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REGULAR CONGRUENCES ON AN IDEMPOTENT-REGULAR-SURJECTIVE SEMIGROUP

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Abstract

A semigroup S is called *idempotent-surjective* (respectively, *regular-surjective*) if whenever ρ is a congruence on S and $a\rho$ is idempotent (respectively, regular) in S/ρ, then there is $e \in E_S \cap a\rho$ (respectively, $r \in \mathrm{Reg}(S) \cap a\rho$), where E_S (respectively, $\mathrm{Reg}(S)$) denotes the set of all idempotents (respectively, regular elements) of S. Moreover, a semigroup S is said to be *idempotent-regular-surjective* if it is both idempotent-surjective and regular-surjective. We show that any regular congruence on an idempotent-regular-surjective (respectively, regular-surjective) semigroup is uniquely determined by its kernel and trace (respectively, the set of equivalence classes containing idempotents). Finally, we prove that all structurally regular semigroups are idempotent-regular-surjective.

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1. Introduction and preliminaries

The main result of this work has been very useful in the description of some fundamental congruences on an idempotent-regular-surjective semigroup; see [4].

An element a of a semigroup S is called *regular* if there is $x \in S$ such that $a = axa$, and a semigroup is *regular* if each of its elements is regular. In [3], Edwards introduced the notions of (a) an *idempotent-surjective* and (b) an *eventually regular* semigroup S (that is, for every $a \in S$ there exists a positive integer n such that a^n is regular). The famous Lallement's lemma says that all regular semigroups are idempotent-surjective. In fact, in [3] it has been shown that all eventually regular semigroups have this property. Note that the class of eventually regular semigroups is wide. For example, all finite and all group bound semigroups are eventually regular. On the other hand, in [5] Kopamu introduced the notion of a *structurally regular* semigroup (see below for the definition), and showed that the classes of eventually regular and structurally regular semigroups are incomparable, that is, neither contains the other. Recall that structurally regular semigroups are also idempotent-surjective [6]. In Section 3 we will prove that they are also regular-surjective. (Note that from [3] it follows that any eventually regular semigroup possesses this property.) Further, it is known that any *regular* congruence (for which the quotient semigroup induced by this congruence is regular) on an eventually regular semigroup is uniquely determined by: (i) the set of equivalence classes containing idempotents; (ii) its *kernel* and *trace* (for the definitions, see after Theorem 2.2) [7, 9]. The main aim of this paper is to show that each regular congruence on a regular-surjective semigroup is uniquely determined by the set of equivalence classes containing idempotents, and each regular congruence on an idempotent-regular-surjective semigroup is uniquely determined by its kernel and trace. In particular, the conditions (i), (ii) are valid for any regular congruence on an arbitrary structurally regular semigroup. The conclusion is that these two mentioned incomparable classes of semigroups have common properties concerning regular congruences.

Whenever possible the notation and conventions of Clifford and Preston are used [1, 2]. Let S be a semigroup and $A \subseteq S$. Denote by E_A the set of all *idempotents* of A, that is, $E_A = \{a \in A : a^2 = a\}$, and by $\mathrm{Reg}(A)$ the set of *regular elements* of A (in S), that is, $\mathrm{Reg}(A) = \{a \in A : a \in aSa\}$. Unless otherwise stated, S denotes an arbitrary semigroup. Let $a \in S$. Denote by $V(a)$, $W(a)$ the set of all *inverses* and *weak inverses* of a, respectively, that is, $V(a) = \{x \in S : a = axa,\ x = xax\}$, $W(a) = \{x \in S : x = xax\}$. Recall that if $a \in S$ is regular, say $a = axa$ for some $x \in S$, then $xax \in V(a)$. Also, S is called *regular* if $V(a) \neq \emptyset$ for every $a \in S$, and S is said to be *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$. Clearly, in such a case $xax \in W(a)$. It follows that S is E-inversive if and only if $W(a) \neq \emptyset$ for every a in S (see [8]).

In [5], Kopamu defined a countable family of congruences on a semigroup S, as follows: for each ordered pair of nonnegative integers (m, n), he put

$$\theta_{m,n} = \{(a, b) \in S \times S : \forall x \in S^m, y \in S^n\ [xay = xby]\},$$

and he made the convention that $S^1 = S$ and S^0 denotes the set containing only the empty word. In particular, $\theta_{0,0}$ is the identity relation on S. Recall that a semigroup S is called *structurally regular* if $S/\theta_{m,n}$ is regular for some nonnegative integers m, n [6]. Note that structurally regular semigroups are idempotent-surjective [6, Corollary 3.4]).

