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The effect of a perturbation on Brezis-Nirenberg's problem

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In this article, we consider some critical Brézis-Nirenberg problems in dimension $N \geq 3$ that do not have a solution. We prove that a supercritical perturbation can lead to the existence of a positive solution. More precisely, we consider the equation:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ \quad u &> 0 & \text{in} \quad B, \\ \quad u &= 0 & \text{on} \quad \partial B, \end{cases}$$

where $B \subset \mathbb{R}^N$ is a unit ball centred at the origin, $N \geq 3$, $r = |x|$, $\alpha \in (0, \min\{N/2, N-2\})$, λ is a fixed real parameter and $q \in [2, 2^*]$. This class of problems can be interpreted as a perturbation of the classical Brézis–Nirenberg problem by the term r^α at the exponent, making the problem supercritical when $r \in (0,1)$. More specifically, we study the effect of this supercritical perturbation on the existence of solutions. In particular, when $N = 3$, an interesting and unexpected phenomenon occurs. We obtain the existence of solutions for λ in a range where the Brézis–Nirenberg problem has no solution.

Keywords: Laplacian operator; supercritical elliptic problems; Brézis-Nirenberg problem; positive solutions

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1. Introduction and main results

In 1983, Brézis and Nirenberg in [1.1] studied the following problem:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, λ is a fixed real parameter, $q \in [2, 2^*)$ and $2^* = 2N/(N-2)$ is the critical exponent in the sense of Sobolev's embedding.

Brézis and Nirenberg proved the following results:

(a) For $q = 2$ and $N \geq 4$, problem (1.1) has a solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$. Moreover, it has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is star-shaped.
(b) When $q = 2$, $N = 3$, and Ω is a ball, problem (1.1) has a solution if and only if $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$.
(c) For $q \in (2, 2^*)$ and $N \geq 4$, problem (1.1) has a solution for every $\lambda > 0$.
(d) When $N = 3$ and $4 < q < 6$, problem (1.1) has a solution for every $\lambda > 0$.
(e) When $N = 3$ and $2 < q \leq 4$, problem (1.1) has a solution only for sufficiently large values of λ.

Recently, do Ó, Ruf, and Ubilla in [5] studied the following problem:

$$\begin{cases} -\Delta u &= u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{1.2}$$

where $B \subset \mathbb{R}^N$ is the unit ball centred at the origin, $N \geq 3$, $r = |x|$, and $\alpha \in (0, \min\{N/2, N-2\})$.

The authors demonstrated that problem (1.2) has a radial solution, which is surprising because it corresponds to a supercritical perturbation of the equation $-\Delta u = u^{2^*-1}$, which has no solution due to the known Pohozaev identity. In this same line of reasoning, in the context of the situation of item (b), we studied the effect of a supercritical perturbation for the case of non-existence $\lambda \in (0, \frac{\lambda_1}{4}]$, which also generated the existence of a positive solution. We will also have the same conclusion for situation (e), in which, due to the supercritical perturbation, we will obtain a solution for all positive λ and not just for sufficiently large λ. Motivated by the results of [1.1] and [5], we studied this problem in a more general context, more precisely, let us consider the following problem:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{1.3}$$

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where $B \subset \mathbb{R}^N$ is a unit ball centred at the origin, $N \geq 3$, $r = |x|$, and $\alpha \in (0, \min\{N/2, N-2\})$ and λ is a fixed real parameter and $q \in [2, 2^*]$.

We will now present the main result of this article.

THEOREM 1.1 *If $q = 2$, $\lambda \in [0, \lambda_1)$ and $N \geq 3$, then the problem (1.3) has a radial weak solution. If $q \in (2, 2^*]$, problem (1.3) has a radial weak solution for every $\lambda \geq 0$ and $N \geq 3$.*

We would like to highlight that in the case $N = 3$ we obtain a solution for the perturbed problem for each $\lambda \in [0, \lambda_1)$, that is, the perturbation solves the non-existence interval $[0, \lambda_1/4]$.

Let $H_0^1(B) := \{u \in L^2(B)\colon \nabla u \in L^2(B)\colon u = 0 \,\text{on}\, \partial B\}$ be the usual Sobolev space equipped with the gradient norm, or let $\|u\|_{H_0^1(B)} = \|\nabla u\|_{L^2(B)}$. We say that $u \in H_0^1(B)$ is a weak solution to problem (1.3) if $u > 0$ in B and it holds:

$$\int_B \nabla u \nabla \varphi \,\mathrm{d}x = \lambda \int_B u^{q-1} \varphi \,\mathrm{d}x + \int_B u^{2^*-1+r^\alpha} \varphi \,\mathrm{d}x, \ \forall\, \varphi \in H_0^1(B). \tag{1.4}$$

REMARK 1.2. It is important to emphasize that the Eq. (1.4) is well defined due to the results obtained in proposition 2.2 and corollary 2.3. Note that (1.4) is not well-defined for $q > 2^*$.

Theorem 1.1 shows (see (b) and (e)) that there are critical equations without solutions that have a solution when a non-negative term is added to them, converting them into supercritical equations. Note that this phenomenon was already observed in [5].

We also consider some perturbations of problem (1.1) that become superlinear on the ball and subcritical for $r \in (0, \delta)$, for some small δ. However, it can be supercritical away from $r = 0$, as in the following equation:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+f(r)-1} & \text{in} \quad B, \\ \quad u &> 0 & \text{in} \quad B, \\ \quad u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{1.5}$$

where $B \subset \mathbb{R}^N$ is a unit ball centred at the origin, $N \geq 3$, $r = |x|$, λ is a fixed real parameter, $q \in [2, 2^*)$ and $f\colon [0,1) \to \mathbb{R}$ is a continuous function satisfying:

(f) $\quad f(0) < 0$ and $\inf_{r \in [0,1)} (2^* + f(r)) > 2$.

