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THE EICHLER TRACE OF $\mathbb{Z}_p$ ACTIONS ON RIEMANN SURFACES

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ABSTRACT. We study $\mathbb{Z}_p$ actions on compact connected Riemann surfaces via their associated Eichler traces. We determine the set of possible Eichler traces and determine the relationship between 2 actions if they have the same trace.

1. **Introduction.** In this paper we study group actions of $\mathbb{Z}_p$, the cyclic group of odd prime order p, on compact connected Riemann surfaces S. If the genus of S is g then the vector space V of holomorphic differentials on S has dimension g and any action of $\mathbb{Z}_p$ on S determines a representation $\rho\colon \mathbb{Z}_p \longrightarrow \mathrm{GL}(V)$. If T is a preferred generator of $\mathbb{Z}_p$ then this representation yields a matrix $\rho(T) \in \mathrm{GL}(V)$. The trace of this matrix, $\chi = \mathrm{tr}(T)$, is referred to as the Eichler trace. Clearly $\chi \in \mathbb{Z}[\zeta]$, where $\zeta = e^{\frac{2\pi i}{p}}$.

One of the goals of this paper is to determine how much information about the action of $\mathbb{Z}_p$ is captured by the Eichler trace. There are actually two questions here.

QUESTION 1. What elements $\chi \in \mathbb{Z}[\zeta]$ can be realized as the trace of some action?

QUESTION 2. What is the relationship between two actions, not necessarily on the same surface, if they have the same trace?

We give complete answers to both questions. To explain our results we need to develop some notation. Let T be an automorphism of order p on a compact connected Riemann surface S. Suppose there are t fixed points $P_1, \ldots, P_t$. In a sufficiently small neighbourhood of a fixed point P_j the automorphism will have the form $T\colon z \longrightarrow \zeta^{k_j} z$ for some integer k_j, $1 \leq k_j \leq p-1$. This integer is defined to be the rotation number at P_j. The Eichler Trace Formula, see [7], is

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}. \tag{1}$$

Let A denote the set of all Eichler traces of all possible actions, that is

$$A = \{\chi \in \mathbb{Z}[\zeta] \mid \chi = \mathrm{tr}(T)\}, \tag{2}$$

where T is any automorphism of order p on any compact connected Riemann surface S. A simple calculation with the Eichler Trace Formula 1 shows that $\chi + \overline{\chi} = 2 - t$ for any $\chi \in A$, where $\overline{\chi}$ denotes the complex conjugate of χ. Thus $A \subset B$, where

$$B = \{\chi \in \mathbb{Z}[\zeta] \mid \chi + \overline{\chi} \in \mathbb{Z}\}. \tag{3}$$

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In Section 3 we show that B is a free abelian subgroup of $\mathbb{Z}[\zeta]$ of rank $(p+1)/2$ and determine a basis. Thus a reasonable first step in describing A is to determine the "index" of A in B. Unfortunately, it turns out that A is not a subgroup of B, so this does not make sense. On the other hand, the quotient set $\hat{A} = A/\mathbb{Z}$, that is the elements of A modulo the integers, is a group, in fact a subgroup of $\hat{B} = B/\mathbb{Z}$. We prove that $\hat{B}$ is a free abelian group of rank $(p-1)/2$ and that the index of $\hat{A}$ in $\hat{B}$ is finite.

A slightly different version of the following theorem was first announced, but not proved, in Ewing's paper [6]. An equivalent result, stated in terms of Witt classes and G-signatures, first appeared in Ewing [5]. At the end of Section 3 we briefly indicate how to translate Ewing's results into ours.

THEOREM 1. *The index of $\hat{A}$ in $\hat{B}$ is h_1, the first factor of the class number h of $\mathbb{Z}[\zeta]$.*

This theorem gives a partial answer to Question 1. In Section 4 we find free generators of $\hat{A}$, thereby answering completely Question 1. See Theorems 4, 5 and Corollary 1 in this section, and Proposition 1 in Section 2.

Implicit in these theorems is an answer to Question 2. To an automorphism $T: S \longrightarrow S$ of order p we associate a "vector" $[g; k_1, \ldots, k_t]$, where g is the genus of the orbit surface $S/\mathbb{Z}_p$, t is the number of fixed points, and the k_j are the rotation numbers. The rotation numbers are unique modulo p, but their order is not determined. From the Eichler Trace Formula 1 it is clear that $\chi = \operatorname{tr}(T)$ does not depend on g or on the order of the k_j. If a cancelling pair $\{k, p-k\}$, where $1 \leq k \leq p-1$, appears amongst the set of rotation numbers $\{k_1, \ldots, k_t\}$, then an easy calculation shows that their contribution to the Eichler trace is

$$\frac{1}{\zeta^k - 1} + \frac{1}{\zeta^{p-k} - 1} = -1. \tag{4}$$

Thus we can replace the cancelling pair $\{k, p-k\}$ by any other cancelling pair $\{l, p-l\}$ and not change the Eichler trace.

Given two such automorphisms

$$T_1: S_1 \longrightarrow S_1, \quad T_2: S_2 \longrightarrow S_2$$

we have two "vectors" $[g; k_1, \ldots, k_t]$, $[h; l_1, \ldots, l_u]$. Let χ_1 and χ_2 denote the respective Eichler traces.

THEOREM 2. *$\chi_1 = \chi_2$ if, and only if, $t = u$ and the set of rotation numbers $\{k_1, \ldots, k_t\}$ agrees with $\{l_1, \ldots, l_u\}$ up to permutations and replacements of cancelling pairs.*

As far as the set of Eichler traces is concerned there is no loss of generality in assuming that the orbit surface $S/\mathbb{Z}_p$ is the extended complex plane $\hat{\mathbb{C}}$. In other words, if χ is the Eichler trace of some action with orbit genus $g > 0$ (the orbit genus is defined to be the genus of $S/\mathbb{Z}_p$) then there will be some other action, on a different Riemann surface, with the same Eichler trace χ and orbit genus $g = 0$. There is also no loss of generality in considering actions up to topological conjugacy since an easy consequence of the Eichler Trace Formula 1 is that conjugate actions have the same Eichler trace.

In Section 2 we recall how $\mathbb{Z}_p$ actions with orbit genus $g = 0$ can be parametrized up to topological conjugacy by sequences

$$(5)\quad [a_1, \dots, a_t], \quad \text{where} \quad 1 \le a_1 \le \cdots \le a_t \le p-1, \text{ and } \sum_{j=1}^{t} a_j \equiv 0 \pmod{p}.$$

The next result is due to Nielsen [11], and is stated here only for reference.

THEOREM 3. *There is a one-to-one correspondence between the set of topological conjugacy classes of automorphisms $T: S \longrightarrow S$ of order p and orbit genus 0, where S is an arbitrary compact connected Riemann surface, and sequences satisfying the conditions in 5. The integer t is the number of fixed points and the rotation numbers k_j are determined by the equations $k_j a_j \equiv 1 \pmod{p}$, $1 \le j \le t$.*

We can also parametrize these actions by sequences $[a_1, \dots, a_t]$, where $1 \le a_j \le p-1$ for $1 \le j \le t$, and $\sum_{j=1}^{t} a_j \equiv 0 \pmod{p}$. Two such sequences $[a_1, \dots, a_t]$, $[b_1, \dots, b_u]$ are the same if $t = u$ and they agree up to a permutation.

