by any operation of the group, then the orbit of w is r-dimensional, and, because the order of the group is at least 2^r (with equality if and only if the generating reflections commute in pairs), the orbit contains at least 2^r points.

The different orbits into which the vertices of P fall thus give rise to parallel r-dimensional subspaces of E^d, each containing at least 2^r points. Consider the orthogonal projection of these subspaces onto a $(d-r)$-dimensional subspace of E^d orthogonal to them. Since P is d-dimensional, the projection of the different orbits must be $(d-r)$-dimensional, and so there are at least $d-r+1$ distinct orbits. Thus, if P has v vertices, then

$$v \geqq v(d, r) = 2^r(d-r+1).$$

Since

$$v(d, r) - v(d, r-1) = 2^{r-1}(d-r), \qquad r = 1, \ldots, d,$$

it follows that for $0 \leqq r < s \leqq d$,

$$v(d, r) \leqq v(d, s),$$

with strict inequality unless $r = d-1$, $s = d$.

This has the following implication. If $d \geqq 3$, and we put $r = 2$, then the minimal number of vertices is

$$v(d, 2) = 4(d-1).$$

We may express this as follows.

(14) THEOREM. *A d-polytope ($d \geqq 3$) with $v < 4(d-1)$ vertices has at most one hyperplane of symmetry containing no vertices.*

Less obvious results of a similar nature can be obtained by considering Gale transforms. As in the last section, we choose the Gale transform $\bar{V}$ of the set of vertices of the polytope P in such a way that every symmetry of P corresponds to a symmetry of $\bar{V}$ (possibly intrinsic), and vice-versa.

We restrict our attention to axes of symmetry A of P containing no vertices, so that all points of $\bar{V}$ on the corresponding axis of symmetry $\bar{A}$ must occur in coincident pairs. If $\dim A = a$ and P has $2n$ vertices, then $\dim \bar{A} = b = n - a - 1$. We now see what happens if b takes certain values; firstly, those corresponding to the special cases discussed in Corollaries (8) and (9) of Theorem (5). If $b = 0$ or $2n - d - 1$ (the dimension of $\bar{V}$), then $\bar{A}$ is unique, and gives rise to an essentially unique axis of symmetry A of P. This proves the following.

(15) THEOREM. *Let P be a d-polytope with $2n$ vertices. If*

$$a = n - 1 \quad or \quad a = d - n,$$

then P has essentially at most one a-dimensional axis of symmetry containing no vertices.

Similarly, if $b = 1$ or $2n - d - 2$ ($= \dim \bar{V} - 1$), then (as in the case of polytopes discussed before Theorem (12)), there is a corresponding finite group generated by at most $2n - d - 1$ reflections. The axes of symmetry of P corresponding to each line or hyperplane of symmetry of $\bar{V}$ again differ only by symmetries of P, and so we have proved the following.

(16) THEOREM. *Let P be a d-polytope with $2n$ vertices. If*

$$a = n - 2 \quad or \quad a = d - n + 1,$$

then P has at most $2n - d - 1$ essentially distinct a-dimensional axes of symmetry containing no vertices.

To illustrate some of the ideas we have discussed in this section, let us consider another example. Let P be the 5-polytope which is the join of two squares [**4**, p. 119]: it is constructed by taking the convex hull of two squares, one in each of two independent planes (2-dimensional subspaces) of E^5. The coordinates of its eight vertices may be taken to be

$$v_{\pm,\pm} = (\pm 1, \pm 1, 0, 0, 1), \qquad w_{\pm,\pm} = (0, 0, \pm 1, \pm 1, -1),$$

and the Gale transform $\bar{V}$ (whose dimension is $8 - 5 - 1 = 2$) is depicted in Figure 2. It consists of eight points forming four coincident pairs, as shown.

Since the origin o is the centroid of the vertices of P, any axis of symmetry of P must pass through o. The various axes of symmetry of P can be classified by considering the easier problem of finding the axes of symmetry of the Gale transform $\bar{V}$, together with the possible sets of fixed points on these axes. $\bar{V}$ has four essentially distinct axes of symmetry: the origin $\{o\}$ alone, the whole plane lin $\bar{V}$, and the two lines $\bar{A}$ and $\bar{B}$ marked in Figure 2.

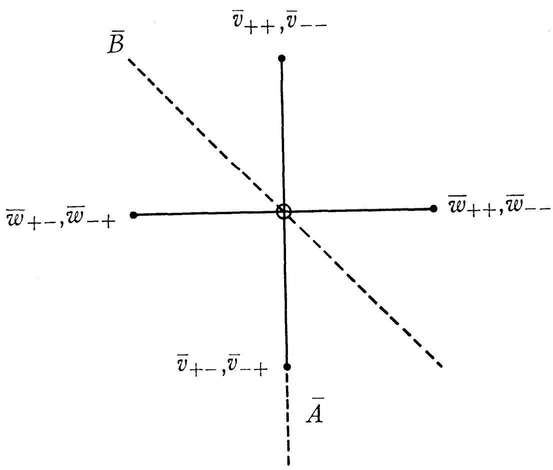

FIGURE 2

The central reflection in o leaves no points of $\bar{V}$ fixed, and so corresponds to an (essentially unique) 3-dimensional axis of symmetry of P, for example, $x_1 + x_3 = 0$. This contains no vertices of P.