The next result involves the assumption (f):

THEOREM 1.3 *Let $q \in [2, 2^*)$, $N \geq 3$, and $f\colon [0,1) \to \mathbb{R}$ a continuous function satisfying condition (f). Then the problem (1.5) has a radial weak solution in the following cases:*

(i) *$q = 2$ and $\lambda \in [0, \lambda_1)$.*
(ii) *$q \in (2, 2^*)$ and $\lambda \geq 0$.*

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Remark 1.4. In theorem 1.3, due to the generality of the growth condition considered for the function *f*, it was not possible to reach the case $q = 2^*$.

The definition of a weak solution for problem (1.5) is analogous to the one we defined in Eq. (1.4). The case $0 < q < 2$, which corresponds to a concave-convex problem, was studied in [3] under a subcritical assumption. Therefore, theorem 1.3 complements the result in [3].

The article is organized as follows: In §2, we present preliminary results, in §3, we prove theorem 1.1, and in §4, we prove theorem 1.3.

2. Preliminaries

First, we define the Sobolev space of radial functions $H^1_{0,\mathrm{rad}}(B) := \{u \in H^1_0(B)\colon u(x) = u(|x|)\}$ equipped with the usual standard $\|u\| = \|\nabla u\|_2$. We will now present the 'radial lemma', which can be found in [5, 8].

Lemma 2.1. *Let $u \in H^1_{0,\mathrm{rad}}(B)$. Then*

$$|u(r)| \leq \frac{1}{(N-2)^{1/2}} \frac{\|\nabla u\|_2}{r^{(N-2)/2}} \tag{2.1}$$

and

$$|u(r)| \leq \frac{(1-r)^{1/2}}{r^{(N-2)/2}} \|\nabla u\|_2. \tag{2.2}$$

For the next result, we refer [5]

Proposition 2.2. *Let $\alpha > 0$; then*

$$\sup\left\{ \int_B |u(x)|^{2^*+r^\alpha}\, dx\colon u \in H^1_{0,\mathrm{rad}}(B),\ \|\nabla u\|_2 = 1\right\} < +\infty. \tag{2.3}$$

Corollary 2.3. *The following embedding is continuous:*

$$H^1_{0,\mathrm{rad}}(B) \hookrightarrow L^{2^*+r^\alpha}(B)\ , \tag{2.4}$$

where $L^{2^*+r^\alpha}(B)$ is defined as follows (see, e.g., [4])

$$L^{2^*+r^\alpha}(B) := \left\{u\colon B \to \mathbb{R} \text{ measurable: } \int_B |u(x)|^{2^*+r^\alpha}\, \mathrm{d}x < \infty\right\}$$

with norm

$$\|u\|_{2^*+r^\alpha} = \inf\left\{\lambda > 0\ ,\ \int_B \left|\frac{u(x)}{\lambda}\right|^{2^*+r^\alpha} \mathrm{d}x \leq 1\right\}.$$

The following proposition follows directly from the definition:

Proposition 2.4. *Let $p : [0,1) \to \mathbb{R}$ be a bounded continuous function and $u \in L^{p(r)}(B)$. Consider $\|u\|_{p(r)} = \mu$. Then we have:*

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(i) *If* $\mu \geq 1$*, then* $\mu^{p_-} \leq \int_B |u(x)|^{p(r)} \mathrm{d}x \leq \mu^{p_+}$,

(ii) *If* $\mu \leq 1$*, then* $\mu^{p_+} \leq \int_B |u(x)|^{p(r)} \mathrm{d}x \leq \mu^{p_-}$,

where $p_+ = \sup_{r\in[0,1)} p(r)$ *and* $p_- = \inf_{r\in[0,1)} p(r)$.

3. Proof of theorem 1.1

To establish a weak solution of problem (1.3), we define the functional $J\colon H^1_{0,\mathrm{rad}}(B) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q - \int_B \frac{1}{2^* + r^\alpha}\,(u^+)^{2^*+r^\alpha}\mathrm{d}x, \tag{3.1}$$

where $u^+(x) = \max\{u(x), 0\}$. By proposition 2.2 and by corollary 2.3, it follows that the functional J is well defined. We also note that J is a functional of class C^1. If $u > 0$ is a critical point of the functional then u is a weak solution to problem (1.3) thanks to the symmetric criticality principle (see [7, 10]). The strategy then consists of obtaining positive critical points of the functional J. For this, we will use the Mountain Pass Lemma, due to Ambrosetti and Rabinowitz [1].

In the next lemmas, we will demonstrate that the functional J has the geometry of the Mountain Pass Theorem.

