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Introduction: basics of QCD perturbation theory

Quantum chromodynamics (QCD) is the theory of strong interactions. This is an exciting physical theory, whose Lagrangian deals with quark and gluon fields and their interactions. At the same time, quarks and gluons do not exist as free particles in nature but combine into bound states (hadrons) instead. This phenomenon, known as *quark confinement*, is one of the most profound puzzles of QCD. Another amazing feature of QCD is the property of *asymptotic freedom*: quarks and gluons tend to interact more weakly over short distances and more strongly over longer distances.

This book is dedicated to another QCD mystery: the behavior of quarks and gluons in high energy collisions. Quantum chromodynamics is omnipresent in high energy collisions of all kinds of known particles. There are vast amounts of high energy scattering data on strong interactions, which have been collected at accelerators around the world. While these data are incredibly diverse they often exhibit intriguingly universal scaling properties, which unify much of the data while puzzling both experimentalists and theorists alike. Such universality appears to imply that the underlying QCD dynamics is the same for a broad range of high energy scattering phenomena.

The main goal of this book is to provide a consistent theoretical description of high energy QCD interactions. We will show that the QCD dynamics in high energy collisions is very sophisticated and often nonlinear. At the same time much solid theoretical progress has been made on the subject over the years. We will present the results of this progress by introducing a universal approach to a broad range of high energy scattering phenomena.

We begin by presenting a brief summary of the tools needed to perform perturbative QCD calculations. Since much of the material in this chapter is covered in standard field theory and particle physics textbooks, we will not derive many results, simply summarizing them and referring the reader to the appropriate literature for detailed derivations.

1.1 The QCD Lagrangian

Quantum chromodynamics is an SU(3) Yang–Mills gauge theory (Yang and Mills 1954) describing the interactions of quarks and gluons. The QCD Lagrangian density is

$$\mathcal{L}_{QCD} = \sum_{\text{flavors } f} \bar{q}_i^f(x) \left[i \gamma^\mu D_\mu - m_f \right]_{ij} q_j^f(x) - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \tag{1.1}$$

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where $q_i^f(x)$ and $\bar{q}_i^f(x)$ are the quark and antiquark spin-$1/2$ Dirac fields of color i, flavor f, and mass m_f, with $\bar{q} = q^\dagger \gamma^0$. A field $A_\mu^a(x)$ describes the gluon, which has spin equal to 1, zero mass, and color index a in the adjoint representation of the SU(3) gauge group. Summation over repeated color and Lorentz indices is assumed, with $i, j = 1, 2, 3$ and $a = 1, \ldots, 8$. The covariant derivative D_μ is defined by

$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igt^a A_\mu^a. \tag{1.2}$$

The t^a are the generators of SU(3) in the fundamental representation ($t^a = \lambda^a/2$, where the λ^a are the Gell-Mann matrices). The non-Abelian gluon field strength tensor $F_{\mu\nu}^a$ is defined by

$$F_{\mu\nu} = t^a F_{\mu\nu}^a = \frac{i}{g}\left[D_\mu, D_\nu\right] \tag{1.3}$$

or, equivalently, by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \tag{1.4}$$

where f^{abc} are the structure constants of the color group SU(3).

We work in natural units, with $\hbar = c = 1$. Our four-vectors are $x^\mu = (t, \vec{x})$, the partial derivatives are denoted $\partial_\mu = \partial/\partial x^\mu$, and the metric in t, x, y, z coordinates is $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

The Lagrangian of Eq. (1.1) was proposed by Fritzsch, Gell-Mann, and Leutwyler (1973), Gross and Wilczek (1973, 1974), and Weinberg (1973). The form of the QCD Lagrangian is based on two assumptions confirmed by experimental observations: all hadrons consist of quarks and quarks cannot be observed as free particles. The first observation leads to a new quantum number for quarks: color. Indeed, without this quantum number we cannot build the wave functions for baryons. For example the Ω^- hyperon has spin $3/2$ and consists of three s-quarks. This means that the spin and flavor parts of its wave function are symmetric with respect to interchange of the identical valence s-quarks. Owing to the Pauli exclusion principle the full wave function of the three identical quarks has to be antisymmetric. If spin and flavor were the only quantum numbers, it would appear that the spatial wave function of the three s-quarks would have to be antisymmetric. However, this would contradict the fact that Ω^- is a stable particle and is, therefore, a ground state of the three s-quark system. The spatial wave function of a ground state has to be symmetric. To resolve this conundrum we need to introduce a new quantum number that should have at least three different values to make the three strange quarks different in the Ω^- hyperon. This quantum number is the quark *color*.

We then need to determine which particle is responsible for interactions between the quarks forming quark bound states, the hadrons. The interactions between the quarks in mesons and baryons have to be attractive, which indicates that they should depend on quark color: if one introduced interactions between quarks using some global (not gauged) non-Abelian color symmetry then one would not be able to obtain attractive interactions between the quark and the antiquark in a meson and between a pair of quarks in a baryon simultaneously, at least not in the lowest nontrivial order in the interaction. One therefore concludes that the non-Abelian color symmetry has to be gauged by introducing a non-Abelian vector boson responsible for quark interactions. Moreover, as we will see below, the high energy scattering data confirms this conclusion as it demonstrates that the particle responsible for quark interactions has spin equal to 1.