The whole plane lin $\bar{V}$ corresponds to four essentially distinct axes of symmetry of P, corresponding to the cases where we consider 0, 1, 2 or 3 of the coincident pairs of points to be fixed. These axes are of dimension 1, 2, 3, and 4, respectively, and the line of symmetry is unique.

The axis of symmetry $\bar{A}$ corresponds to three essentially distinct axes of symmetry of P, corresponding to the cases where we take 0, 1 or 2 of the coincident pairs of points on $\bar{A}$ to be fixed. These axes are of dimension 2, 3, and 4, respectively.

The axis of symmetry $\bar{B}$ contains no fixed points, and corresponds to essentially one 2-dimensional axis of symmetry of P, which contains no vertices of P.

Summarizing this information, we see that P has no centre of symmetry; a unique line of symmetry, which contains no vertices of P (Theorem (15) and also Theorem (13)); three essentially distinct 2-dimensional axes of symmetry, of which two contain no vertices of P (Theorem (16)); three essentially distinct 3-dimensional axes of symmetry, of which just one contains no vertices of P (Theorem (15)), and two essentially distinct hyperplanes of symmetry, both containing vertices of P.

5. Enumeration of polytopes with an axis of symmetry of large dimension. This, and the following three sections, are concerned with enumeration problems, that is, the determination of the number $c^*(2n, d, a)$ defined in the introduction. We begin by discussing three cases in which the axes of the polytopes have dimension $a = d - 1$ or $d - 2$.

(17) $$c^*(2(a + 1), a + 1, a) = 1.$$

We need to consider polytopes with a hyperplane of symmetry, each of whose axis figures is the set of vertices of an a-simplex. Such a polytope therefore has $a + 1$ parallel edges bisected orthogonally by the hyperplane, and so is of the same combinatorial type as a simplicial prism. It is therefore unique, and we obtain the value 1 as stated.

(18) $$c^*(2(a + 2), a + 1, a) = [\tfrac{1}{4}a(3a + 2)].$$

Again we are considering polytopes with a hyperplane of symmetry, but here the number of vertices exceeds the minimum given by (4). (In fact, this is the only case in which we have been able to solve the enumeration problem with more than the minimum number of vertices.) The axis figure is a set of $a + 2$ distinct points in a dimensions. By Radon's Theorem [**6**], there is a unique partition of these $a + 2$ points into three disjoint sets, say $\{u_0, \ldots, u_r\}$, $\{v_0, \ldots, v_s\}$, $\{w_0, \ldots, w_t\}$, where r, s, and t are some non-negative integers with $r + s + t = a - 1$, with the properties:

(i) $\operatorname{conv}\{u_0, \ldots, u_r\} \cap \operatorname{conv}\{v_0, \ldots, v_s\}$ is a single point, which we shall denote by x, and

(ii) the points $w_0, \ldots, w_t$ are affinely independent of $u_0, \ldots, u_r, v_0, \ldots, v_s$.

Each of the two sets $\{u_0, \ldots, u_r\}$, $\{v_0, \ldots, v_s\}$ is also affinely independent, and, since the $a + 2$ points are distinct, we cannot have $r = s = 0$.

Since the subsets $\{u_0, \ldots, u_r, v_0, \ldots, v_s\}$ and $\{w_0, \ldots, w_t\}$ of V_A lie in independent affine subspaces of A, we are led to consider the following more general situation. Let Q and R be two polytopes in E^{a+1} having the same hyperplane A of E^{a+1} as an axis of symmetry, and let the respective axis figures V_A and W_A lie in independent affine subspaces of A. Without loss of generality, we may assume that the polytope $S = \operatorname{conv}(Q \cup R)$ has dimension $a + 1$. Clearly S has A as a hyperplane of symmetry and the corresponding axis figure S_A is the union of the axis figures of Q and R. Let D be a face of S, so that there is a hyperplane H of E^{a+1} such that $D = H \cap S$. Then $F = H \cap Q$ and $G = H \cap R$ are faces of Q and R, respectively, allowing the possibility that either F or G may be empty. If $D = D'$ (the reflection of D in A), then H must contain a line perpendicular to A, and so $F = F'$ and $G = G'$. If $D \neq D'$, then D must lie in one of the closed half-spaces, A^+ say, into which A divides E^{a+1}, and this implies that F and G also lie in A^+. Conversely, if F and G are faces of Q and R such that $F = F'$ and $G = G'$, then, since the axis figures of Q and R lie in independent subspaces, $D = \operatorname{conv}(F \cup G)$ is a face of S such that $D = D'$. On the other hand, if F and G are faces of Q and R both lying in A^+, then $D = \operatorname{conv}(F \cup G)$ is again a face of S.

We have thus obtained a complete description of the faces of $S = \operatorname{conv}(Q \cup R)$, and from this we deduce that the combinatorial type of S does not depend on the relative positions of the axis figures of Q and R in A (so long as they lie in independent affine subspaces), nor on the choices of Q and R in their respective combinatorial equivalence classes. In other words, the construction we have just described is one of combinatorial type.