Now consider the infinitely generated free abelian group F generated by all sequences $[a_1, \dots, a_t]$ as in 5. The goal is to make the Eichler trace into a group isomorphism, so we introduce some relations.

DEFINITION 1. Let A denote the abelian group F/R, where R is the subgroup of F generated by the following relations:

(i) $[a_1, \dots, a_t] + [b_1, \dots, b_u] = [a_1, \dots, a_t, b_1, \dots, b_u]$.

(ii) $[\dots, a, \dots, p-a, \dots] = [\dots, \hat{a}, \dots, \widehat{p-a}, \dots]$.

In relation (i) the sequences $[a_1, \dots, a_t]$, $[b_1, \dots, b_u]$ are arbitrary sequences satisfying condition 5, and in relation (ii) the sequence $[\dots, a, \dots, p-a, \dots]$ is any sequence satisfying condition 5 and having a cancelling pair $\{a, p-a\}$. The first relation says that addition in A is, up to rearrangement, concatenation of sequences, and the second relation allows us to delete a cancelling pair $\{a, p-a\}$. It follows that the identity element of A is represented by the empty sequence [ ], or by any sequence consisting entirely of cancelling pairs. The inverse of $[a_1, \dots, a_t]$ is represented by $[p-a_1, \dots, p-a_t]$, up to rearrangement.

In Section 4 we prove the following two theorems.

THEOREM 4. *The Eichler trace determines a natural group isomorphism $\eta: A \longrightarrow \hat{A}$.*

THEOREM 5. *The abelian group A is free of rank $(p-1)/2$. A free basis is given by the triples $[1, r, s]$, where $1 \le r \le s \le p-1$ and $1 + r + s \equiv 0 \pmod{p}$.*

It follows from these theorems that $\hat{A}$ is a free abelian group of rank of $(p-1)/2$. In the next corollary we give a basis, thereby completely answering Question 1.

COROLLARY 1. *$\hat{A}$ is a free abelian group of rank $(p-1)/2$. It is freely generated by the $\bmod \mathbb{Z}$ representatives of the $(p-1)/2$ elements:*

$$\chi_{r,s} = \frac{1}{\zeta - 1} + \frac{1}{\zeta^r - 1} + \frac{1}{\zeta^s - 1}, \quad \textit{where } 1 \le r \le s \le p-1 \textit{ and } 1 + r + s \equiv 0 \pmod{p}.$$

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Then in Section 5 we give Theorems 4 and 5 geometric content by relating equivariant cobordism of $\mathbb{Z}_p$ actions on compact connected Riemann surfaces to $\hat{A}$. To explain this let Ω denote the group of equivariant cobordism classes of $\mathbb{Z}_p$ actions. In the definition of Ω we do not assume the orbit genus is zero. We show that the Eichler trace induces a natural group homomorphism $\phi: A \longrightarrow \Omega$.

THEOREM 6. *$\phi: A \longrightarrow \Omega$ is a group isomorphism.*

COROLLARY 2. *Ω is a free abelian group of rank $(p-1)/2$. A free basis is given by those cobordism classes of automorphisms $T: S \longrightarrow S$ having order p, orbit genus 0, and three fixed points, at least one of which has rotation number one. If the other two rotation numbers are k_2, k_3 then the only restriction is that $1 + a_2 + a_3 \equiv 0 \pmod{p}$, where $k_2a_2 \equiv k_3a_3 \equiv 1 \pmod{p}$.*

2. **Preliminaries.** In this section we collect some of the preliminaries needed for later sections. First we describe how all group actions on Riemann surfaces occur and then we specialize to the case of the group $\mathbb{Z}_p$.

We will use the notation $\mathrm{Aut}(S)$ for the group of analytic automorphisms of a Riemann surface S. Throughout the paper all surfaces will be connected, orientable and without boundary. By the uniformization theorem the universal covering space $\mathbb{U}$ of S is one of three possibilities: the extended complex plane $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper half plane $\mathbb{H}$. The letter $\mathbb{U}$ will always denote one of these three.

If G is a finite group acting topologically on a surface S by orientation preserving homeomorphisms then the positive solution of the Nielsen Realization Problem guarantees that there exists a complex analytic structure on S for which the action of G is by analytic automorphisms (see [12], [9], [8] or [2]). Thus there is no loss of generality in assuming that the action of G is complex analytic to begin with, and we will tacitly do so.

To any action of G on S we associate a short exact sequence of groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1, \tag{6}$$

with Γ being a discrete subgroup of $\mathrm{Aut}(\mathbb{U})$ and Π a torsion free normal subgroup of Γ, as follows. Let $\pi: \mathbb{U} \longrightarrow S$ denote the covering map. Then Γ is defined by

$$\Gamma = \{\gamma \in \mathrm{Aut}(\mathbb{U}) \mid \pi \circ \gamma = g \circ \pi, g \in G\}. \tag{7}$$

In other words Γ consists of all lifts $\gamma: \mathbb{U} \longrightarrow \mathbb{U}$ of all automorphisms $g: S \longrightarrow S$, $g \in G$. The subgroup Γ is unique up to conjugation in $\mathrm{Aut}(\mathbb{U})$.

The epimorphism $\theta: \Gamma \longrightarrow G$ is defined by $\theta(\gamma) = g$, where γ and g are as in 7. The kernel of $\theta: \Gamma \longrightarrow G$ is Π, the fundamental group of S, and is therefore torsion free. The action of G on $S = \mathbb{U}/\Pi$ is given by $g[z]_\Pi = [\gamma(z)]_\Pi$, where $z \in \mathbb{U}$, $g \in G$, and $\gamma \in \Gamma$ is any element such that $\theta(\gamma) = g$. Here the square brackets denote the orbits under the action of Π. The orbit surface $\overline{S} = \mathbb{U}/\Gamma$, and the branched covering $\pi: S \longrightarrow \overline{S}$ is just the natural map $\mathbb{U}/\Pi \longrightarrow \mathbb{U}/\Gamma$, $[z]_\Pi \longmapsto [z]_\Gamma$.

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Conversely, suppose $1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1$ is a given short exact sequence of groups, where Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{U})$ and Π is torsion free. Then this short exact sequence corresponds to the one arising from the action of G on the Riemann surface $S = \mathbb{U}/\Pi$ defined above.

Thus there is a one-to-one correspondence between analytic conjugacy classes of analytic actions by the finite group G on compact connected Riemann surfaces and short exact sequences 6, where Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{U})$, unique only up to conjugation in $\mathrm{Aut}(\mathbb{U})$, and Π is a torsion free subgroup of Γ.

Now suppose G is the cyclic group $\mathbb{Z}_p$ and $T \in \mathbb{Z}_p$ denotes a fixed generator. Actions of $\mathbb{Z}_p$ on Riemann surfaces correspond to short exact sequences $1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} \mathbb{Z}_p \longrightarrow 1$. Since the kernel of θ is torsion free the signature of Γ must have the form $(g; \overbrace{p, \ldots, p}^{t\text{ times}})$, where g and t are non-negative integers. As an abstract group Γ has the following presentation

(i) $t + 2g$ generators $A_1, \ldots, A_t, X_1, Y_1, \ldots, X_g, Y_g$.

(ii) $t + 1$ relations $A_1^p = \cdots = A_t^p = A_1 \cdots A_t [X_1, Y_1] \cdots [X_g, Y_g] = 1$.

We denote this group by $\Gamma = \Gamma(g; \overbrace{p, \ldots, p}^{t\text{ times}})$. Any such group can be embedded in $\mathrm{Aut}(\mathbb{U})$ as a discrete subgroup in many different ways up to conjugation.