Lemma 3.1. *There exist* $\rho > 0$ *and* $\theta > 0$ *such that*

$$J(u) \geq \rho > 0, \ \text{if } \|\nabla u\|_2 = \theta.$$

Proof. Note that

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q - \int_B \frac{(u^+)^{2^*+r^\alpha}}{2^* + r^\alpha}\,\mathrm{d}x \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \frac{1}{2^*}\int_B |u|^{2^*+r^\alpha}\,\mathrm{d}x. \end{aligned}$$

Let $u \in H^1_{0,\mathrm{rad}}(B)$ be such that $\|\nabla u\|_2 = \theta$ where $\theta \in (0, 1)$ will be chosen. By proposition 2.4 and corollary 2.3 follow that

$$\int_B |u|^{2^*+r^\alpha}\,\mathrm{d}x \leq \|u\|^{2^*}_{2^*+r^\alpha} \leq C\|\nabla u\|_2^{2^*}.$$

Therefore,

$$J(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \frac{C}{2^*}\|\nabla u\|_2^{2^*}. \tag{3.2}$$

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We observe that when $q=2$, we consider $\lambda \in [0,\lambda_1)$, then the expression $\sqrt{\|\nabla u\|^2 - \lambda\|u\|^2}$ defines a norm in $H_0^1(B)$ equivalent to norm $\|\nabla u\|_2$. Since $\|\nabla u\|_2 = \theta$, we have

$$J(u) \geq \frac{C}{2}\theta^2 - \frac{C}{2^*}\theta^{2^*}.$$

So, choosing $\theta_1 \in (0,1)$ small enough we have that for $\theta \in (0,\theta_1)$ fixed there is $\rho_1 > 0$ such that $J(u) \geq \rho_1 > 0$.

If $q \in (2, 2^*]$, by using (3.2) and Sobolev inequality, we get

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{C_1\lambda}{q}\|\nabla u\|_q^q - \frac{C}{2^*}\|\nabla u\|_2^{2^*} \\ &= \frac{1}{2}\theta^2 - \frac{\lambda C_1}{q}\theta^q - \frac{C}{2^*}\theta^{2^*}. \end{aligned}$$

Since $2^* \geq q > 2$, we can choose $\theta_2 \in (0,1)$ small enough such that for any fixed $\theta \in (0,\theta_2)$, there exists $\rho_2 > 0$ such that $J(u) \geq \rho_2 > 0$. □

Now, we will state the second condition of the mountain pass geometry.

Lemma 3.2. *Exist $u \in H^1_{0,\mathrm{rad}}(B)$ such that $\|\nabla u\|_2 > \theta$ and $J(u) < 0$.*

Proof. Let $u \in H^1_{0,\mathrm{rad}}(B) \setminus \{0\}$ such that $u > 0$ in B. We have for $t > 1$ that

$$\begin{aligned} J(tu) &= \frac{t^2}{2}\|\nabla u^+\|_2^2 - \frac{\lambda t^q}{q}\|u\|_q^q - \int_B \frac{t^{2^*+r^\alpha}(u^+)^{2^*+r^\alpha}}{2^*+r^\alpha}\,\mathrm{d}x \\ &\leq \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{\lambda t^q}{q}\|u^+\|_q^q - \frac{t^{2^*}}{2^*+1}\int_B (u^+)^{2^*+r^\alpha}\,\mathrm{d}x. \end{aligned}$$

Therefore, since $2 \leq q \leq 2^*$ we get

$$\lim_{t\to+\infty} J(tu) = -\infty,$$

which proves the lemma. □

We now define S_N as the best constant in the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S_N := \inf\left\{\frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|u\|^2_{L^{2^*}(\mathbb{R}^N)}} : u \in L^{2^*}(\mathbb{R}^N)\setminus\{0\};\ \nabla u \in L^2(\mathbb{R}^N)\right\}. \tag{3.3}$$

We consider

$$\bar{u}(x) = C\left(1+|x|^2\right)^{-\frac{(N-2)}{2}}$$

the standard Sobolev instantons, which satisfy the equation (see [9])

$$-\Delta u = u^{2^*-1}\ , \text{ on } \mathbb{R}^N.$$

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We also consider $u^*(x) = \bar{u}(x/S_N^{1/2})$ and $U_\varepsilon(x) = \varepsilon^{-\frac{(N-2)}{2}} u^*(x/\varepsilon)$. As in [9] and also [10], we know that,

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \, \mathrm{d}x = S_N^{N/2} \text{ and } \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} \mathrm{d}x = S_N^{N/2}.$$

Taking a suitable cut-off function η and setting $u_\varepsilon = \eta\, U_\varepsilon$, it is known that

$$\int_B |\nabla u_\varepsilon|^2 \, \mathrm{d}x = S_N^{N/2} + O(\varepsilon^{N-2}) \;\; , \int_B |u_\varepsilon(x)|^{2^*} \; \mathrm{d}x = S_N^{N/2} + O(\varepsilon^N). \tag{3.4}$$

Do Ó, Ruf, and Ubilla, in [5], demonstrated the following lemma:

Lemma 3.3. *There exists a constant $C>0$ such that for all $\varepsilon>0$ small*

$$\int_B |u_\varepsilon(x)|^{2^*+r^\alpha} \; \mathrm{d}x \geq \int_B |u_\varepsilon(x)|^{2^*} \; \mathrm{d}x + C\,|\log \varepsilon|\varepsilon^\alpha + O(\varepsilon^{N/2}) + O(\varepsilon^{N-2}).$$

Now let's control the min-max level of the mountain pass theorem.

Lemma 3.4. *The level c of the mountain pass of the functional J satisfies $0 < c < \frac{1}{N} S_N^{N/2}$.*

Proof. By lemmas 3.1 and 3.2, J has the geometry of the Mountain Pass lemma. We consider u_ε as before and set

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u)$$

where

$$\Gamma := \Big\{ \gamma : [0, R] \to H_0^1(B) \text{continuous }, \gamma(0) = 0, \gamma(1) = R\, u_\varepsilon \Big\},$$

with $R>0$ sufficiently large such that $J(R\, u_\varepsilon) \leq 0$. By (3.4) and lemma 3.3, we note that R can be chosen independent of ε. The path $\gamma_\varepsilon(t) = t u_\varepsilon$, $t \in [0, R]$, belongs to Γ, and

$$c \leq \max_{t \in [0,R]} J(t\, u_\varepsilon) := J(t_\varepsilon u_\varepsilon). \tag{3.5}$$

We have also that $\frac{d}{dt} J(t\, u_\varepsilon)\Big|_{t=t_\varepsilon} = 0$ and by J satisfying the geometric conditions of the Mountain Pass lemma, we can assume that $t_\varepsilon \in (\delta, R]$ with $\delta>0$ because if $t_\varepsilon \to 0$ by (3.4) and lemma 3.3 we obtain that $J(t_\varepsilon u_\varepsilon) \to 0$. So, for $\varepsilon>0$ small enough, we have $J(t_\varepsilon u_\varepsilon) < S_N^{N/2}/N$.