Returning to the particular problem with which we are concerned, let the vertices of P be denoted by $u_0^{\pm}, \ldots, u_r^{\pm}, v_0^{\pm}, \ldots, v_s^{\pm}, w_0^{\pm}, \ldots, w_t^{\pm}$ corresponding to the points of the axis figure, where those points with superscript $+$ lie in the half-space A^+. The two polytopes

$$Q = \operatorname{conv}\{u_0^{\pm}, \ldots, u_r^{\pm}, v_0^{\pm}, \ldots, v_s^{\pm}\} \quad \text{and} \quad R = \operatorname{conv}\{w_0^{\pm}, \ldots, w_t^{\pm}\}$$

satisfy the conditions described above, and so can be considered separately. Since the axis figure $R_A = \{w_0, \ldots, w_t\}$ is the set of vertices of a t-simplex, as in (6), the polytope R is combinatorially equivalent to a $(t + 1)$-dimensional simplicial prism.

Consider now the $(r + s + 1)$-polytope Q. As before, let x denote the common point of $\operatorname{conv}\{u_0, \ldots, u_r\}$ and $\operatorname{conv}\{v_0, \ldots, v_s\}$, and let the line through x perpendicular to A meet $U^+ = \operatorname{conv}\{u_0^+, \ldots, u_r^+\}$ in y, and $V^+ = \{v_0^+, \ldots, v_s^+\}$ in z. Notice that each of U^+ and V^+ is a simplex, so that each of

$$U = \operatorname{conv}\{u_0^{\pm}, \ldots, u_r^{\pm}\} \quad \text{and} \quad V = \operatorname{conv}\{v_0^{\pm}, \ldots, v_s^{\pm}\}$$

is combinatorially equivalent to a simplicial prism. There are three possibilities:

(a) $y = z$;
(b) y is beyond z with respect to A;
(c) y is beneath z with respect to A.

Reversing the rôles of U and V, we see that (b) and (c) are equivalent, except that if $r = 0$, only (b) (and not (a) or (c)) is possible.

We now investigate the set of faces of $Q = \operatorname{conv}(U \cup V)$ in order to show that its combinatorial type depends only on r, s, and the relative positions of y and z as described by (a), (b), or (c).

If (a) holds, then clearly Q is of the combinatorial type of the prism whose basis is the simplicial polytope $T^{r,s} = \operatorname{conv}\{u_0, \ldots, u_r, v_0, \ldots, v_s\}$. The combinatorial type of $T^{r,s}$, and therefore the combinatorial type of Q, depends only on the integers r and s [**2**, 6.1].

If (b) holds, so that $r \geqq 0, s \geqq 1$, the faces of Q are of two types. If such a face meets the axis figure $\{u_0, \ldots, u_r, v_0, \ldots, v_s\}$, then it is a prism on one of the proper (simplicial) faces of $T^{r,s}$. If not, the face is a simplex, namely the convex hull of a face of U^+ (or U^-) and a proper face of V^+ (or V^- respectively). Conversely, any proper face of $T^{r,s}$, of a face of U^+ and a proper face of V^+, give rise to a face of Q in this way.

We deduce that for given $k \geqq 1$, the number of distinct combinatorial types with $r + s = k$ is, in the various cases,

$$\text{(a)} \quad [\tfrac{1}{2}k], \qquad \text{(b) and (c)} \quad \sum_{r \geqq 0, s \geqq 1} 1 = k.$$

Thus the overall total (remembering that $t = -1$ is possible) is

$$\sum_{k=1}^{a} [\tfrac{3}{2}k] = [\tfrac{1}{4}a(3a + 2)].$$

Since, by Theorem (14), when $a \geqq 3$ the polytope can have at most one hyperplane of symmetry, we see immediately that the above description of P, in terms of its axis figure, is unique, and so the total we have obtained is the number of possible combinatorial types. For $a = 1$ and 2, it is trivial to check the result numerically, and so the assertion (18) is true in all cases.

$$(19)\quad c^*(2(a+1), a+2, a) = \frac{1}{2(a+1)} \sum_{s|a+1} 2^s \phi\left(\frac{a+1}{s}\right) - 2 + \begin{cases} 2^{\frac{1}{2}a}, & a \text{ even}, \\ 3\cdot 2^{\frac{1}{2}(a-3)}, & a \text{ odd}, \end{cases}$$

where the summation is over all positive integer divisors s of $a + 1$, and ϕ is Euler's function: $\phi(r)$ is the number of positive integers less than r and prime to r.

Let P be an $(a + 2)$-polytope with $2(a + 1)$ vertices and an a-dimensional axis of symmetry. The axis figure V_A is the set of vertices of a simplex, and so P is completely determined by its coaxis figure V_C. V_C is a 2-dimensional centrally symmetric set of $2(a + 1)$ points, none of which coincide with o. We need to investigate which coaxis figures correspond to the same, or distinct, combinatorial types of polytope.