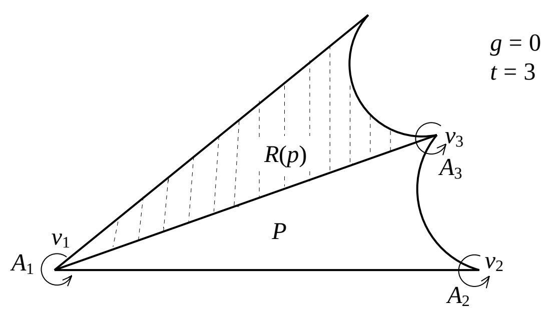

FIGURE 1: Fundamental Domain

Figure 1 illustrates a fundamental domain for a particular embedding when $g = 0$ and $t = 3$. It consists of a regular 3-gon P, all of whose angles are π/p, together with a copy of P obtained by reflection in one of its sides. The generators A_1, A_2, A_3 are the rotations by $2\pi/p$ about consecutive vertices, ordered in the counterclockwise sense.

Let Γ be any Fuchsian group of signature $(g; \overbrace{p, \ldots, p}^{t\text{ times}})$. Then an epimorphism $\theta \colon \Gamma \longrightarrow \mathbb{Z}_p$ is determined by the images of the generators. The relations in Γ must be preserved and the kernel of θ must be torsion free, so θ is determined by the equations

$$\theta(A_j) = T^{a_j}, \quad 1 \le j \le t; \ \theta(X_k) = T^{b_k}, \ \theta(Y_k) = T^{c_k}, \quad 1 \le k \le g.$$

The following restrictions must hold:

(i) The a_j are integers such that $1 \le a_j \le p-1$ and $\sum_{j=1}^{t} a_j \equiv 0 \pmod p$.

(ii) The b_k, c_k are arbitrary integers mod p, except that at least one of them is non-zero if $t = 0$ (this guarantees that θ is an epimorphism).

It follows from the first restriction that the only possible values of t are $t = 0, 2, 3, \ldots$.

Conversely, given integers a_j, b_k, c_k satisfying conditions (i) and (ii), there is an epimorphism $\theta\colon \Gamma \longrightarrow \mathbb{Z}_p$ with torsion free kernel Π and a corresponding $\mathbb{Z}_p$ action $T\colon S \longrightarrow S$, where $S = \mathbb{U}/\Pi$.

The integer t equals the number of fixed points of $T\colon S \longrightarrow S$ and g is the genus of the orbit surface $S/\mathbb{Z}_p$. A well known result of Nielsen [11] says that the topological conjugacy class of $T\colon S \longrightarrow S$ is completely determined by g and the unordered sequence $(a_1, \ldots, a_t)$. We use the notation $[g \mid a_1, \ldots, a_t]$ to denote the topological conjugacy class of the homeomorphism $T\colon S \longrightarrow S$ determined by this data. If $g = 0$ we use the notation $[a_1, \ldots, a_t]$, and usually order the a_j so that $1 \le a_1 \le \cdots \le a_t \le p-1$.

Of particular interest is the case $g = 0$. Then the orbit surface $S/\mathbb{Z}_p$ is the extended complex plane $\hat{\mathbb{C}}$ and Γ has the presentation

(i) t generators $A_1, \ldots, A_t$.

(ii) $t+1$ relations $A_1^p = \cdots = A_t^p = A_1 \cdots A_t = 1$.

The epimorphism θ is given by the equations

$$\theta(A_j) = T^{a_j}, \quad \text{where } 1 \le a_1 \le \cdots \le a_t \le p-1, \text{ and } \sum_{j=1}^{t} a_j \equiv 0 \pmod p. \tag{8}$$

With these preliminaries it is now straight forward to give an answer to Question 1 in the introduction. This is just a matter of determining the possible sets of rotation numbers. Thus let $\{k_1, \ldots, k_t\}$ be any set of t numbers satisfying $1 \le k_j \le p-1$, $1 \le j \le t$, and let a_j denote that number such that $k_j a_j \equiv 1 \pmod p$ and $1 \le a_j \le p-1$.

PROPOSITION 1. *$1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j}-1} \in A$, if, and only if, $\sum_{j=1}^{t} a_j \equiv 0 \pmod p$.*

3. **The Eichler trace.** In this section we prove Theorems 1 and 2. Recall that the class number h of the ring of integers $\mathbb{Z}[\zeta]$ is the number of equivalence classes of non-zero integral ideals I in $\mathbb{Z}[\zeta]$, where the equivalence relation is fractional equivalence:

$$I \sim J \quad \text{if there exist non-zero elements } r, s \in \mathbb{Z}[\zeta], \text{ such that } rI = sJ.$$

In fact the collection of equivalence classes of integral ideals forms a finite abelian group $\mathcal{C}$ of order h, where the group structure is given by multiplication of ideals. See [10]. If I is an ideal then so is its complex conjugate $\bar{I}$ and it is easy to see that

$$\mathcal{C}_1 = \{I \mid I\bar{I} \text{ is a principal ideal}\}$$

is a subgroup of $\mathcal{C}$. The order of this subgroup is by definition the first factor h_1 of the class number.

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We begin by observing that the set A is not a subgroup of $\mathbb{Z}[\zeta]$. To see this suppose that $\chi \in A$, that is

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}$$

is the Eichler trace of some automorphism $T\colon S \longrightarrow S$. The possible values for the number of fixed points are $t = 0, 2, 3, \ldots$, and therefore the possible values of $\chi + \overline{\chi} = 2 - t$ are $2, 0, -1, -2, \ldots$. We also have $\overline{\chi} \in A$ since

$$\overline{\chi} = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{-k_j} - 1}$$

is the trace of $T^{-1}\colon S \longrightarrow S$. Therefore, if A were a subgroup we would have $\chi + \overline{\chi} = 2 - t \in A$, and hence $\mathbb{Z}$ would be a subgroup of A. But if $n \in A$ is an integer, $n \geq 2$, then $n + \overline{n} = 2n \geq 4$ is not of the form $2 - t$ for an admissible t. Therefore A is not a subgroup.

Recall that $\hat{A}$ is the set of realizable Eichler traces modulo $\mathbb{Z}$.

PROPOSITION 2. *$\hat{A}$ is a subgroup of $\widehat{\mathbb{Z}[\zeta]}$.*

PROOF. Suppose $\chi_1 = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j}-1}$ and $\chi_2 = 1 + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j}-1}$ are in A. Therefore $\widehat{\chi_1} + \widehat{\chi_2} = \hat{\chi}$, where $\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j}-1} + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j}-1}$. If χ_1 and χ_2 are represented by $T_1\colon S_1 \longrightarrow S_1$ and $T_2\colon S_2 \longrightarrow S_2$ respectively, then χ can be represented by the equivariant connected sum of T_1 and T_2. Thus $\hat{A}$ is closed under sums.

If $\chi \in A$ then also $\overline{\chi} \in A$ and $\chi + \overline{\chi} = 2 - t$. Therefore $\overline{\chi}$ is the inverse of χ once we reduce modulo the integers. The identity element of $\hat{A}$ is represented by any fixed point free action. ∎

To determine the index of $\hat{A}$ in $\hat{B}$ we need a basis for $\hat{B}$, but first we find a basis for B.