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Now, let's consider the following auxiliary functional:

$$J_A(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_B \frac{(u^+)^{2^*+r^\alpha}}{2^*+r^\alpha}\,\mathrm{d}x,\ \ u \in H^1_{0,\mathrm{rad}}(B).$$

So, for $t_\varepsilon \in (\delta, R]$ we have the estimate

$$\begin{aligned}
J(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|\nabla u_\varepsilon\|_2^2 - \frac{\lambda t_\varepsilon^q}{q}\|u_\varepsilon\|_q^q - \int_B \frac{t_\varepsilon^{2^*+r^\alpha}}{2^*+r^\alpha}(u_\varepsilon)^{2^*+r^\alpha}\,\mathrm{d}x\\
&\le J_A(t_\varepsilon u_\varepsilon)\\
&= \frac{t_\varepsilon^2}{2}\|\nabla u_\varepsilon\|_2^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_B u_\varepsilon^{2^*}\,\mathrm{d}x\\
&\quad + t_\varepsilon^{2^*}\int_B\left(\frac{1}{2^*} - \frac{1}{2^*+r^\alpha}\right)u_\varepsilon^{2^*}\,\mathrm{d}x\\
&\quad + \int_B \frac{1}{2^*+r^\alpha}\left((t_\varepsilon u_\varepsilon)^{2^*} - (t_\varepsilon u_\varepsilon)^{2^*+r^\alpha}\right)\mathrm{d}x\\
&\le \max_{t\in[0,R]}\left(\frac{t^2}{2}\|\nabla u_\varepsilon\|_2^2 - \frac{t^{2^*}}{2^*}\|u_\varepsilon\|_{2^*}^{2^*}\right) + c\varepsilon^\alpha - c\varepsilon^\alpha|\log\varepsilon|\\
&\quad + O(\varepsilon^{N/2}) + O(\varepsilon^{N-2})\\
&= \frac{1}{2}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^{2^*}}\right)^{2/(2^*-2)}\|\nabla u_\varepsilon\|_2^2 - \frac{1}{2^*}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^{2^*}}\right)^{2^*/(2^*-2)}\|u_\varepsilon\|_{2^*}^{2^*}\\
&\quad + c\varepsilon^\alpha - c\varepsilon^\alpha|\log\varepsilon|\\
&= \frac{1}{N}\frac{\left(\|\nabla u_\varepsilon\|_2^2\right)^{2^*/(2^*-2)}}{\left(\|u_\varepsilon\|_{2^*}^{2^*}\right)^{2/(2^*-2)}} + c\varepsilon^\alpha - c\varepsilon^\alpha|\log\varepsilon|, \qquad (3.6)
\end{aligned}$$

where we use that $\alpha \in (0, \min\{N/2, N-2\})$, the lemma 3.3, and the estimate

$$\begin{aligned}
\int_B\left(\frac{1}{2^*} - \frac{1}{2^*+r^\alpha}\right)|u_\varepsilon|^{2^*}\mathrm{d}x &= \int_B \frac{r^\alpha}{2^*(2^*+r^\alpha)}|u_\varepsilon|^{2^*}r^{N-1}\,\mathrm{d}x\\
&\le c\int_0^\varepsilon r^\alpha\varepsilon^{-N}r^{N-1}\,\mathrm{d}r + c\int_\varepsilon^1 r^\alpha\frac{\varepsilon^N}{r^{2N}}r^{N-1}\,\mathrm{d}r \qquad (3.7)\\
&\le c\,\varepsilon^\alpha + c\,(\varepsilon^\alpha - \varepsilon^N) = c\,\varepsilon^\alpha.
\end{aligned}$$

Therefore, by using (3.4) and (3.6), we obtain

$$\begin{aligned}
J(t_\varepsilon u_\varepsilon) \le J_A(t_\varepsilon u_\varepsilon) &\le \frac{1}{N}\frac{\left(S_N^{N/2} + O(\varepsilon^{N-2})\right)^{2^*/(2^*-2)}}{\left(S_N^{N/2} + O(\varepsilon^N)\right)^{2/(2^*-2)}} + c\varepsilon^\alpha - c\varepsilon^\alpha|\log\varepsilon|,\\
&= \frac{1}{N}S_N^{N/2} + O(\varepsilon^{N-2}) + c\varepsilon^\alpha - c\varepsilon^\alpha|\log\varepsilon|,\\
&< \frac{1}{N}S_N^{N/2},\ \text{for } \varepsilon > 0 \text{ small enough and for all } N \ge 3.
\end{aligned}$$