It is convenient, and interesting, to discuss a more general problem: the relationship between the combinatorial type of an $(a + k)$-polytope P (with $2(a + 1)$ vertices and an a-dimensional axis of symmetry) and its k-dimensional coaxis figure, where $1 \leqq k \leqq a + 1$. The Gale transform $\bar{V}$ of the set of vertices of such a polytope P is, by Corollary (8) of § 2, a centrally symmetric set of $2(a + 1)$ points in E^{a+1-k}. First we notice that the combinatorial type of P does not depend on the distance of the points of the coaxis figure V_C from the origin o. For, if we write the vertices of P in the form (1), and then replace $\pm y_j$ by $\pm \alpha_j y_j$ ($\alpha_j > 0$, $j = 1, \ldots, a + 1$), remembering that the Gale transform is also a c.s. transform of V_C, we see that in $\bar{V}$ we must replace $\pm \bar{y}_j$ by $\pm \alpha_j^{-1} \bar{y}_j$ ($j = 1, \ldots, a + 1$). This new Gale transform is isomorphic to $\bar{V}$ [**2**, 5.4.5] and so the corresponding polytopes are combinatorially equivalent. We may therefore assume, without loss of generality, that the points of the coaxis figure V_C lie on the unit sphere in lin V_C.

Now let us consider which sets of vertices of P determine faces of P. If $\bar{y}_j$ is any point of $\bar{V}$, then

$$o \in \text{relint conv}\{\bar{y}_j, -\bar{y}_j\}$$

(the relative interior of the convex hull), from which it follows by [**2**, 5.4.1], that the subset vert $P \backslash \{w_j, w_j'\}$ of vert P is the set of vertices of a face of P. Since the intersection of any two faces of P is a face of P, the set of faces of P, and therefore the combinatorial type of P, is completely determined by those faces of P which contain at least one of every pair of vertices symmetrical about A. Let F be such a face, and for convenience of notation suppose that F contains all the vertices of P except $w_1, \ldots, w_r$. By [**2**, 5.4.1] again, this implies that

$$o \in \text{relint conv}\{\bar{y}_1, \ldots, \bar{y}_r\},$$

that is, there are positive numbers $\bar{\nu}_1, \ldots, \bar{\nu}_r$ such that

$$\bar{\nu}_1\bar{y}_1 + \ldots + \bar{\nu}_r\bar{y}_r = o.$$

Thus the vector $(\bar{\nu}_1, \ldots, \bar{\nu}_r, 0, \ldots, 0)$ is a linear dependence of the points $\bar{V}^+ = \{\bar{y}_1, \ldots, \bar{y}_{a+1}\}$. Because of the symmetry between a centrally symmetric set of points and a c.s. transform [**3**, Chapter 4], this implies that there is a vector c of lin V_C such that

$$\begin{aligned} \langle c, y_j \rangle &= \bar{\nu}_j > 0, && j = 1, \ldots, r, \\ \langle c, y_j \rangle &= 0, && j = r + 1, \ldots, a + 1, \\ \langle c, -y_j \rangle &\leqq 0, && j = 1, \ldots, a + 1. \end{aligned}$$

(Compare the proof of [**2**, 5.4.1] or [**3**, 2(13)].) In other words, all the points of V_C except $y_1, \ldots, y_r$ lie in the closed half-space $\{y | \langle c, y \rangle \leqq 0\}$ of lin V_C. This argument is reversible, leading to the following conclusion.

(20) THEOREM. *Let P be an $(a + k)$-polytope $(1 \leqq k \leqq a + 1)$ in E^{a+k}, with an a-dimensional axis of symmetry A, and $a + 1$ pairs of vertices symmetric with respect to A. Then any set of vertices of P, which contains at least one vertex of every pair, is the set of vertices of a proper face of P if and only if it lies in some closed half-space of E^{a+k} whose boundary contains A.*

We now return to the special case $k = 2$. As we have seen, we may suppose that the points of the coaxis figure V_C lie at the ends of a number of diameters of the unit circle in E^2; it is possible for each end of a diameter to contain more than one point of V_C. From the theorem, it follows that we may not separate or coalesce different diameters, nor move diameters with different end multiplicities across one another, without altering the combinatorial type. However, other modifications to V_C are permissible and these allow us to assume that the diameters of the figure are evenly spaced. In Figure 3 we give examples of coaxis figures corresponding to different combinatorial types in the cases $a = 1, 2$, and 3.

The above discussion shows that $c^*(2(a + 1), a + 2, a)$ is precisely the number of ways of arranging $2(a + 1)$ points, in a centrally symmetric manner, at the ends of r diameters of the unit circle $(r \geqq 2)$, each end carrying at least one point. Only *essentially* distinct arrangements are to be counted, that is, arrangements which are not equivalent under the symmetry group of the unit circle. This number may be determined using Pólya's Theorem [**5**]. We do not reproduce details of the calculation, but assert that we are led to the value of

$$c^*(2(a + 1), a + 2, a)$$

stated above.