Notice that if a semigroup S is regular-surjective and $a \in S$, then from the definition of a Rees congruence on S it follows that SaS has at least one regular element, so SaS contains an idempotent of S, say $e = xay$, where $x, y \in S$. One can easily check that $yex \in W(a)$. Thus every regular-surjective semigroup is E-inversive. Also, by the above each idempotent-surjective semigroup is E-inversive, too.

Let $\mathcal{A}(S)$ denote the set of all reflexive and symmetric (binary) relations on S, $\mathcal{E}(S)$ the set of all equivalences of S and $C(S)$ the set of all (two-sided) congruences on S. Recall that both lattices $\mathcal{E}(S)$, $C(S)$ are complete, where

$$\bigvee\{\rho_i : i \in I\} = \Big(\bigcup\{\rho_i : i \in I\}\Big)T$$

(ρT denotes the *transitive closure* of ρ).

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We now define the two (inclusion-preserving) maps C, C^* of $\mathcal{B}(S)$ into $\mathcal{B}(S)$, where $\mathcal{B}(S)$ denotes the set of all binary relations on S, putting henceforth $\rho \in \mathcal{B}(S)$:

$$\rho C = \{(x, y) \in S \times S : \forall s, t \in S^{(1)} \ [(sxt, syt) \in \rho]\},$$
$$\rho C^* = \{(x, y) \in S \times S : \exists\, u, v \in S,\ s, t \in S^{(1)} \ [x = sut,\ y = svt,\ (u, v) \in \rho]\},$$

where $S^{(1)}$ denotes the monoid obtained from S by adjoining an identity if necessary. Observe that if $\rho \in \mathcal{B}(S)$, then $\rho C \subseteq \rho \subseteq \rho C^*$.

We then get the following well-known result which is a part of [2, Lemma 10.3].

Result 1.1. *Let S be a semigroup.*

(i) *If $\rho \in \mathcal{E}(S)$, then ρC is the largest congruence contained in ρ.*

(ii) *If $\rho \in \mathcal{A}(S)$, then $\rho C^* T$ is the least congruence containing ρ.*

Let $\mathcal{A}$ be a collection of disjoint subsets of a semigroup S. We shall say that $\mathcal{A}$ is *admissible* if $\mathcal{A}$ is a subset of the set of equivalence classes for some congruence on S. Also, if ρ is an equivalence on S such that elements of $\mathcal{A}$ are ρ-classes (not necessarily all ρ-classes of ρ), then ρ is said to *admit* $\mathcal{A}$.

We easily see that the intersection and join of any set of equivalences which admit $\mathcal{A}$ also admit $\mathcal{A}$. Hence the set of all congruences which admit $\mathcal{A}$ forms a complete sublattice of the lattice of all congruences on S.

We start from the 0-element and 1-element on the lattice of equivalences of S which admit $\mathcal{A}$. Let $\mathcal{A} = \{A_i : i \in I\}$ be a disjoint set of subsets of S. Put

$$A = \bigcup \{A_i : i \in I\},$$
$$\alpha(\mathcal{A}) = \alpha = \bigcup \{A_i \times A_i : i \in I\} \cup 1_S,$$
$$\beta(\mathcal{A}) = \beta = \bigcup \{A_i \times A_i : i \in I\} \cup (S \setminus A \times S \setminus A).$$

Clearly, α is the smallest and β is the largest equivalence admitting $\mathcal{A}$. Also, it is easy to see that an equivalence ρ has the property that each A_i, where $i \in I$, is a union of ρ-classes if and only if $\rho \subseteq \beta$. Similarly, $\alpha \subseteq \rho$ if and only if each A_i is contained in some ρ-class. Hence, by Result 1.1, βC is the largest congruence on S such that each A_i is the union of congruence classes and $\alpha C^* T$ is the least congruence on S with the property that each A_i is contained in a single congruence class.

Suppose that $\mathcal{A}$ is admissible and ρ is a congruence on S. It is immediate that the set A_i is a ρ-class if and only if it is both a union of ρ-classes and it is contained in a ρ-class.