DEFINITION 2. Let $m = (p-1)/2$ and define elements $\theta_1, \theta_2, \ldots, \theta_m$ in $\mathbb{Z}[\zeta]$ by $\theta_1 = \zeta + \sum_{j=m+1}^{p-2} \zeta^j$ and $\theta_k = \zeta^k - \zeta^{-k}$, $2 \leq k \leq m$.

The proof of the following proposition is elementary.

PROPOSITION 3. *A basis of B is given by the $m+1$ elements* $1, \theta_1, \theta_2, \ldots, \theta_m$.

REMARK. Every integer $n \in B$, and in fact $\theta_1 + \overline{\theta_1} = -1$. We also have $\zeta - \zeta^{-1} \in B$; since

$$\zeta - \zeta^{-1} = 1 + 2\theta_1 + \theta_2 + \cdots + \theta_m.$$

It follows that $1, \zeta - \zeta^{-1}, \zeta^2 - \zeta^{-2}, \ldots, \zeta^m - \zeta^{-m}$ form a basis for an index 2 subgroup of B.

An immediate corollary of Proposition 3 is

COROLLARY 3. *$\hat{B}$ is a free abelian group of rank $(p-1)/2$ with basis*

$$\widehat{\theta_1}, \widehat{\theta_2}, \ldots, \widehat{\theta_m}.$$

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Before completing the calculation of the index of $\hat{A}$ in $\hat{B}$ we first consider Question 2 from Section 1. Thus suppose two elements from A have the same Eichler trace, say

$$1+\sum_{j=1}^{t}\frac{1}{\zeta^{k_j}-1}=1+\sum_{j=1}^{u}\frac{1}{\zeta^{l_j}-1}.$$

This leads us into consideration of equations $\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1}=0$ in integers x_k. To solve this equation we need the next lemma, whose proof is omitted.

If s is any integer relatively prime to p then let $R(s)$ denote that integer q such that $1\le q\le p-1$ and $q\equiv s \pmod p$, that is, $s=[s/p]p+R(s)$. In what follows $\sum_{jk\equiv n}$ denotes the sum over all ordered pairs (j,k) such that $jk\equiv n \pmod p$ and $1\le j\le p-1$.

LEMMA 1.

$$\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1}=-\frac{1}{p}\sum_{jk\equiv -1}jx_k+\frac{1}{p}\sum_{n=1}^{p-2}\Big(\sum_{jk\equiv n}jx_k-\sum_{jk\equiv -1}jx_k\Big)\zeta^n.$$

As a corollary we get

COROLLARY 4. $\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1}=0$ *if, and only if,* $\sum_{jk\equiv n}jx_k=0$, *for* $1\le n\le p-1$.

Now it is convenient to change the variables $x_1,\ldots,x_{p-1}$ to new variables $y_1,\ldots,y_{p-1}$ according to the equation

$$y_l=x_k,\quad \text{where } kl\equiv 1 \pmod p. \tag{9}$$

Then Corollary 4 becomes

COROLLARY 5. $\sum_{k=1}^{p-1}\frac{x_k}{\zeta^k-1}=0$ *if, and only if,* $\sum_{k=1}^{p-1}R(nk)y_k=0$, *for* $1\le n\le p-1$.

The coefficient matrix of this linear system is the $(p-1)\times(p-1)$ matrix M whose (i,j) entry is $M_{(i,j)}=R(ij)$. To solve this system of $p-1$ equations in $p-1$ unknowns we apply a sequence of row and column operations. The details are left to the reader. Recall that $m=(p-1)/2$.

1. Add the i-th row to the $(p-i)$-th row, $1\le i\le m$.
2. Add the j-th column to the $(p-j)$-th column, $1\le j\le m$.
3. Subtract the $(m+1)$-st row from rows $m+2,\ldots,p-1$, and then subtract the $(m+1)$-st column from columns $m+2,\ldots,p-1$.

The variables z_k for the new coefficient matrix are related to the y_k by the equations

$$z_k=y_k-y_{p-k},\quad 1\le k\le m,\ z_{m+1}=y_{m+1}+\cdots+y_{p-1},\ z_{m+j}=y_{m+j},\quad 2\le j\le p-1.$$

It turns out that $z_{m+2},\ldots,z_{p-1}$ are completely independent; whereas, $z_1,\ldots,z_{m+1}$ satisfy

$$\begin{bmatrix}1&2&\cdots&m&p\\2&4&\cdots&2m&p\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ i&R(2i)&\cdots&R(mi)&p\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ m&R(2m)&\cdots&R(m^2)&p\\ p&p&\cdots&p&2p\end{bmatrix}\begin{bmatrix}z_1\\z_2\\ \vdots\\ z_i\\ \vdots\\ z_m\\ z_{m+1}\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\0\\ \vdots\\0\\0\end{bmatrix}$$

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Now we apply another sequence of row and column operations to this last coefficient matrix.

1. Subtract i times the first row from the i-th row, $2 \le i \le m$.
2. Subtract j times the first column from the j-th column, $2 \le j \le m$.

The new variables w_j are related to the z_j by $w_1 = z_1 + 2z_2 + \cdots + mz_m$ and $w_j = z_j$, $2 \le j \le m+1$, and the new equations are $w_1 = w_{m+1} = 0$, $w_2 + 2w_3 + \cdots + (m-1)w_m = 0$ and

$$\begin{bmatrix} -[9/p]p & \cdots & -[3j/p]p & \cdots & -[3m/p]p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[3i/p]p & \cdots & -[ij/p]p & \cdots & -[im/p]p \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -[3m/p]p & \cdots & -[mj/p]p & \cdots & -[m^2/p]p \end{bmatrix} \begin{bmatrix} w_3 \\ \vdots \\ w_j \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The coefficient matrix of this system can be row reduced to the matrix whose (i,j) entry, $3 \le i,j \le m$, is $[ij/p]p - [(i-1)j/p]p$, by first subtracting row $m-3$ from row $m-2$, then row $m-4$ from row $m-3$, *etc.*, and then changing all signs. The resulting matrix is invertible, in fact its determinant equals $\pm p^{m-2}h_1$, where h_1 is the first factor of the class number [1]. Thus $w_j = 0$, $1 \le j \le m+1$.

This proves that $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k-1} = 0$ if, and only if, $y_k = y_{p-k}$ for $1 \le k \le p-1$, and

$$y_m = -y_{m+2} - \cdots - y_{p-1},$$

where $y_{m+2}, \ldots, y_{p-1}$ are completely arbitrary. Translating back to the x_k variables we have:

COROLLARY 6. *$\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k-1} = 0$ if, and only if, $x_k = x_{p-k}$ for $1 \le k \le p-1$, and*

$$x_m = -x_{m+2} - \cdots - x_{p-1},$$

where $x_{m+2}, \ldots, x_{p-1}$ are completely arbitrary.

We can now complete the proof of Theorem 2.