□

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3.1. Proof of theorem 1.1

By lemmas 3.1 and 3.2, we know that the functional J satisfies the geometric conditions of the Mountain Pass lemma; by lemma 3.4, it follows that there is a sequence of Palais-Smale $\{u_n\} \subset H^1_{0,\mathrm{rad}}(B)$ such that:

$$J(u_n) \to c < \frac{1}{N} S_N^{N/2}, \text{ and } J'(u_n) \to 0.$$

Let's show that the sequence $\{u_n\}$ is bounded in $H^1_{0,\mathrm{rad}}(B)$. Indeed, for n sufficiently large and $q \in (2, 2^*]$, we have:

$$\begin{aligned} c + 1 + \|\nabla u_n\|_2 &\geq J(u_n) - \frac{1}{q} J'(u_n) u_n \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \|\nabla u_n\|_2^2 + \int_B \left(\frac{1}{q} - \frac{1}{2^* + r^\alpha}\right) (u_n^+)^{2^*+r^\alpha} \, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \|\nabla u_n\|_2^2. \end{aligned}$$

It follows that $\{u_n\}$ is bounded in $H^1_{0,\mathrm{rad}}(B)$. If $q = 2$, we recall that $\lambda \in [0, \lambda_1)$ and in this case the expression $(\|\nabla u\|_2^2 - \lambda \|u\|_2^2)^{1/2}$ defines a norm in $H^1_{0,\mathrm{rad}}(B)$ equivalent to the usual norm $\|\nabla u\|_2$. Thus, we will also have for n sufficiently large that:

$$\begin{aligned} c + 1 + \|\nabla u_n\|_2 &\geq J(u_n) - \frac{1}{2^*} J'(u_n) u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \left(\|\nabla u_n\|_2^2 - \lambda \|u\|_2^2\right) \\ &\quad + \int_B \left(\frac{1}{2^*} - \frac{1}{2^* + r^\alpha}\right) (u_n^+)^{2^*+r^\alpha} \, \mathrm{d}x \\ &\geq c_1 \|\nabla u_n\|_2^2. \end{aligned}$$

It follows that $\{u_n\}$ is bounded in $H^1_{0,\mathrm{rad}}(B)$. So there exists $u \in H^1_{0,\mathrm{rad}}(B)$ such that $u_n \rightharpoonup u$ in $H^1_{0,\mathrm{rad}}(B)$. We have two possibilities:

If $u \not\equiv 0$, then u is a non-trivial non-negative solution to problem (1.3). By the maximum principle, we guarantee that u is positive, thus proving the theorem.

If $u = 0$, we have $u_n \rightharpoonup 0$, and for every $\varepsilon > 0$ and n sufficiently large, the following inequality holds:

$$\int_B |u_n|^{2^*+r^\alpha} \, \mathrm{d}x - \int_B |u_n|^{2^*} \, \mathrm{d}x \leq \varepsilon. \tag{3.8}$$

Indeed, note that for all $\eta > 0$, we have $H^1_{0,\mathrm{rad}}(B \setminus B_\eta) \subset\subset L^s(B \setminus B_\eta)$ for all $s \geq 1$. Therefore,

$$u_n \to 0 \text{in } L^s(B \setminus B_\eta) \text{for all } s \geq 1,$$

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and consequently,

$$\int_{B\setminus B\eta} |u_n|^{2^*+r^\alpha}\,\mathrm{d}x \to 0 \text{and} \int_{B\setminus B\eta} |u_n|^{2^*}\,\mathrm{d}x \to 0 \text{as } n\to\infty. \tag{3.9}$$

By (3.9), we can write

$$\begin{aligned}
\int_{B\eta} |u_n|^{2^*+r^\alpha}\,\mathrm{d}x &= \omega_{N-1}\int_0^\eta |u_n(r)|^{2^*+r^\alpha}\, r^{N-1}\mathrm{d}r && (3.10)\\
&= \omega_{N-1}\int_0^\eta |u_n(r)|^{2^*}\left(|u_n(r)|^{r^\alpha}-1\right) r^{N-1}\mathrm{d}r && (3.11)\\
&\quad + \omega_{N-1}\int_0^\eta |u_n(r)|^{2^*}\, r^{N-1}\mathrm{d}r\\
&= \omega_{N-1}\int_0^\eta |u_n(r)|^{2^*}\left(|u_n(r)|^{r^\alpha}-1\right) r^{N-1}\mathrm{d}r && (3.12)\\
&\quad + \int_B |u_n(x)|^{2^*}\,\mathrm{d}x + o(1). && (3.13)
\end{aligned}$$

Using lemma 2.1 (Radial Lemma), we can estimate

$$\begin{aligned}
&\int_0^\eta |u_n(r)|^{2^*}\left(|u_n(r)|^{r^\alpha}-1\right) r^{N-1}\mathrm{d}r\\
&\le \int_0^\eta |u_n(r)|^{2^*}\left[\left(\frac{1}{r^{(N-2)/2}}\right)^{r^\alpha}-1\right] r^{N-1}\mathrm{d}r\\
&\le \int_0^\eta |u_n(r)|^{2^*}\left[\exp\left(r^\alpha\log\left(\frac{1}{r^{(N-2)/2}}\right)\right)-1\right] r^{N-1}\mathrm{d}r\\
&\le \int_0^\eta |u_n(r)|^{2^*} r^\alpha \left|\log r^{(N-2)/2}\right| r^{N-1}\mathrm{d}r\\
&\le C_1\eta^\alpha\left|\log\eta\right|\int_0^1 |u_n(r)|^{2^*}\, r^{N-1}\mathrm{d}r\\
&\le C_2\eta^\alpha\left|\log\eta\right|,
\end{aligned}$$

where C_1 and C_2 are constants.