Result 1.2 (A part of [2, Theorem 10.5]). *Let $\mathcal{A} = \{A_i : i \in I\}$ be a disjoint collection of subsets of a semigroup S. Then a congruence ρ on S admits $\mathcal{A}$ if and only if $\alpha C^* T \subseteq \rho \subseteq \beta C$. Thus $\mathcal{A}$ is an admissible set of subsets if and only if $\alpha C^* T \subseteq \beta C$.*

We shall say that an admissible set $\mathcal{A}$ is *normal* in a semigroup S if there is a unique (two-sided) congruence on S which admits $\mathcal{A}$. From Result 1.2 we immediately have the following useful result.

Result 1.3 [2]. *Let $\mathcal{A}$ be an admissible set of subsets of a semigroup S. Then $\mathcal{A}$ is normal if and only if $\alpha C^*T = \beta C$, that is, if and only if $\beta C \subseteq \rho$, where ρ is any congruence which admits $\mathcal{A}$.*

2. The main results

Notice that in [9] the authors showed that any regular congruence ρ on an eventually regular semigroup is uniquely determined by the set of ρ-classes containing idempotents. Using different methods, we generalise this result for regular congruences on a regular-surjective semigroup (see Theorem 2.2, below). Denote by $\mathcal{RC}(S)$ the set of regular congruences on a semigroup S. First, we prove the following proposition.

Proposition 2.1. *Let S be a regular-surjective semigroup, $\rho \in \mathcal{RC}(S)$. Then the set $\mathcal{A} = \{e\rho : e \in E_S\}$ is normal.*

Proof. In view of Result 1.3, it is sufficient to show that $\beta C \subseteq \rho$. Since $\rho \subseteq \beta C$ (see Result 1.2) and ρ is regular, then $\beta C \in \mathcal{RC}(S)$. Thus $\beta C \subseteq \rho$ if and only if $a\beta C \subseteq a\rho$ for every $a \in \mathrm{Reg}(S)$. Let $a \in \mathrm{Reg}(S)$. Suppose by way of contradiction that $(x, a) \in \beta C$ and $(x, a) \notin \rho$ for some $x \in S$. Then there is $b \in \mathrm{Reg}(S)$ such that $(b, x) \in \rho \subseteq \beta C$, $(a, b) \notin \rho$. It follows that $(a, b) \in \beta C$. Let now $a^* \in V(a)$, $b^* \in V(b)$. Then $(aa^*, ba^*), (b^*a, b^*b) \in \beta C \subseteq \beta = \beta(\mathcal{A})$. Thus, since $aa^*, b^*b \in E_S$,

$$(aa^*, ba^*) \in \rho \tag{2.1}$$

$$(b^*a, b^*b) \in \rho. \tag{2.2}$$

By (2.1), $(a, ba^*a) \in \rho$, so $(bb^*ba^*a, bb^*a) = (ba^*a, bb^*a) \in \rho$. Further, by (2.2), $(bb^*a, b) \in \rho$. Consequently, $(a, ba^*a), (ba^*a, bb^*a), (bb^*a, b) \in \rho$, that is, $(a, b) \in \rho$, a contradiction with $(a, b) \notin \rho$. Thus $\beta C \subseteq \rho$, as required. □

Theorem 2.2. *Let S be a regular-surjective semigroup, $\rho, \sigma \in \mathcal{RC}(S)$. The following conditions are equivalent:*

(i) *$e\rho \subseteq e\sigma$ for every $e \in E_S$;*

(ii) *$\rho \subseteq \sigma$.*

Thus $\rho = \sigma$ if and only if $e\rho = e\sigma$ for every $e \in E_S$.

Proof. (i) $\Longrightarrow$ (ii). Let $\mathcal{A} = \{e\rho : e \in E_S\}$, $\mathcal{B} = \{e\sigma : e \in E_S\}$. Since $(x, y) \in \alpha(\mathcal{A})$ if and only if either $x = y$ or there exists $e \in E_S$ such that $x, y \in e\rho$, then $\alpha(\mathcal{A}) \subseteq \alpha(\mathcal{B})$. Thus $\alpha(\mathcal{A})C^*T \subseteq \alpha(\mathcal{B})C^*T$ and so $\rho \subseteq \sigma$ (by Proposition 2.1 and Result 1.2).

(ii) $\Longrightarrow$ (i). This is obvious. □

Note that from [7] it follows that any regular congruence on an eventually regular semigroup is uniquely determined by its trace and kernel. We show that an analogous result is valid for each regular congruence on an arbitrary idempotent-regular-surjective semigroup. Firstly, by the *kernel* $\ker(\rho)$ of a congruence ρ on a

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semigroup S we shall mean the set $\{x \in S : (x, x^2) \in \rho\}$, and by the *trace* $\mathrm{tr}(\rho)$ of ρ we shall mean the restriction of ρ to E_S. Clearly, if S is idempotent-surjective, then $\ker(\rho) = \{x \in S : \exists\, e \in E_S\ [(x, e) \in \rho]\}$.