6. Enumeration of polytopes with an axis of symmetry of small dimension.

Here we are concerned with the determination of $c^*(v, d, a)$, where $a \leqq \frac{1}{2}(d - 1)$ and $v = 2(d - a)$. We shall prove a general enumeration theorem

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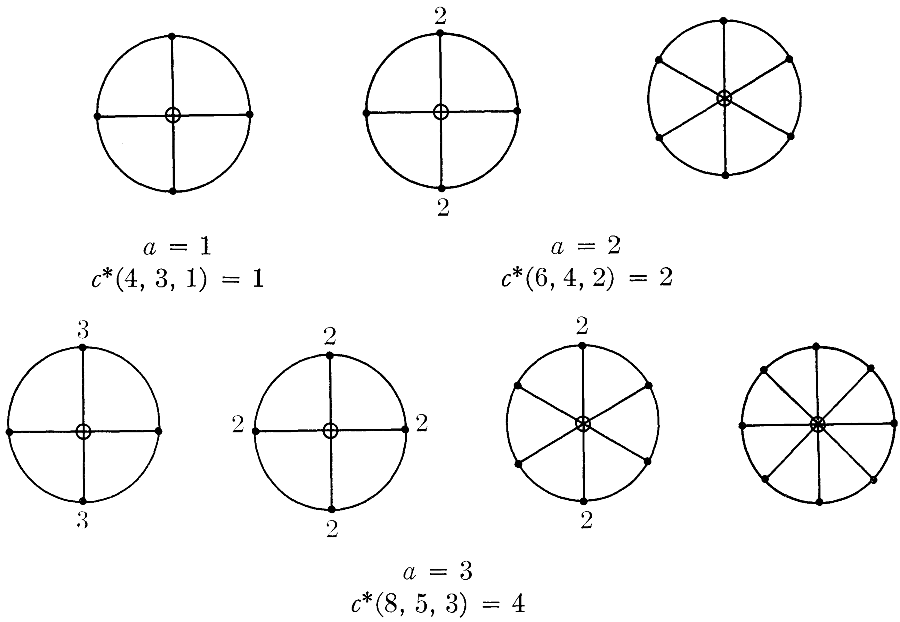

FIGURE 3

for polytopes of this type, from which explicit numerical values of $c^*(v, d, a)$ can be found in the cases $a = 0$ and $a = 1$.

Let P be a d-polytope with an a-dimensional axis of symmetry A $(a \leqq \frac{1}{2}(d-1))$, and $v = 2(d-a)$ vertices paired with respect to A. The axis figure consists of $d - a$ points in A, and $P_A = \text{conv } V_A$ is an a-polytope in A. Every pair of vertices $\{w, w'\}$ of P with respect to A is *associated* with a point $w'' = \frac{1}{2}(w + w')$ of V_A. If $w'' \in \text{relint } P_A$, then it is clear that P is a bipyramid with opposite vertices w and w', and its basis is the $(d-1)$-polytope $P_1 = \text{conv}(V \backslash \{w, w'\})$, where as usual we write $V = \text{vert } P$. In this case the combinatorial type of P is completely determined by the combinatorial type of P_1, and hence, by an obvious extension, it is completely determined by

(I) the number t of points of V_A lying in relint P_A, and

(II) the $(d-t)$-polytope whose a-dimensional axis figure is $V_A \cap \text{bd } P_A$ (bd P_A is the boundary of P_A).

For simplicity therefore, in the following, we may restrict our attention to those polytopes P for which $V_A \subset \text{bd } P_A$. Let us consider the faces of such a polytope. Let H be a supporting hyperplane of P, and write $H_A = H \cap A$. It is clear that if H_A contains a point w'' of V_A, then H contains both of the pair of vertices w and w' of P associated with w''. Thus the following possibilities arise:

(i) H_A supports P_A, $H_A \cap P_A = F_A$ which is a j-dimensional face of P_A, and H contains no vertices of P other than those pairs associated with points of $V_A \cap F_A$.

If $\operatorname{card}(V_A \cap F_A) = k$, then the face $F = H \cap P$ is easily seen to be the $(j + k)$-polytope with $2k$ vertices whose axis figure is $V_A \cap F_A$. Such a face we shall say is associated with the face F_A of P_A.

(ii) H_A does not support P_A.

If H_A contains s vertices of P, then because these points are affinely independent, $H \cap P$ is an $(s - 1)$-simplex. Notice that, of these s vertices, at most one can belong to each pair of vertices of P, and conversely, if we are given any s vertices of P with at most one belonging to each pair, then the convex hull of these vertices is a face of P.

(iii) H_A supports P_A, $H_A \cap P_A = F_A$ which is a j-dimensional face of P_A, and H contains s vertices of P other than those associated with the points $V_A \cap F_A$.

In this case, $H \cap P$ is an s-fold pyramid whose basis is the $(j + k)$-face F of P associated with the face F_A of P_A as described in (i). Again, of the s vertices, at most one can belong to any pair of vertices of P, and conversely, if we are given any such selection of s vertices which are not associated with the points $V_A \cap F_A$, then the convex hull of these vertices and F is a face of P.

From this description of the set of faces of P it is clear that the combinatorial type of P is determined completely by the type and arrangement of the faces F of P that are associated with the faces F_A of P_A as described in (i). Since these have axis figures of dimension strictly less than a, we can use an obvious inductive argument on a.