Therefore, for all $\varepsilon>0$, we can choose $\eta=\eta(\varepsilon)>0$ sufficiently small such that

$$C_2\eta^\alpha\left|\log\eta\right| \le \frac{\varepsilon}{2},$$

which implies

$$\omega_{N-1}\int_0^\eta |u_n(r)|^{2^*}\left(|u_n(r)|^{r^\alpha}-1\right) r^{N-1}\mathrm{d}r \le \frac{\varepsilon}{2}. \tag{3.14}$$

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From (3.9), (3.10), and (3.14), we obtain that for sufficiently large n and for all $\varepsilon > 0$,

$$\int_B |u_n|^{2^*+r^\alpha}\,\mathrm{d}x - \int_B |u_n|^{2^*}\,\mathrm{d}x \le \varepsilon.$$

Therefore, we have proven (3.8).

Now, for sufficiently large n, we obtain the inequality

$$-\frac{1}{2^*}\int_B |u_n|^{2^*}\,\mathrm{d}x \le -\int_B \frac{|u_n|^{2^*+r^\alpha}}{2^*+r^\alpha}\,\mathrm{d}x.$$

Thus, for sufficiently large n, we get

$$J_0(u_n) \le J_A(u_n),$$

where

$$J_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q - \frac{1}{2^*}\int_B (u^+)^{2^*}\,\mathrm{d}x, \quad u \in H^1_{0,\mathrm{rad}}(B).$$

Then, we have

$$J_0(u_n) \to d \le c < \frac{1}{N}S^{N/2}.$$

Since $u_n \rightharpoonup 0$ also in $L^{2^*}(B)$, it follows that $\langle J_0'(u_n), \varphi\rangle \to 0$ for all $\varphi \in H^1_{0,\mathrm{rad}}(B)$. Indeed, by the embedding $H^1_{0,\mathrm{rad}}(B) \hookrightarrow L^s(B)$ for all $s \in [1, 2^*]$, we have

$$\int_B \nabla u_n \cdot \nabla\varphi\,\mathrm{d}x \to 0, \quad \int_B (u_n^+)^{q-1}\varphi\,\mathrm{d}x \to 0, \quad \text{and}$$
$$\int_B (u_n^+)^{2^*-1}\varphi\,\mathrm{d}x \to 0 \text{as } n \to \infty.$$

Therefore, $\{u_n\}$ is a Palais-Smale sequence for the functional J_0 at the level $d < \frac{1}{N}S^{N/2}$. According to [1.1, 10], the functional J_0 satisfies the Palais–Smale condition for levels $d < \frac{1}{N}S^{N/2}$. Thus, we have $u_n \to 0$ strongly in $H^1_{0,\mathrm{rad}}(B)$, and by the continuity of the functional J, it follows that $J(u_n) \to 0$, which leads to a contradiction.

Therefore, we have $u \not\equiv 0$. Choosing $\varphi = u^-$ as the test function in the equation $\langle J'(u), \varphi\rangle = 0$, we get that $u = u^+ \ge 0$. By the strong maximum principle (see [6, theorem 4, pp. 333]), it follows that $u > 0$ in B. Therefore, u is a weak solution of the problem (1.3), and this completes the proof of theorem 1.1.

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4. Proof of theorem 1.3

For problem (1.5), we will follow a similar strategy to the one we used in the proof of theorem 1.1. We define the functional $J\colon H^1_{0,\mathrm{rad}}(B) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q - \int_B \frac{(u^+)^{2^*+f(r)}}{2^*+f(r)}\,\mathrm{d}x, \tag{4.1}$$

where $u^+(x) = \max\{u(x), 0\}$, $f\colon [0,1) \to \mathbb{R}$ is a continuous function satisfying condition (f) and $q \in [2, 2^*)$. The parameter λ is considered in two cases: if $q = 2$ then $\lambda \in [0, \lambda_1)$, if $q \in (2, 2^*)$ then $\lambda \geq 0$. We will show in the following lemma that the functional J is well defined and by standard arguments, we will obtain that J is of class C^1. We also know that positive critical points of J are weak solutions to the problem (1.5).

LEMMA 4.1. *Let J be the functional given in (4.1). Then J is well-defined.*

Proof. We only have to demonstrate that the variable integral is finite. Let $u \in H^1_{0,\mathrm{rad}}(B)$, then we write

$$\int_B |u|^{2^*+f(r)}\,\mathrm{d}x = \int_{B_{\rho_1}} |u|^{2^*+f(r)}\,\mathrm{d}x + \int_{B_{\rho_2}\setminus B_{\rho_1}} |u|^{2^*+f(r)}\,\mathrm{d}x + \int_{B\setminus B_{\rho_2}} |u|^{2^*+f(r)}\,\mathrm{d}x \tag{4.2}$$

where ρ_1 and ρ_2 will be chosen later. By hypothesis (f), it follows that there exists $\rho_1 > 0$ such that $2 < 2^* + f(r) < 2^*, \forall r \in [0, \rho_1]$. From Hölder's inequality and proposition 2.4, it follows that

$$\int_{B_{\rho_1}} |u|^{2^*+f(r)}\,\mathrm{d}x \leq C(\|u\|_{2^*}^{F_+} + \|u\|_{2^*}^{F_-}) < +\infty \tag{4.3}$$

where $F_+ = \sup_{r\in[0,\rho_1]}(2^*+f(r))$ and $F_- = \inf_{r\in[0,\rho_1]}(2^*+f(r))$. Now, we consider $\rho_2 > 0$ sufficiently close to 1. By lemma 2.1, we know that, for $r \in [\rho_1, \rho_2]$

$$|u(r)| \leq \frac{(1-r)^{1/2}}{r^{(N-2)/2}}\|\nabla u\|_2 \leq \frac{(1-\rho_1)^{1/2}}{\rho_1^{(N-2)/2}}\|\nabla u\|_2 := C_{\rho_1}\|\nabla u\|_2. \tag{4.4}$$