THEOREM 2.3. *Let S be an idempotent-regular-surjective semigroup and $\rho, \sigma \in \mathcal{RC}(S)$. Then the following conditions are equivalent:*

(i) $\mathrm{tr}(\rho) \subseteq \mathrm{tr}(\sigma)$ *and* $\ker(\rho) \subseteq \ker(\sigma)$*;*
(ii) $\rho \subseteq \sigma$.

Thus $\rho = \sigma$ if and only if $\mathrm{tr}(\rho) = \mathrm{tr}(\sigma)$ *and* $\ker(\rho) = \ker(\sigma)$.

PROOF. (i) $\Longrightarrow$ (ii). Let $x \in e\rho$, where $e \in E_S$. Then $x \in \ker(\rho) \subseteq \ker(\sigma)$. Therefore, $(x, x^2) \in \rho \cap \sigma$. Since S is idempotent-surjective, then $(x, g) \in \rho \cap \sigma$ for some $g \in E_S$. It follows that $g \in e\rho$. Hence $(g, e) \in \sigma$, so $x \in g\sigma = e\sigma$. Consequently, $e\rho \subseteq e\sigma$ for all $e \in E_S$. Thus $\rho \subseteq \sigma$ (Theorem 2.2).

(ii) $\Longrightarrow$ (i). This is clear. □

3. Structurally regular semigroups

In this section we will show that all structurally regular semigroups are regular-surjective. As we have mentioned above they are idempotent-surjective.

To begin, we prove the following two propositions.

PROPOSITION 3.1. *Let S be a semigroup, (m, n) be an ordered pair of nonnegative integers and let $a, b \in S$. Suppose also that $a\theta_{m,n} \in V(b\theta_{m,n})$. Then there are elements $c, d \in S$ such that $c \in a\theta_{m,n}$, $d \in b\theta_{m,n}$ and $c \in V(d)$.*

PROOF. We remark that if $(s, t) \in \theta_{m,n}$, $p \geqslant m$, $q \geqslant n$, then $xsy = xty$ for $x \in S^p$, $y \in S^q$. Let $a\theta_{m,n} \in V(b\theta_{m,n})$, $u, v \in S^q$, where $q > \max\{m, n\}$. Then $uav = u(aba)v = (uab)av$. Hence $uav = (uab)(aba)v = u(ab)^2av$, so $uav = u(ab)^k a(ba)^l v$ for $k, l = 1, 2, \ldots$. By symmetry, $ubv = u(ba)^k b(ab)^l v$ for $k, l = 1, 2, \ldots$. Put $c = (ab)^q a(ba)^q \in a\theta_{m,n}$ and $d = (ba)^q b(ab)^q \in b\theta_{m,n}$. Then

$$cdc = (ab)^q a \cdot (ba)^{2q} b(ab)^{2q} \cdot a(ba)^q = (ab)^q aba(ba)^q = c.$$

We may equally well show that $dcd = d$. Consequently, $c \in a\theta_{m,n}$, $d \in b\theta_{m,n}$ and $c \in V(d)$, as required. □

PROPOSITION 3.2. *Let S be a structurally regular semigroup. Then* $\mathrm{Reg}(S)$ *is a regular subsemigroup of S.*

PROOF. Let $S/\theta_{m,n}$ be a regular semigroup for some nonnegative integers m, n. In view of Proposition 3.1, $\mathrm{Reg}(S)$ (or equivalently E_S) is nonempty. Take $e, f \in E_S$. Then $(ef)\theta_{m,n}$ is regular, so $(efxef)\theta_{m,n} = (ef)\theta_{m,n}$ for some $x \in S$. Hence $e^m(efxef)f^n = e^m(ef)f^n$. Thus $efxef = ef$, so the product of any two idempotents of S is regular. It is easy to see that the product LR of any $\mathcal{L}$-class L and any $\mathcal{R}$-class R of S is contained in a single $\mathcal{D}$-class D of S (see [1, Theorem 2.4]), so $\mathrm{Reg}(S)$ is a semigroup. □

Further, the following special version of [5, Theorem 2.4] will be useful.