If $a = 1$, the problem is particularly simple: the $d - 1$ points of V_A lie on the line segment P_A. If r points of V_A coincide with one end point (vertex) of P_A, and s points of V_A with the other end, then $d - r - s - 1$ points of V_A will lie in relint P_A. In this case P is a $(d - r - s - 1)$-fold bipyramid whose basis is the free join of an r-cross-polytope and s-cross-polytope, see [**4**, p. 119].

The above discussion motivates the introduction of the concept of a labelled polytope. An a-polytope Q is said to be *labelled* if, with every face of Q, including Q itself but excluding the empty face $\emptyset$, is associated an integer, positive in the case of the vertices of Q, and non-negative in the case of the other faces. Q is said to be *v-labelled* if the sum of the integer labels is v. Two labelled polytopes Q_1 and Q_2 are combinatorially equivalent if there is a one-to-one inclusion preserving correspondence between the set of faces of Q_1 and the set of faces of Q_2, for which corresponding faces carry the same label. Otherwise, Q_1 and Q_2 will be called *distinct*. Using this terminology, we can now state the main enumeration theorem for polytopes with axes of symmetry of small dimension.

(21) THEOREM. *The number of combinatorial types of d-polytopes with an a-dimensional axis of symmetry* ($a \leqq \frac{1}{2}(d - 1)$), *and* $2(d - a)$ *vertices, none of which lie on the axis, is equal to the number of distinct* $(d - a)$*-labelled a-polytopes.*

The only cases in which we have been able to apply this theorem to yield numerical results are $a = 0$ and $a = 1$. The former case is trivial, for there is only one type of centrally symmetric d-polytope with $2d$ vertices, namely the cross-polytope X^d. Hence:

$$c^*(2d, d, 0) = 1. \tag{22}$$

If $a = 1$, then by the theorem, $c^*(2(d-1), d, 1)$ is the number of distinct $(d-1)$-labelled 1-polytopes. This is, using r and s as above, the number of integer solutions of the inequalities

$$1 \leqq r \leqq s, \qquad r + s \leqq d - 1.$$

It follows immediately that

$$c^*(2(d-1), d, 1) = [\tfrac{1}{4}(d-1)^2]. \tag{23}$$

7. Enumeration of polytopes with an axis of symmetry of dimension near $\frac{1}{2}d$. If the axis of symmetry of a polytope P has dimension near $\frac{1}{2}d$, and the number of vertices is minimal, then the corresponding Gale transform will be of small dimension. When this dimension is 0, 1 or 2, the enumeration problem is soluble.

$$c^*(2(a+1), 2a+1, a) = 1. \tag{24}$$

The axis figure is the set of vertices of an a-simplex, and the coaxis figure is the set of vertices of an $(a+1)$-cross-polytope. Hence the Gale transform consists of $2(a+1)$ points coinciding with the origin, and the corresponding polytope is unique, namely, the $(2a+1)$-simplex.

$$c^*(2(a+1), 2a, a) = a. \tag{25}$$

Any Gale transform of the set of vertices of a $2a$-polytope P with $2(a+1)$ vertices is 1-dimensional, and if P has an a-dimensional axis of symmetry containing no vertices, then by Corollary (8) the Gale transform is centrally symmetric. Hence it consists of $2(a+1)$ points on a line arranged symmetrically about the origin o. By [**2**, 5.4.2], there must be at least two points on each side of o, and so, to within isomorphism of the Gale transforms [**2**, 5.4.5], their number is a. Hence this is the value of $c^*(2(a+1), 2a, a)$.

$$c^*(2(a+1), 2a-1, a) = \sum_{r=1}^{a+1} \frac{1}{2r} \sum_{s\mid r} 2^s \phi\left(\frac{r}{s}\right) - 3a - \frac{9}{2} + \begin{cases} 7 \cdot 2^{\frac{1}{2}a-1}, & a \text{ even}, \\ 5 \cdot 2^{\frac{1}{2}(a-1)}, & a \text{ odd}. \end{cases} \tag{26}$$

Any Gale transform of the set of vertices of a $(2a-1)$-polytope P with $2(a+1)$ vertices is 2-dimensional, and in this case, by Corollary (8), it is also centrally symmetric. Thus $c^*(2(a+1), 2a-1, a)$ is the number of non-isomorphic, centrally symmetric, 2-dimensional Gale transforms. Some of the points may coincide with the origin (in pairs), and hence $c^*(2(a+1), 2a-1, a)$ is the number of essentially different ways of labelling the vertices of a $2n$-gon (for each $n \geqq 2$) in a centrally symmetric way, so that each label is at least 1,

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and the sum of the labels is at most $2(a + 1)$. (If $n = 2$, then the labels must be at least 2, since otherwise the diagram does not represent a polytope [**2**, 5.4.2].) This, in turn, is equal to the number of essentially different ways of labelling the vertices of an n-gon ($n \geqq 2$), so that each label is at least one and the sum of the labels is at most $a + 1$. (Again, each label must be at least 2 if $n = 2$.) The number of such arrangements can be found using Pólya's Theorem [**5**]. The details of the calculation are omitted, but it yields the number stated above.