PROOF. Suppose $\chi_1 = \chi_2$ are the Eichler traces of two actions, say

$$\chi_1 = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1} = 1 + \sum_{k=1}^{p-1} \frac{u_k}{\zeta^k - 1},$$
$$\chi_2 = 1 + \sum_{j=1}^{u} \frac{1}{\zeta^{l_j} - 1} = 1 + \sum_{k=1}^{p-1} \frac{v_k}{\zeta^k - 1},$$

where u_k is the number of times k appears as a rotation number in χ_1, and v_k is defined similarly. We immediately get $t = u$ since $\chi_1 + \overline{\chi}_1 = 2-t$ and $\chi_2 + \overline{\chi}_2 = 2-u$. The equation $\chi_1 - \chi_2 = 0$ gives the linear relation $\sum_{k=1}^{p-1} \frac{x_k}{\zeta^k-1} = 0$, where $x_k = u_k - v_k$. It follows from Corollary 6 that the vector $\vec{x} = (x_1, \ldots, x_{p-1})$ is an integral linear combination of the vectors

$$\vec{e}_j = (\ldots, 1, \ldots, -1, -1, \ldots, 1, \ldots), \quad 1 \le j \le m-1,$$

where the 1's are in positions $j, p-j$; the -1's are in positions $m, m+1$; and the other entries are zero.

For argument's sake suppose $\vec{x} = \vec{e_j}$ for some j. This means we can move from the vector of rotation numbers $[u_1, \ldots, u_{p-1}]$ to the vector $[v_1, \ldots, v_{p-1}]$ by replacing the cancelling pair $\{j, p-j\}$ by the cancelling pair $\{m, m+1\}$. Taking linear combinations of the $\vec{e_j}$ corresponds to a sequence of such moves. This completes the proof of Theorem 2. ∎

The remainder of this section is concerned with the proof of Theorem 1. According to Proposition 1 the set of Eichler traces is given by

$$A = \Big\{\chi \in \mathbb{Z}[\zeta] \mid \chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}\Big\},$$

where the only restriction on the rotation numbers k_j is that $\sum_{j=1}^{t} R(k_j^{-1}) \equiv 0 \pmod p$. If we define x_k to be the number of j, $1 \le j \le t$, such that $k_j = k$, then we can characterize A by

$$A = \Big\{\chi \in \mathbb{Z}[\zeta] \mid \chi = 1 + \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1}, x_k \ge 0 \text{ and } \sum_{k=1}^{p-1} R(k^{-1}) x_k \equiv 0 \pmod p\Big\}. \tag{10}$$

In the next lemma we show that by passing to $\hat{A}$ we can remove the restriction that the x_k be non-negative integers.

LEMMA 2. *The set of Eichler traces modulo $\mathbb{Z}$ is given by*

$$\hat{A} = \Big\{\hat{\chi} \in \mathbb{Z}\hat{[\zeta]} \mid \chi = \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1}, \sum_{k=1}^{p-1} R(k^{-1}) x_k \equiv 0 \pmod p\Big\}.$$

PROOF. First note that by choosing all $x_k = 1$ in Equation 10 we get an element $\chi \in A$. In fact a short calculation using Lemma 1 gives $\chi = 1 - (p-1)/2$, and thus this element represents 0 in $\hat{A}$. By adding χ sufficiently many times to an element in A we can ensure that all the coefficients x_k become positive, and this does not change its value in $\hat{A}$. ∎

This description of $\hat{A}$ contains a lot of redundancy, as the next lemma shows.

LEMMA 3. *The set of Eichler traces modulo $\mathbb{Z}$ is given by*

$$\hat{A} = \Big\{\hat{\chi} \mid \chi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1}, \sum_{k=1}^{m} R(k^{-1}) z_k \equiv 0 \pmod p\Big\}.$$

PROOF. According to Lemma 2 a typical element $\hat{\chi} \in \hat{A}$ can be represented by

$$\chi = \sum_{k=1}^{p-1} \frac{x_k}{\zeta^k - 1} = \sum_{k=1}^{m} \frac{x_k}{\zeta^k - 1} + \sum_{k=1}^{m} \frac{x_{p-k}}{\zeta^{-k} - 1},$$

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where the x_k are integers satisfying $\sum_{k=1}^{p-1} R(k^{-1})x_k \equiv 0 \pmod p$. Using Equation 4 we see that $\hat{\chi} = \hat{\psi}$, where $\psi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1}$ and $z_k = x_k - x_{p-k}$. The restriction on the integers z_k is easily seen to be $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0 \pmod p$. ∎

In Definition 2 we introduced elements $\theta_1, \theta_2, \ldots, \theta_m$ and then in Corollary 3 we showed that the corresponding classes modulo $\mathbb{Z}$, that is $\widehat{\theta_1}, \widehat{\theta_2}, \ldots, \widehat{\theta_m}$, formed a basis of $\hat{B}$. To determine the index of $\hat{A}$ in $\hat{B}$ we want to express a typical element of $\hat{A}$ in terms of this basis. But first we need a definition.

DEFINITION 3. For integers k, n define $C(k, n) = R(k^{-1}n) + R(k^{-1}) - p$. The following properties of the coefficients $C(k, n)$ are easy to verify:

(i) $C(k, n) + C(p - k, n) = 0$ and $C(k, n) + C(k, p - n) = 2R(k^{-1}) - p$.

(ii) $C(1, n) = n+1-p$, $C(k, 1) = 2R(k^{-1})-p$, $C(p-1, n) = p-n-1$, and $C(k, p-1) = 0$.

LEMMA 4. *The elements of $\hat{A}$ are those elements $\hat{\chi} \in \widehat{\mathbb{Z}[\zeta]}$ of the form*

$$\hat{\chi} = \frac{1}{p} \sum_{n=1}^{m} \Big(\sum_{k=1}^{m} C(k, n) z_k \Big) \widehat{\theta_n},$$

where the only restriction on the integers z_k is $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0 \pmod p$.

PROOF. By Lemma 3 a typical Eichler trace modulo $\mathbb{Z}$ is given by $\hat{\chi}$, where $\chi = \sum_{k=1}^{m} \frac{z_k}{\zeta^k - 1}$, and $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0 \pmod p$. Using Lemma 1 we have

$$\chi = -\frac{1}{p} \sum_{jk \equiv -1} jz_k + \frac{1}{p} \sum_{n=1}^{p-2} \Big(\sum_{jk \equiv n} jz_k - \sum_{jk \equiv -1} jz_k \Big) \zeta^n.$$

The condition $\sum_{k=1}^{m} R(k^{-1})z_k \equiv 0 \pmod p$ can be written as $\sum_{jk \equiv 1} jz_k \equiv 0 \pmod p$, and so $\sum_{jk \equiv -1} jz_k = \sum_{jk \equiv 1} (p - j)z_k \equiv 0 \pmod p$. Therefore, modulo $\mathbb{Z}$ we have

$$\chi \equiv \frac{1}{p} \sum_{n=1}^{p-2} \Big(\sum_{jk \equiv n} jz_k - \sum_{jk \equiv -1} jz_k \Big) \zeta^n \equiv \frac{1}{p} \sum_{n=1}^{p-1} \Big(\sum_{jk \equiv n} jz_k - \sum_{jk \equiv -1} jz_k \Big) \zeta^n.$$

Note that the term corresponding to $n = p - 1$ contributes 0 to the sum. Also note that

$$\sum_{jk \equiv n} jz_k - \sum_{jk \equiv -1} jz_k = \sum_{k=1}^{m} C(k, n) z_k$$

and therefore $\chi \equiv \frac{1}{p} \sum_{n=1}^{p-1} \big(\sum_{k=1}^{m} C(k, n) z_k \big) \zeta^n$.