Result 3.3. *Let ψ be an epimorphism between the semigroups S and T, (m, n) be an ordered pair of nonnegative integers. Then there exists an epimorphism $\psi_{m,n}$ from the semigroup $S/\theta_{m,n}$ onto the semigroup $T/\theta_{m,n}$, given by the rule: $[a\theta_{m,n}]\psi_{m,n} = (a\psi)\theta_{m,n}$ $(a \in S)$.*

Thus if S is structurally regular, then every homomorphic image of S is structurally regular, too.

Remark 3.4. Note that a semigroup S is called *structurally eventually regular* if $S/\theta_{m,n}$ is eventually regular for some nonnegative integers m, n, see [6]. In the second part of the above result we may replace 'structurally regular' by 'structurally eventually regular'.

Proposition 3.5. *Let S be a structurally eventually regular semigroup (that is, $S/\theta_{m,n}$ is eventually regular, say), $\rho \in C(S)$. If $(a\rho)\theta_{m,n} \in V((b\rho)\theta_{m,n})$ $(a, b \in S)$, then there exist $c, d \in S$ such that $c\rho \in (a\rho)\theta_{m,n}$, $d\rho \in (b\rho)\theta_{m,n}$ and $c \in V(d)$.*

In particular, if $a\rho \in V(b\rho)$ $(a, b \in S)$, then there exist elements $c, d \in S$ such that $c\rho \in (a\rho)\theta_{m,n}$, $d\rho \in (b\rho)\theta_{m,n}$ and $c \in V(d)$.

Proof. Let $\rho \in C(S)$. Then the mapping $\psi_{m,n} : S/\theta_{m,n} \to (S/\rho)/\theta_{m,n}$, where

$$[a\theta_{m,n}]\psi_{m,n} = (a\rho)\theta_{m,n} \quad (a \in S)$$

is an epimorphism (see Result 3.3). Next, suppose that $(a\rho)\theta_{m,n} \in V((b\rho)\theta_{m,n})$ where $a, b \in S$. Then

$$(a\theta_{m,n})\mathrm{Ker}(\psi_{m,n}) \in V((b\theta_{m,n})\mathrm{Ker}(\psi_{m,n})).$$

Since $S/\theta_{m,n}$ is eventually regular, then (by [3, Theorem 1]) there exist $x, y \in S$ such that $x\theta_{m,n} \in (a\theta_{m,n})\,\mathrm{Ker}(\psi_{m,n})$, $y\theta_{m,n} \in (b\theta_{m,n})\,\mathrm{Ker}(\psi_{m,n})$ and $x\theta_{m,n} \in V(y\theta_{m,n})$. Therefore, $(a\theta_{m,n}, x\theta_{m,n}), (b\theta_{m,n}, y\theta_{m,n}) \in \mathrm{Ker}(\psi_{m,n})$ and there are $c \in x\theta_{m,n}, d \in y\theta_{m,n}$ such that $c \in V(d)$ (Proposition 3.1). This implies the thesis of the proposition. □

From Proposition 3.5 we obtain the following important corollary.

Corollary 3.6. *Every structurally regular semigroup S is regular-surjective.*

Proof. Let $S/\theta_{m,n}$ be a regular semigroup, $\rho \in C(S)$ and suppose that $a\rho$ is regular $(a \in S)$. Then $a\rho \in V(b\rho)$ for some $b \in S$. Therefore, there exist elements $c, d \in S$ such that $c \in V(d)$ and $c\rho \in (a\rho)\theta_{m,n}$. Hence $((ab)\rho)^m(c\rho)((ba)\rho)^n = ((ab)\rho)^m(a\rho)((ba)\rho)^n$, so $(abcba)\rho = a\rho$. On the other hand, $(ab)\rho, (ba)\rho$ are idempotent elements of S/ρ. Thus there exist $e, f \in E_S$ such that $e \in (ab)\rho$, $f \in (ba)\rho$, so $a\rho = (ecf)\rho$, $ecf \in \mathrm{Reg}(S)$ (Proposition 3.2). □

Consequently, structurally regular semigroups are idempotent-regular-surjective. From Theorems 2.2, 2.3 we obtain the main result of this section.

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Theorem 3.7. *Let S be a structurally regular semigroup, $\rho \in \mathcal{RC}(S)$. Then:*

(i) *ρ is uniquely determined by the set of ρ-classes containing idempotents;*

(ii) *ρ is uniquely determined by its kernel and trace.*

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