Since f is continuous in $[\rho_1, \rho_2]$ it follows that $f \in L^\infty[\rho_1, \rho_2]$ and therefore the second integral in (4.2) is finite. For $r \in [\rho_2, 1)$, again by lemma 2.1, we get

$$|u(r)| \leq \frac{(1-r)^{1/2}}{r^{(N-2)/2}}\|\nabla u\|_2 \leq \frac{(1-\rho_2)^{1/2}}{\rho_2^{(N-2)/2}}\|\nabla u\|_2 := C_{\rho_2}\|\nabla u\|_2 \leq 1 \tag{4.5}$$

since ρ_2 was chosen sufficiently close to 1. Therefore, we obtain that the third integral in (4.2) is also finite. Therefore, we conclude that the J functional is well-defined. □

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As previously mentioned, we must ensure that the functional J has a positive critical point, for this, we will use the Mountain Pass Theorem, due to Ambrosetti and Rabinowitz [1]. We will show now that the functional J has the geometry of the Mountain Pass Theorem.

LEMMA 4.2. *There exist $\rho > 0$ and $\theta > 0$ such that*

$$J(u) \geq \rho > 0, \text{ if } \|\nabla u\|_2 = \theta.$$

Proof. Let $u \in H^1_{0\mathrm{rad}}(B)$ be such that $\|\nabla u\|_2 = \theta < 1$. By (4.3) and Sobolev's inequality, we have for ρ_1 small enough that

$$\int_{B_{\rho_1}} |u|^{2^*+f(r)}\,\mathrm{d}x \leq C(\|u\|_{2^*}^{F_+} + \|u\|_{2^*}^{F_-}) \leq C_1(\|\nabla u\|_2^{F_+} + \|\nabla u\|_2^{F_-}) \leq C_2\|\nabla u\|_2^{F_-}, \tag{4.6}$$

where $F_+ = \sup_{r\in[0,\rho_1]}(2^* + f(r))$ and $F_- = \inf_{r\in[0,\rho_1]}(2^* + f(r))$. Let $\rho_2 > 0$ be sufficiently close to 1 as in lemma 4.1. By (4.4) and (4.5), and choosing $\theta > 0$ small enough such that $\mathrm{Max}\{C_{\rho_1}, C_{\rho_2}\}\|\nabla u\|_2 < 1$, we obtain

$$\int_{B\setminus B_{\rho_1}} |u|^{2^*+f(r)}\,\mathrm{d}x \leq \int_B \left(\mathrm{Max}\{C_{\rho_1}, C_{\rho_2}\}\|\nabla u\|_2\right)^{2^*+f(r)}\,\mathrm{d}x \leq C_3\|\nabla u\|_2^{F_-} \tag{4.7}$$

where $C_3 = |B|\left(\mathrm{Max}\{C_{\rho_1}, C_{\rho_2}\}\right)^{F_-}$. Then, by (4.6) and (4.7), we get

$$\int_B |u|^{2^*+f(r)}\,\mathrm{d}x \leq C\|\nabla u\|_2^{F_-}, \tag{4.8}$$

where $C = \mathrm{Max}\{C_2, C_3\}$. Therefore, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - \int_B \frac{|u|^{2^*+f(r)}}{2^* + f(r)}\mathrm{d}x \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{q}\|u\|_q^q - C\|\nabla u\|_2^{F_-}. \end{aligned}$$

When $q = 2$, we consider $\lambda \in [0, \lambda_1)$ and then the expression $\sqrt{\|\nabla u\|^2 - \lambda\|u\|^2}$ define a norm in $H^1_0(B)$ equivalent to usual norm. Since $2^* > q \geq 2$ and $F_- > 2$ due to the above inequality and by Sobolev inequality follows that for $\|\nabla u\|_2 = \theta$ with θ sufficiently small, that there exists $\rho > 0$ such that $J(u) \geq \rho > 0$. □

LEMMA 4.3. *Exist $u \in H^1_{0,\mathrm{rad}}(B)$ such that $\|\nabla u\|_2 > \theta$ and $J(u) < 0$.*

Proof. Let $u \in H^1_{0,\mathrm{rad}}(B) \setminus \{0\}$ such that $u > 0$ in B. We have for $t > 1$ that

$$\begin{aligned} J(tu) &= \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{\lambda t^q}{q}\|u\|_q^q - \int_B \frac{t^{2^*+f(r)}|u|^{2^*+f(r)}}{2^* + r^\alpha}\,\mathrm{d}x \\ &\leq \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{\lambda t^q}{q}\|u\|_q^q - \frac{t^{F_-}}{2^* + F_+}\int_B |u|^{2^*+f(r)}\,\mathrm{d}x. \end{aligned}$$

Therefore, since $F_{-} > 2$ and $q \in [2, 2^*)$, we get

$$\lim_{t\to+\infty} J(tu) = -\infty,$$

which proves the lemma. □

Now we will show that the functional J satisfies the (PS) condition.