To complete the proof we break the last sum up into two pieces, one piece for $1 \leq n \leq m$, the other for the remaining values of n, and then use properties of the coefficients $C(k, n)$. ∎

DEFINITION 4. Let K be the collection of m-tuples $\vec{v} = [z_1, \ldots, z_m]$ satisfying the condition

$$\sum_{k=1}^{m} R(k^{-1}) z_k \equiv 0 \pmod p.$$

Thus K is a free abelian group of rank m. We can write $z_1 = lp - \sum_{k=2}^{m} R(k^{-1})z_k$, for some integer l, and therefore a basis of K is given by the vectors

$$\vec{v_1} = [p, 0, \dots, 0], \vec{v_k} = [-R(k^{-1}), \dots, 1, \dots], \quad 2 \le k \le m,$$

where the 1 is in the k-th entry, and all other entries, except the first, are zero.

Now consider the group homomorphism $L\colon K \longrightarrow \hat{A}$ defined by

$$L(\vec{v}) = \frac{1}{p} \sum_{n=1}^{m} \Big(\sum_{k=1}^{m} C(k,n) z_k \Big) \widehat{\theta_n}.$$

Lemma 4 implies that L is an epimorphism.

PROPOSITION 4. *L is a group isomorphism.*

PROOF. We first compute the images of the basis elements $\vec{v_1}$ and $\vec{v_k}$, $2 \le k \le m$, using properties of the coefficients $C(k,n)$. By a routine calculation we have:

$$L(\vec{v_1}) = \sum_{n=1}^{m} (n+1-p)\widehat{\theta_n}$$

$$L(\vec{v_k}) = \sum_{n=1}^{m} \Big(-\Big[\frac{jn}{p}\Big] + j - 1 \Big) \widehat{\theta_n},$$

where $j = R(k^{-1})$.

Now consider the $m \times m$ matrix M whose (k,n) entry is given by

$$M_{(k,n)} = \begin{cases} n+1-p & \text{if } k = 1, \\ -[\frac{jn}{p}] + j - 1 & \text{if } k \ge 2. \end{cases}$$

To complete the proof of the proposition we need only show that $\det(M) \neq 0$. In fact we will show that the determinant of this matrix is $\pm h_1$, thereby completing the proof of Theorem 1.

There are two cases to consider. The first case concerns those values of k, $2 \le k \le m$, for which $m+1 \le j \le p-1$. For each such value of k we add the first row of M to the k-th row, and then change signs. The resulting entries of the new k-th row are

$$-\Big(n+1-p-\Big[\frac{jn}{p}\Big]+j-1\Big) = -\Big(n+1+\Big[-\frac{jn}{p}\Big]-(p-j)\Big) = -\Big[\frac{(p-j)n}{p}\Big]+(p-j)-1.$$

Notice that the form of these entries is the same as that of the matrix M and that now $1 \le p-j \le m$.

In the second case, that is for those values of k such that $2 \le k \le m$ and $1 \le j \le m$, we leave the k-th row as it is.

Applying these elementary row operations to M results in a matrix which agrees, up to rearrangement of rows, with the matrix N whose entries are given by

$$N_{(k,n)} = \begin{cases} n+1-p & \text{if } k = 1, \\ -[\frac{kn}{p}] + k - 1 & \text{if } k \ge 2. \end{cases}$$

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By a sequence of operations, similar to those used in the proof of Corollary 6, the determinant of N is $\pm$ that of the following matrix, where $3 \leq k, n \leq m$:

$$\begin{bmatrix} & \vdots & \\ \cdots & -[\frac{kn}{p}] + [\frac{(k-1)n}{p}] & \cdots \\ & \vdots & \end{bmatrix}$$

According to [1] the determinant of this matrix is $\pm h_1$. This proves the proposition since the determinant of M has only changed by a $\pm$ sign in the course of the above elementary row and column operations. ∎

The proof of Theorem 1 follows from the fact that $\det(M) = \pm h_1$ since the matrix M is the coefficient matrix for expressing the basis elements of $\hat{A}$ in the basis elements of $\hat{B}$.

As mentioned in the introduction, Ewing proves our Theorem 1, but in a different setting. See Theorem 3.2 in [5]. To explain how Ewing's results relate to ours we need some notation.

Let W denote the Witt group of equivalence classes $[V, \beta, \rho]$, where V is a finitely generated free abelian group, β is a skew symmetric non-degenerate bilinear form on V, and ρ is a representation of $\mathbb{Z}_p$ into the group of β-isometries of V. To an automorphism of order p, $T \colon S \longrightarrow S$, we assign the Witt class $[V, \beta, \rho]$, where V is the first cohomology group, β is the cup product form, and ρ is the induced representation on cohomology. This assignment is well defined up to cobordism and so defines a group homomorphism $\mathrm{ab} \colon \Omega \longrightarrow W$, the so-called Atiyah-Bott map.

The G-signature of Atiyah and Singer defines a group homomorphism from the group of Witt classes to the complex representation ring of $\mathbb{Z}_p$, $\mathrm{sig} \colon W \longrightarrow R(\mathbb{Z}_p)$. Let $e \colon R(\mathbb{Z}_p) \longrightarrow \mathbb{Z}[\zeta]$ be the homomorphism that evaluates the character of a representation at the generator $T \in \mathbb{Z}_p$. Let $s \colon \Omega \longrightarrow \mathbb{Z}[\zeta]$ denote the composite $e \circ \mathrm{sig} \circ \mathrm{ab} \colon \Omega \longrightarrow \mathbb{Z}[\zeta]$.

Ewing proves that s is a monomorphism whose image has index h_1 in the subgroup R of $\mathbb{Z}[\zeta]$ defined by

$$R = \Big\{ \sum_{i=1}^{m} a_i(\zeta^i - \zeta^{-i}) \mid a_1 \equiv a_2 \equiv \cdots \equiv a_m \pmod 2 \Big\}.$$

From the Remark earlier in this section it follows that $\hat{R}$ has index 2^m in $\hat{B}$. If $\langle g \mid a_1, \ldots, a_t \rangle$ denotes the cobordism class of T, see Section 5 for the notation, then the G signature σ is given by

$$\sigma = s\langle g \mid a_1, \ldots, a_t \rangle = \sum_{j=1}^{t} \frac{\zeta^{k_j} + 1}{\zeta^{k_j} - 1}. \tag{11}$$

The relationship between the G-signature σ and the Eichler trace χ is given by $\sigma = 2\chi + t - 2$, and from this it is not too difficult to translate Ewing's results into ours.

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4. **The Eichler isomorphism.** We start this section with some preliminaries needed for the proofs of Theorems 4 and 5. Any sequence $[a_1, \dots, a_t]$, as in 5, determines uniquely up to topological conjugacy, a compact connected Riemann surface S and an analytical automorphism $T: S \longrightarrow S$ having order p, orbit genus 0, and whose Eichler trace is given by the equation

$$\chi = 1 + \sum_{j=1}^{t} \frac{1}{\zeta^{k_j} - 1}, \quad \text{where } k_j a_j \equiv 1 \pmod p), \text{ for } 1 \le j \le t. \tag{12}$$

Let $\chi[a_1, \dots, a_t]$ denote this Eichler trace.

By Theorem 2, if $[a_1, \dots, a_t]$ and $[b_1, \dots, b_u]$ are two such sequences then $\chi[a_1, \dots, a_t] = \chi[b_1, \dots, b_u]$ if, and only if, $t = u$ and the sequences agree up to re-arrangement and cancelling pairs.

Define a group homomorphism η from the abelian group $\mathcal{A}$ in Definition 1 to the free abelian group $\mathbb{Z}[\zeta]/\mathbb{Z}$ by:

$$\eta: \mathcal{A} \longrightarrow \mathbb{Z}[\zeta]/\mathbb{Z}, \eta: [a_1, \dots, a_t] \longrightarrow \chi[a_1, \dots, a_t] \pmod{\mathbb{Z}}. \tag{13}$$

Now we prove Theorem 4.