(27) $$c^*(2(a + 2), 2a + 2, a) = [\tfrac{1}{4}(a + 2)^2].$$

If P is a $(2a + 2)$-polytope with $2(a + 2)$ vertices, then any Gale transform of the set of its vertices is 1-dimensional. The fact that P has an a-dimensional axis of symmetry containing none of the vertices implies, by Corollary (9), that the Gale transform consists of $a + 2$ pairs of coincident points. Considering the case where r pairs lie on one side of the origin, and s pairs on the other, we see that the number of non-isomorphic Gale transforms of the required type is equal to the number of integer solutions of the inequalities,

$$1 \leqq r \leqq s, \qquad r + s \leqq a + 2.$$

This number is $[\frac{1}{4}(a + 2)^2]$, yielding the stated result.

(28) $$c^*(2(a + 3), 2a + 3, a) = c(a + 3, a) + \chi(a),$$

where

$$\chi(a) = \frac{1}{2}\binom{a+5}{5} + \frac{5}{8}\binom{a+1}{3} + \frac{1}{2} - \frac{1}{8}(a-3)\left[\frac{1}{2}(3a+1)\right].$$

Let P be a $(2a + 3)$-polytope with $2(a + 3)$ vertices paired with respect to an a-dimensional axis of symmetry. Then any Gale transform $\bar{V}$ of vert P is 2-dimensional, and consists of $a + 3$ pairs of coincident points. Let us consider the diagram $\bar{W}$ which is obtained from $\bar{V}$ by halving the multiplicity of each point of $\bar{V}$. If every open half-plane in lin $\bar{V}$ with o on its boundary contains at least two points of $\bar{W}$, then $\bar{W}$ is a Gale transform of the set of vertices of an a-polytope with $a + 3$ vertices [**2**, 5.4.2]. If this condition does not hold, then we shall call $\bar{W}$ an *exceptional* transform. Writing $\chi(a)$ for the number of non-isomorphic exceptional transforms, we see that $c^*(2(a + 3), 2a + 3, a) = c(a + 3, a) + \chi(a)$, as stated.

We must now determine $\chi(a)$. Remembering that in the exceptional transforms, each open half-plane with o on its boundary contains at least one point of $\bar{W}$, we see that the exceptional transforms must be of the form shown in Figure 4, where the multiplicities m, n, p, q, r, and s satisfy the equality and inequalities stated.

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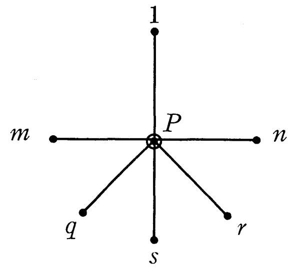


$m + n + p + q + r + s = a + 2, m + q \geqq 1, n + r \geqq 1, q + r + s \geqq 1;$
$m = 0$ if and only if $n = 0$.

FIGURE 4

Noticing that the diagrams in Figure 5 are isomorphic,

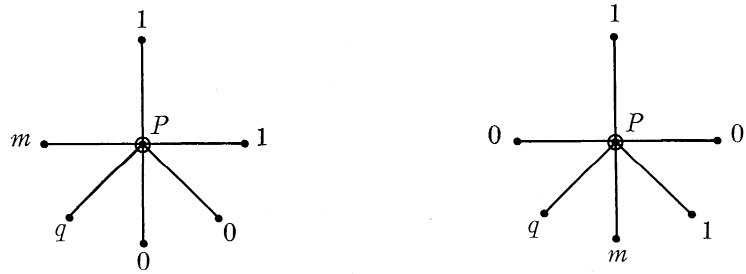


FIGURE 5

it is a straightforward matter to enumerate the possibilities. We do not reproduce the calculations here, but merely assert that they lead to the value of $\chi(a)$ stated above.

8. Enumeration of simplicial polytopes with an axis of symmetry. It is not surprising that enumeration problems can be solved more completely if we restrict ourselves to simplicial polytopes. We shall prove the following.

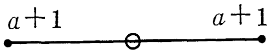
(29) THEOREM.

$$c_s^*(2(a + 1), d, a) = 0 \text{ if } a > \tfrac{1}{2}d,$$
$$c_s^*(2(a + 1), 2a, a) = 1,$$
$$c_s^*(2(d - a), d, a) = \sum_{v \leqq d-a} c_s(v, a) \text{ if } a \leqq \tfrac{1}{2}(d - 1).$$

In each case, the number of vertices is the smallest possible number permitted by the inequality (4).

