ON AN INTEGRAL EQUATION

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(Received 27 February, 1973)

1. We shall solve the equation

$$\frac{1}{\pi}\int_a^b g(t)\ln\frac{|x-t|}{x+t}\,dt = f(x) \qquad (a < x < b), \tag{1}$$

where $0 < a < b$, and $f(x)$ is a continuous function on the interval (a, b).

We are interested in solving this equation since it appears in the study of the steady supersonic motion of an airfoil with subsonic attack edges [**2**]. In the case $a=0$ the equation was considered by Williams [**6**] and Cooke [**1**].

The equation (1) can also be used to solve the following dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \frac{\sin xt}{t}h(t)\,dt = f(x) && (c_1 < x < c_2),\\ &\int_0^\infty \sin xt\, h(t)\,dt = 0 && (x\in(0, c_1)\cup(c_2, +\infty)) \end{aligned}\right\} \tag{2}$$

and dual trigonometric series

$$\left.\begin{aligned} &\sum_{n=1}^\infty \frac{a_n \sin nx}{n} = f(x) && (c_1 < x < c_2),\\ &\sum_{n=1}^\infty a_n \sin nx = 0 && (x\in(0, c_1)\cup(c_2, \pi)). \end{aligned}\right\} \tag{3}$$

These problems are generalizations of some cases considered by Tranter in [**4**] and [**5**].

2. Let us consider the function

$$G(z) = \frac{1}{\pi}\int_a^b g(t)\log\frac{z-t}{z+t}\,dt, \tag{4}$$

with the logarithm determination that is real for $z=x>b$. $G(z)$ is a holomorphic function of the complex variable $z=x+iy$ in the upper half-plane.

On the x-axis we have

$$G(x+i0) = \frac{1}{\pi}\int_a^b g(t)\ln\frac{|x-t|}{x+t}\,dt + \left\{\begin{array}{lll} 0 & \text{for} & |x| > b,\\ i\int_{|x|}^b g(t)\,dt & \text{for} & a < |x| < b,\\ ik & \text{for} & |x| < a, \end{array}\right\} \tag{5}$$

where $k = \int_a^b g(t)\,dt.$

https://doi.org/10.1017/S0017089500002251 Published online by Cambridge University Press

If in (1) we put $z=-x'$, this equation becomes

$$\frac{1}{\pi}\int_a^b g(t)\ln\left|\frac{x'-t}{x'+t}\right|dt = -f(-x') \qquad \text{for} \quad -b<x'<-a. \tag{6}$$

From the relationships (1), (4), (5), (6), it follows that $G(z)$ is a holomorphic function in the half-plane $y>0$, vanishes at infinity, is imaginary on the y-axis and satisfies the following boundary conditions:

$$\left.\begin{aligned} &\operatorname{Im}\{G(z)\}_{y=+0}=0 && \text{for } |x|>b,\\ &\operatorname{Re}\{G(z)\}_{y=+0}=(\operatorname{sgn} x)f(|x|) && \text{for } a<|x|<b,\\ &\operatorname{Im}\{G(z)\}_{y=+0}=k && \text{for } |x|<a. \end{aligned}\right\} \tag{7}$$

Then the function $G(z)$ will be the solution of a Volterra boundary value problem **[3]**.

Let us consider on the upper half-plane the holomorphic function.

$$H(z)=\frac{iG(z)}{\sqrt{\{(z^2-a^2)(z^2-b^2)\}}},$$

with the radical determination which is negative for $z=0$. We have

$$\operatorname{Re}\{H(z)\}_{y=+0}=0 \qquad \text{for} \quad |x|>b,$$

$$\operatorname{Re}\{H(z)\}_{y=+0}=\frac{i\operatorname{sgn} x\,.f(|x|)}{\sqrt{\{(x^2-a^2)(x^2-b^2)\}}} \qquad \text{for} \quad a<|x|<b,$$

$$\operatorname{Re}\{H(z)\}_{y=+0}=-\frac{k}{\sqrt{\{(x^2-a^2)(x^2-b^2)\}}} \qquad \text{for} \quad |x|<a.$$