LEMMA 4.4. Palais–Smale condition *Let $q \in [2, 2^*)$, $\lambda \geq 0$, and $f\colon [0, 1) \to \mathbb{R}$ a continuous function satisfying condition (f). Then the functional J given in (4.1) satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\} \subset H^1_{0,\text{rad}}(B)$ be a Palais–Smale sequence. So, we get

$$J(u_n) \to c > 0 \text{ and } J'(u_n) \to 0. \tag{4.9}$$

Since $2^* + f(r) > 1$ for $r \in [0, 1)$ by standard calculations we know that the sequence $\{u_n\}$ is bounded in $H^1_{0,\text{rad}}(B)$, So, up to a subsequence, there exists $u \in H^1_{0,\text{rad}}(B)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H^1_{0,\text{rad}}(B),\\ &u_n \to u \text{ in } L^s(B)\ \forall\ s \in [1, 2^*). \end{aligned} \tag{4.10}$$

From $J'(u_n) \to 0$, we can choose $\varphi = u_n - u$ as the test function and obtain the following inequality:

$$\begin{aligned} \left| \int_B \nabla u_n \nabla(u_n - u)\, \mathrm{d}x - \lambda \int_B (u_n^+)^{q-1}(u_n - u)\, \mathrm{d}x - \int_B (u_n^+)^{2^*-1+f(r)}(u_n - u)\, \mathrm{d}x \right| \\ \leq \varepsilon_n \|\nabla(u_n - u)\|_2 \leq C\varepsilon_n, \end{aligned} \tag{4.11}$$

where $\varepsilon_n \to 0$. As $q \in [2, 2^*)$, by Hölder inequality and (4.10), we obtain

$$\begin{aligned} \int_B (u_n^+)^{q-1}(u_n - u)\, \mathrm{d}x &\leq \left(\int_B |(u_n^+)^q\, \mathrm{d}x\right)^{(q-1)/q} \left(\int_B |u_n - u|^q\, \mathrm{d}x\right)^{1/q} \\ &\leq C\|u_n - u\|_q \to 0. \end{aligned}$$

Therefore, by (4.11), the lemma will be proved if we check that

$$\int_B (u_n^+)^{2^*-1+f(r)}(u_n - u)\, \mathrm{d}x \to 0. \tag{4.12}$$

Indeed,

$$\begin{aligned} \frac{1}{\omega_{N-1}} \int_B (u_n^+)^{2^*-1+f(r)}(u_n - u)\, \mathrm{d}x = &\int_0^{\rho_1} (u_n^+)^{2^*-1+f(r)}(u_n - u) r^{N-1}\, \mathrm{d}r \\ &+ \int_{\rho_1}^{\rho_2} (u_n^+)^{2^*-1+f(r)}(u_n - u) r^{N-1}\, \mathrm{d}x \\ &+ \int_{\rho_2}^{1} (u_n^+)^{2^*-1+f(r)}(u_n - u) r^{N-1}\, \mathrm{d}r, \end{aligned}$$

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where ρ_1 and ρ_2 will be chosen later. We will estimate each integral above separately. First, for $r>0$ small enough, we know that $2-2^* < f(r) < 0$ because f is continuous at $r=0$ and $f(0)<0$. Therefore, for r small enough, we have that $2 < 2^* + f(r) < 2^*$. So, we can choose $\rho_1 > 0$ small enough such that $2 < 2^* + f_+(\rho^1) < 2^*$, where $f_+(\rho^1) = \sup_{r\in[0,\rho_1]} f(r)$. Then, we get

$$\begin{aligned}
&\omega_{N-1}\int_0^{\rho_1} (u_n^+)^{2^*-1+f(r)}(u_n-u)r^{N-1}\,\mathrm{d}r\\
&\quad\le 2\|(u_n^+)^{2^*-1+f(r)}\|_{\frac{2^*+f(r)}{2^*-1+f(r)}}\|u_n-u\|_{2^*+f(r)}\\
&\quad\le C\|u_n-u\|_{2^*+f_+(\rho^1)}\to 0.
\end{aligned}$$

To estimate the second integral, we need to choose $\rho_2 = 1-\rho_1^{N-2}$ sufficiently close to 1. So, by inequality (2.1) of the lemma 2.1, we get

$$\begin{aligned}
\int_{\rho_1}^{\rho_2} (u_n^+)^{2^*-1+f(r)}(u_n-u)r^{N-1}\,\mathrm{d}x &\le \left(\frac{1}{\rho_1^{(N-2)/2}}\right)^{2^*-1+f_+(\rho)}\int_{\rho_1}^{\rho_2}(u_n-u)r^{N-1}\,\mathrm{d}r\\
&\le C\|u_n-u\|_{L^1(B)}\to 0,
\end{aligned}$$

where $f_+(\rho) = \sup_{r\in[\rho_1,\rho_2]} f(r)$.

To estimate the last integral. Note that for $\rho_2 = 1-\rho_1^{N-2}$ and $\rho_2 < r < 1$, we have

$$\frac{(1-r)^{1/2}}{\rho_1^{(N-2)/2}} \le 1.$$

By inequality (2.2) from lemma 2.1, we get

$$\begin{aligned}
\int_{\rho_2}^{1} (u_n^+)^{2^*-1+f(r)}(u_n-u)r^{N-1}\,\mathrm{d}r &\le \int_{\rho_1}^{1}\left(\frac{(1-r)^{1/2}}{\rho_1^{(N-2)/2}}\right)^{2^*-1+f(r)}(u_n-u)r^{N-1}\,\mathrm{d}r\\
&\le \|u_n-u\|_{L^1(B)}\to 0.
\end{aligned}$$

Therefore, (4.12) is verified and the proof of the lemma is concluded. □

From lemmas 4.2, 4.3, and 4.4, we conclude that the functional J has a non-trivial critical point u. Using $\varphi = u_-$ as a test function in equation $\langle J'(u),\varphi\rangle = 0$, we obtain that $u = u_+ \ge 0$ and by the strong maximum principle (see [6, theorem 4, pp. 333]) it follows that u is positive, thus finishing the proof of the theorem 1.3.

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