Proof. Let us consider first the case of axes of symmetry of large dimension, that is, $a \geqq \frac{1}{2}(d - 1)$. As we have already shown (Corollary (8)), a polytope P satisfying this condition corresponds to a centrally symmetric Gale transform $\bar{V}$

of $2a - d + 1$ dimensions. In particular, if $2a - d + 1 \geqq 2$, then P has cofaces consisting of just two vertices (corresponding to any point of $\bar{V}$ distinct from o, and its reflection in o), and so has faces with $2a \geqq d + 1$ vertices. This is inconsistent with the supposition that P is simplicial. Consequently, if $a > \frac{1}{2}d$, then $c_s^*(2(a + 1), d, a) = 0$. On the other hand, if $d = 2a$ or $2a + 1$, then the dimension $2a - d + 1$ of the Gale transform is either 1 or 0. The latter case corresponds to the simplex alone. In the former case we must enumerate centrally symmetric 1-dimensional Gale transforms with $2(a + 1)$ points, none of which coincide with o. Again, up to isomorphism, there is only one possibility, namely:



The corresponding polytope is $T^a \oplus T^a$ (in the notation of [**4**, p. 119]), and we therefore have $c_s^*(2(a + 1), 2a, a) = 1$.

When the axis of symmetry of the polytope P is of small dimension ($a \leqq \frac{1}{2}(d - 1)$), the enumeration can be carried out by a slight modification of Theorem (21) of § 6. The fact that P is simplicial places a number of restrictions on the possible axis figure V_A and the corresponding polytope P_A:

(i) *P_A must be a simplicial polytope.* For if F_A were a non-simplicial proper face of P_A, then the face F of P associated with F_A would not be a simplex.

(ii) *The label attached to each vertex of P_A must be at most* 1, *and so exactly* 1. For a label r at a vertex of P_A would imply that P had an r-cross-polytope as one of its r-faces.

(iii) *Every proper face of P_A, other than a vertex, must carry the label* 0. For if a face F_A of P_A carried a non-zero label, then it is easy to see that the face F of P, associated with F_A, would not be a simplex. The polytope P_A itself can, of course, carry a non-zero label.

The number of distinct $(d - a)$-labelled a-polytopes satisfying these conditions is easily seen to be

$$\sum_{v \leqq d-a} c_s(v, a),$$

and so we obtain the result stated in the theorem.

Actual numerical values can be found only when $d = 2a + 1$, $2a + 2$ or $2a + 3$. These yield:

$$c_s^*(2(a + 1), 2a + 1, a) = 1,$$

$$c_s^*(2(a + 2), 2a + 2, a) = [\tfrac{1}{2}a] + 1,$$

$$c_s^*(2(a + 3), 2a + 3, a) = 2^{[\frac{1}{2}a]} - 1 + \frac{1}{4(a + 3)} \sum_{\substack{h \mid a+3; \\ h \text{ odd}}} \phi(h) 2^{(a+3)/h},$$

the last value being obtained from Perles' formula for $c_s(a + 3, a)$, see [**2**, 6.3].

TABLE. *Summary of results*

The symbols $c^*(v, d, a)$, $c_s^*(v, d, a)$, $c(v, d)$, and $c_s(v, d)$ are defined in § 1.

(A) Axes of large dimension (§ 5).

(17) $c^*(2(a+1), a+1, a) = 1.$

(18) $c^*(2(a+2), a+1, a) = [\frac{1}{4}a(3a+2)].$

(19) $$c^*(2(a+1), a+2, a) = \frac{1}{2(a+1)} \sum_{s|a+1} 2^s \phi\left(\frac{a+1}{s}\right) - 2 + \begin{cases} 2^{\frac{1}{2}a}, & a \text{ even}, \\ 3 \cdot 2^{\frac{1}{2}(a-3)}, & a \text{ odd}. \end{cases}$$

(B) Axes of small dimension (§ 6).

(22) $$c^*(2d, d, 0) = 1.$$

(23) $$c^*(2(d-1), d, 1) = [\tfrac{1}{4}(d-1)^2].$$

(C) Axes of dimension near $\frac{1}{2}d$ (§ 7).

(24) $c^*(2(a+1), 2a+1, a) = 1.$

(25) $c^*(2(a+1), 2a, a) = a.$

(26) $$c^*(2(a+1), 2a-1, a) = \sum_{r=1}^{a+1} \frac{1}{2r} \sum_{s|r} 2^s \phi\left(\frac{r}{s}\right) - 3a - \frac{9}{2} + \begin{cases} 7 \cdot 2^{\frac{1}{2}a-1}, & a \text{ even}, \\ 5 \cdot 2^{\frac{1}{2}(a-1)}, & a \text{ odd}. \end{cases}$$

(27) $c^*(2(a+2), 2a+2, a) = [\frac{1}{4}(a+2)^2].$

(28) $c^*(2(a+3), 2a+3, a) = c(a+3, a) + \chi(a),$

where

$$\chi(a) = \frac{1}{2}\binom{a+5}{5} + \frac{5}{8}\binom{a+1}{3} + \frac{1}{2} - \frac{1}{8}(a-3)\left[\frac{1}{2}(3a+1)\right].$$

(D) Simplicial polytopes (§ 8).

(29) $$\begin{cases} c_s^*(2(a+1), d, a) = 0 \quad \text{if } a > \tfrac{1}{2}d, \\ c_s^*(2(a+1), 2a, a) = 1, \\ c_s^*(2(d-a), d, a) = \sum_{v \leq d-a} c_s(v, a) \quad \text{if } a \leq \tfrac{1}{2}(d-1). \end{cases}$$

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