The solution of the Dirichlet problem corresponding to these boundary conditions is the following:

$$H(z)=\frac{i}{\pi}\left\{\int_{-b}^{-a}\frac{-if(-t)}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}-k\int_{-a}^{a}\frac{1}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}\right.$$
$$\left.+\int_a^b\frac{if(t)}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}\right\}.$$

Hence

$$G(z)=\frac{i}{\pi}\sqrt{\{(z^2-a^2)(z^2-b^2)\}}\left\{\int_{-b}^{-a}\frac{-f(-t)}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}\right.$$
$$\left.+\int_{-a}^{a}\frac{ik}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}+\int_a^b\frac{f(t)}{\sqrt{\{(t^2-a^2)(t^2-b^2)\}}}\frac{dt}{z-t}\right\}. \tag{8}$$

In order that $G(z)$ should be zero at infinity, it is necessary that

$$k\int_0^a\frac{dt}{\sqrt{\{(b^2-t^2)(a^2-t^2)\}}}=-\int_a^b\frac{f(t)\,dt}{\sqrt{\{(b^2-t^2)(a^2-t^2)\}}}. \tag{9}$$

This relationship will define the constant k.

From (8) for $z = x \in (a, b)$, we have

$$G(x+i0) = f(x) + 2x\sqrt{\{(b-x)(x-a)\}}\frac{i}{\pi}\left\{k\int_0^a \frac{1}{\sqrt{\{(a^2-t^2)(b^2-t^2)\}}}\frac{dt}{x^2-t^2} + \int_a^b \frac{f(t)}{\sqrt{\{(b^2-t^2)(t^2-a^2)\}}}\frac{dt}{x^2-t^2}\right\}.$$

(The integral on the right-hand side of this relationship is the principal value in Cauchy's sense.) From this relationship we eventually obtain

$$\int_x^b g(t)\,dt = \frac{2}{\pi}x\sqrt{\{(b^2-x^2)(x^2-a^2)\}}\left\{k\int_0^a \frac{1}{\sqrt{\{(a^2-t^2)(b^2-t^2)\}}}\frac{dt}{x^2-t^2} + \int_a^b \frac{f(t)}{\sqrt{\{(a^2-t^2)(b^2-t^2)\}}}\frac{dt}{x^2-t^2}\right\}. \tag{10}$$

The solution of the equation (1) follows by differentiation of this relationship with respect to x.

3. The above method can also be applied to solve the equation (1) when $a = 0$. In this case the last condition in (7) disappears and the solution of the corresponding Volterra's-type boundary value problem is

$$G(z) = \frac{1}{\pi}\sqrt{(z^2-b^2)}\left\{\int_{-b}^0 \frac{-if(-t)}{\sqrt{(t^2-b^2)}}\frac{dt}{z-t} + \int_0^b \frac{if(t)}{\sqrt{(t^2-b^2)}}\frac{dt}{z-t}\right\}, \tag{11}$$

with the radical determination that is positive for $z = x > b$. Hence we have

$$\int_x^b g(t)\,dt = \frac{2}{\pi}\sqrt{(b^2-x^2)}\int_0^b \frac{tf(t)}{\sqrt{(b^2-t^2)}}\frac{dt}{x^2-t^2}. \tag{12}$$

The solution that follows by differentiation with respect to x agrees with that given in [**1**].

Indeed, from (12) after integration by parts we have

$$\int_x^b g(t)\,dt = -\frac{1}{\pi}f(0)\ln\frac{b+\sqrt{(b^2-x^2)}}{b-\sqrt{(b^2-x^2)}} - \frac{1}{\pi}\int_0^b f'(t)\ln\left|\frac{\sqrt{(b^2-t^2)}+\sqrt{(b^2-x^2)}}{\sqrt{(b^2-t^2)}-\sqrt{(b^2-x^2)}}\right|dt.$$

Hence

$$g(x) = -\frac{2x}{\pi\sqrt{(b^2-x^2)}}\int_0^b f'(t)\frac{\sqrt{(b^2-t^2)}}{x^2-t^2}\,dt - \frac{2b}{\pi}\frac{f(0)}{x\sqrt{(b^2-x^2)}}.$$

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