PROOF. To prove that η is well defined recall that the relations used to define $\mathcal{A}$ are:

$$[a_1, \dots, a_t] + [b_1, \dots, b_u] = [a_1, \dots, a_t, b_1, \dots, b_u],$$
$$[\dots, a, \dots, p - a, \dots] = [\dots, \hat{a}, \dots, \widehat{p - a}, \dots].$$

The corresponding equations for the Eichler trace are

$$\chi[a_1, \dots, a_t] + \chi[b_1, \dots, b_u] = \chi[a_1, \dots, a_t, b_1, \dots, b_u] + 1,$$
$$\chi[\dots, a, \dots, p - a, \dots] = \chi[\dots, \hat{a}, \dots, \widehat{p - a}, \dots] - 1.$$

This follows from the Eichler Trace Formula 12. Thus the Eichler trace is not additive, but reducing modulo $\mathbb{Z}$ we see that η is a well defined group homomorphism.

By definition the image of η is $\hat{A}$, the set of Eichler traces modulo $\mathbb{Z}$. It remains to show that η is a monomorphism. If there is an element in the kernel of η we may assume it is a generator, say $\eta[a_1, \dots, a_t] = 0$. This follows from the nature of the defining relations in $\mathcal{A}$. Therefore $\chi = \chi[a_1, \dots, a_t] = n$ for some integer n. From the Eichler Trace Formula 12 we get $\chi + \overline{\chi} = 2 - t = 2n$, and so $n \le 1$. A short calculation then gives $\chi = \chi[1, p-1, \dots, 1, p-1]$, where there are $1 - n$ cancelling pairs $\{1, p-1\}$. Now Theorem 2 implies that $[a_1, \dots, a_t]$ consists entirely of cancelling pairs, and so represents 0 in $\mathcal{A}$. This completes the proof of Theorem 4. ∎

Now we begin the proof of Theorem 5. First we show that $\mathcal{A}$ is generated by all triples. The argument used in the following lemma is analogous to an argument used by Symonds in [13].

LEMMA 5. *The abelian group $\mathcal{A}$ is generated by the triples $[q, r, s]$, where $1 \le q \le r \le s \le p - 1$ and $q + r + s \equiv 0 \pmod p$.*

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PROOF. We will show that any generator $[a_1, \ldots, a_t]$ can be expressed as a linear combination of triples. We can assume that the sequence $[a_1, \ldots, a_t]$ does not have any subsequence $[q, r, s]$ such that $q + r + s \equiv 0 \pmod p$, and does not contain any cancelling pairs. Therefore $t \geq 4$. The following equation is valid because of the defining relations in A:

$$[a_1, a_2, a_3, a_4, \ldots] = [a_1, a_2, b] + [a_1 + a_2, a_3, \ldots], \quad \text{where } b \equiv p - a_1 - a_2 \pmod p).$$

Arguing by induction on the length of the sequence completes the proof. ∎

The generators in Lemma 5 are not independent as the next Example shows.

EXAMPLE. Let r be any integer such that $1 \leq r \leq p - 3$. Then

$$[1, r, p-r-1]+[1, r+1, p-r-2] = [1, 1, r, p-r-2] = [1, 1, p-2]+[2, r, p-r-2].$$

Now we complete the proof of Theorem 5, that is we show that the abelian group A is freely generated by the triples $[1, r, s]$, where $1 \leq r \leq s \leq p-1$ and $1+r+s \equiv 0 \pmod p$.

PROOF. Let G denote the subgroup generated by these triples. The first equation in the example shows that all 4-tuples $[1, 1, r, s] \in G$, where

$$1 \leq r \leq s \leq p - 1 \text{ and } 2 + r + s \equiv 0 \pmod p.$$

We now set up an induction. To reduce the amount of notation we omit mentioning some of the restrictions that the following sequences must satisfy.

Assume that we have shown that for some integer $q \geq 1$ all 3-tuples of the form $[q, r, s] \in G$ and all 4-tuples of the form $[1, q, r, s] \in G$. The Example above establishes the initial case, $q = 1$, of the induction. Now consider the equations

$$[1, q, p-q-1]+[q+1, r, s] = [1, q, r, s], \quad [q+1, r, s]+[1, r+q+1, s-1] = [1, q+1, r, s-1].$$

The first equation shows that all 3-tuples of the form $[q + 1, r, s] \in G$, and then the next equation shows that all 4-tuples of the form $[1, q + 1, r, s - 1] \in G$. The induction ends when q is so large that there are no triples satisfying the conditions stated in Lemma 5.

This proves that A is generated by the triples $[1, r, s]$, where

$$1 \leq r \leq s \leq p - 1 \quad \text{and} \quad 1 + r + s \equiv 0 \pmod p.$$

There are $(p - 1)/2$ such triples. To complete the proof we show that A is free abelian of rank $(p - 1)/2$.

To do this recall that $\hat{B}$ is a free abelian group of rank $(p - 1)/2$, see Corollary 3. But $\hat{A}$ is a subgroup of finite index in $\hat{B}$, see Theorem 1, and therefore $\hat{A}$ is also a free abelian group of rank $(p-1)/2$. Theorem 4 now implies that A is free abelian of rank $(p-1)/2$, and so the generators $[1, r, s]$ freely generate A. This completes the proof of Theorem 5. ∎

5. **Equivariant cobordism.** In this section we prove Theorem 6. To begin with suppose $T_1\colon S_1 \longrightarrow S_1$ and $T_2\colon S_2 \longrightarrow S_2$ are automorphisms of order p on compact connected Riemann surfaces. We do not assume that the orbit genus of either S_1 or S_2 is 0. We start with a standard definition.

DEFINITION 5. We say that T_1 is equivariantly cobordant to T_2, written $T_1 \sim T_2$, if there exists a smooth, compact, connected 3-manifold W and a smooth $\mathbb{Z}_p$ action $T\colon W \longrightarrow W$ such that

(i) The boundary of W is the disjoint union of S_1 and S_2, $\partial(W) = S_1 \sqcup S_2$.

(ii) T restricted to $\partial(W)$ agrees with $T_1 \sqcup T_2$.

The cobordism class of an automorphism $T\colon S \longrightarrow S$ depends only upon its topological conjugacy class $[g \mid a_1, \ldots, a_t]$. We denote this cobordism class by $\langle g \mid a_1, \ldots, a_t\rangle$, and if the orbit genus $g = 0$, we denote it by $\langle a_1, \ldots, a_t\rangle$.

The set of all cobordism classes of $\mathbb{Z}_p$ actions on compact connected Riemann surfaces is denoted by Ω. Addition of the cobordism classes of the automorphisms $T_1\colon S_1 \longrightarrow S_1$, $T_2\colon S_2 \longrightarrow S_2$ is defined by equivariant connected sum.

$$\langle g \mid a_1, \ldots, a_t\rangle + \langle h \mid b_1, \ldots, b_u\rangle = \langle g + h \mid a_1, \ldots, a_t, b_1, \ldots, b_u\rangle. \tag{14}$$

The next two lemmas show that Ω is an abelian group generated by the cobordism classes $\langle a_1, \ldots, a_t\rangle$. The identity is represented by any fixed point free action, or by any cobordism class consisting entirely of cancelling pairs, and the inverse of $\langle g \mid a_1, \ldots, a_t\rangle$ is represented by $\langle g \mid p - a_1, \ldots, p - a_t\rangle$. The proofs are not original, but are presented here to emphasize the relationship with $\mathcal{A}$.

LEMMA 6. $\langle g \mid a_1, \ldots, a_t\rangle = \langle a_1, \ldots, a_t\rangle$.

PROOF. Let $T\colon S \longrightarrow S$ represent the class $\langle a_1, \ldots, a_t\rangle$. First we take the product cobordism $W_1 = S \times [0, 1]$, where T is extended over W_1 in the obvious way. Next we modify W_1 on the top end $S \times \{1\}$ as follows. Take a disc D in S such that $D, T(D), \ldots, T^{p-1}(D)$ are mutually disjoint, and then to each disc $T^k(D)$ in $S \times \{1\}$, $k = 0, 1, \ldots, p - 1$, attach a copy of a handlebody H of genus g by identifying the disc $T^k(D)$ with some disc $D' \subset \partial(H)$. Let W_2 denote the resulting 3-manifold. See Figure 2. The action of $\mathbb{Z}_p$ can be extended to W_2 by permuting the handlebodies. The manifold W_2 provides the cobordism showing that $\langle g \mid a_1, \ldots, a_t\rangle = \langle a_1, \ldots, a_t\rangle$. ∎

LEMMA 7. $\langle a, p - a, a_3, \ldots, a_t\rangle = \langle 1 \mid a_3, \ldots, a_t\rangle = \langle a_3, \ldots, a_t\rangle$.

PROOF. The proof of this lemma is similar to the proof of the last one. Start with a product cobordism W_1. Suppose P_0, P_1 are the fixed points corresponding to the cancelling pair $\{a, p-a\}$. Choose small invariant discs D_0, D_1 around P_0, P_1 respectively, and then modify the cobordism at the top end by adding a solid tube $D \times [0, 1]$ so that $D \times \{0\} = D_0$ and $D \times \{1\} = D_1$. The automorphism T can be extended over this tube, and the resulting cobordism shows that

$$\langle a, p - a, a_3, \ldots, a_t\rangle = \langle 1 \mid a_3, \ldots, a_t\rangle.$$

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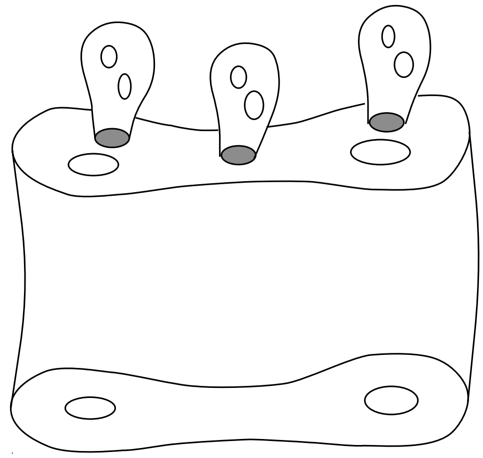

FIGURE 2:

See Figure 3. Lemma 6 completes the proof. ∎

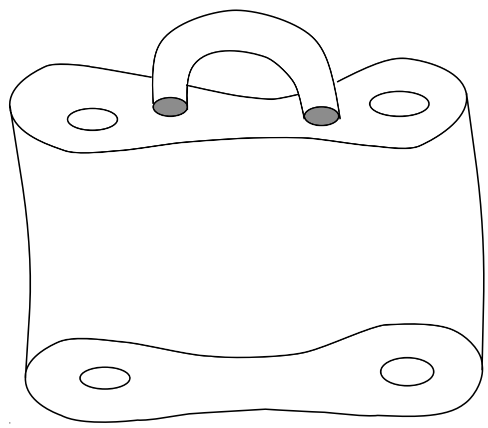

FIGURE 3:

Define the isomorphism of Theorem 6, $\phi: A \longrightarrow \Omega$, by $\phi[a_1, \ldots, a_t] = \langle a_1, \ldots, a_t \rangle$. The defining relations of A are

(i) $[a_1, \ldots, a_t] + [b_1, \ldots, b_u] = [a_1, \ldots, a_t, b_1, \ldots, b_u]$.

(ii) $[\ldots, a, \ldots, p - a, \ldots] = [\ldots, \hat{a}, \ldots, \widehat{p - a}, \ldots]$.

The same relations hold for cobordism classes, see Equation 14 and Lemma 7, and therefore the mapping ϕ is a well defined group homomorphism.

Now we complete the proof of Theorem 6. The argument is analogous to one used in [3].

PROOF. From the remarks above we know that $\phi: A \longrightarrow \Omega$ is a well defined group homomorphism. Lemma 6 implies that it is an epimorphism. It only remains to prove

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that ϕ is a monomorphism.

If there is an element in the kernel of ϕ we can assume it is a generator, say $[a_1, \ldots, a_t]$. Suppose $T: S \longrightarrow S$ represents $[a_1, \ldots, a_t]$. Then there is a compact, connected, smooth 3-manifold W such that $\partial(W) = S$, and an extension of T to a smooth homeomorphism $T: W \longrightarrow W$ of order p, also denoted by T. The fixed point set of $T: W \longrightarrow W$ must consist of disjoint, properly embedded arcs joining fixed points in S to fixed points in S. The fixed points at the end of each arc will form a cancelling pair $\{a, p-a\}$. In this way we see that $[a_1, \ldots, a_t]$ consists entirely of cancelling pairs, and hence $[a_1, \ldots, a_t] = 0$ in A. ∎

REFERENCES

1. Carlitz and Olson, *Maillet's Determinant*. Proc. Amer. Math. Soc. **6**(1955), 265–269.
2. Eckmann and Müeller, *Plane Motion Groups and Virtual Poincaré Duality in Dimension Two*. Invent. Math. **69**(1982), 293–310.
3. Edmonds and Ewing, *Remarks on the cobordism group of Surface diffeomorphisms*. Math. Ann. **259**(1982), 497–504.
4. ———, *Surface Symmetry and Homology*. Math. Proc. Cambridge Philos. Soc. **99**(1986), 73–77.
5. Ewing, *The Image of the Atiyah-Bott Map*. Math. Z. **165**(1979), 53-71.
6. ———, *Automorphisms of Surfaces and Class Numbers: An Illustration of the G-Index Theorem*. In: Topological Topics, (ed., I. M. James), London Math. Soc. Lecture Notes Series **86**, Cambridge Univ. Press, 1983, 120–127.
7. Farkas and Kra, *Riemann Surfaces*. Graduate Texts in Math. **71**, Springer-Verlag, 1980.
8. Gabai, *Convergence Groups are Fuchsian*. Ann. of Math. **136**(1992), 447–510.
9. Kerckhoff, *The Nielsen Realization Problem*. Ann. of Math. **117**(1983), 235–265.
10. Lang, *Cyclotomic Fields I and II*. Springer-Verlag, New York, 1990.
11. Nielsen, *Die Struktur periodischer Transformationen von Flächen*. Mat.-Fys. Medd. Danske Vid. Selsk. **15**(1937), 1–77.
12. ———, *Abbildungsklassen endlicher Ordnung*. Acta Math. **75**(1942), 23–115.
13. Symonds, *The Cohomology Representation of an action of $\mathbb{Z}_p$ on a surface*. Trans. Amer. Math. Soc. **306**(1988), 389–400.