The second experimental observation needed for the construction of the QCD Lagrangian, that quarks are never seen as free particles, means that the forces between quarks should be stronger at longer distances to prevent quarks from leaving a hadron. For point-like particles our best chance of getting such forces is by assuming that quark interactions are mediated by a massless particle. For such a particle the lowest-order quark–antiquark interaction potential decreases at long distances roughly as to $1/r$, where r is the distance between the quarks. (Indeed in a full QCD calculation this behavior changes to $\sim r$, that of a confining potential.) Massive particles would give an exponentially decreasing potential, which would have a shorter range than the potential in the massless case. We therefore conclude that the particle responsible for quark interactions is a non-Abelian massless vector boson, a gluon.

However, particle interactions may generate a mass even for a particle that is massless at the Lagrangian level. To protect the zero mass of the gluon from higher-order corrections we have to assume the existence of gauge symmetry in our Lagrangian. Namely, the Lagrangian should be invariant with respect to

$$q(x) \to S(x)\,q(x), \tag{1.5a}$$

$$\bar{q}(x) \to \bar{q}(x)\,S^{-1}(x), \tag{1.5b}$$

$$A_\mu(x) \to S(x)A_\mu(x)S^{-1}(x) - \frac{i}{g}\left[\partial_\mu S(x)\right]S^{-1}(x), \tag{1.5c}$$

where we have defined a unitary 3×3 matrix

$$S(x) = e^{i\alpha^a(x)\,t^a}, \tag{1.6}$$

where the $\alpha^a(x)$ are arbitrary real-valued functions; summation over repeated color indices a is again implied. The form of the Yang–Mills Lagrangian (1.1) can be derived directly from the gauge symmetry in Eqs. (1.5) (see e.g. Peskin and Schroeder (1995)).

1.2 A review of Feynman rules for QCD

To derive the Feynman rules from the Lagrangian (1.1) we need to define the functional integral (the QCD partition function)

$$Z_{QCD} = \int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\;\exp\left\{i\int d^4x\,\mathcal{L}_{QCD}(A, q, \bar{q})\right\}. \tag{1.7}$$

One can see that this integral is divergent since its integrand has the same value for an infinite set of fields related to each other by all possible gauge transformations (1.5). However, the values of physical observables are given by the expectation values of operators. For an arbitrary gauge-invariant operator $\mathcal{O}$ we have the vacuum expectation value

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O} \exp\{i \int d^4x \mathcal{L}_{QCD}\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q} \exp\{i \int d^4x\, \mathcal{L}_{QCD}\}} \tag{1.8}$$

The divergences caused by integrations over gauge directions in the numerator and in the denominator of Eq. (1.8) cancel each other. Faddeev and Popov (1967) suggested a procedure allowing one to see such cancellations in the most economic way by multiplying the definition (1.7) with the functional integral identity[1]

$$1 = \int \mathcal{D}\alpha\, \delta(\alpha) = \int \mathcal{D}\alpha\, \delta(G(A^\alpha))\ \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right), \tag{1.9}$$

where the integral runs over all gauge transformations labeled by α^a (see Eq. (1.6)), A^α is a gauge field related to the original one by the gauge transformation defined by α^a, and $G(A) = 0$ is the gauge-fixing condition. (For instance, $G(A) = \partial_\mu A^\mu$ in a covariant gauge.) Let us restrict ourselves to gauges in which the functional determinant $\det[\delta G(A^\alpha)/\delta\alpha]$ is independent of α^a for a given A^α. Using Eq. (1.9) the expectation values of the operators can be written as

$$\langle \mathcal{O} \rangle = \frac{\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O}\ \delta(G(A))\ \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}{\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \delta\left(G(A)\right)\ \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}, \tag{1.10}$$

where we have relabeled the integration variable A^α as A everywhere except in the determinants, in which one should put $\alpha^a = 0$ after differentiation thus turning A^α into A. The infinities in the numerator and the denominator of Eq. (1.10) are clearly identifiable as being due to the integration over α^a. As nothing else in the integrands of Eq. (1.10) depends on α we can simply cancel the $\mathcal{D}\alpha$ integrations, writing

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O}\ \delta(G(A))\ \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \delta(G(A))\ \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}. \tag{1.11}$$

To obtain the Feynman rules we have to put all the A-dependence in the integrands in Eq. (1.11) into the exponents. We start with the delta functions and first note that making the replacement in Eq. (1.11)

$$\delta(G(A)) \to \delta(G(A) - r(x)), \tag{1.12}$$

where $r(x)$ is some arbitrary function of x^μ, would not change the values of the functional integrals in the numerator and the denominator and would therefore leave $\langle \mathcal{O} \rangle$ unchanged. Indeed different choices of $r(x)$ correspond to different choices of the gauge defined by the $G(A) = r(x)$ gauge condition. Thus the replacement (1.12) simply modifies the function defining the gauge condition: $G(A) \to G(A) - r(x)$. Since our initial gauge-defining function $G(A)$ is arbitrary, and as neither of the integrals in the numerator and the denominator of Eq. (1.11) depends on $G(A)$, we conclude that nothing in the numerator

[1] In discussing the Faddeev–Popov method we will follow closely the presentations in Peskin and Schroeder (1995) and in Sterman (1993).

or the denominator of Eq. (1.11) changes if we perform the replacement (1.12). Moreover, the resulting expression,

$$\langle\mathcal{O}\rangle = \frac{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{O}\;\delta(G(A)-r(x))\;\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\;\exp\left\{i\int d^4x\,\mathcal{L}_{QCD}\right\}}{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\delta(G(A)-r(x))\;\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\;\exp\left\{i\int d^4x\,\mathcal{L}_{QCD}\right\}}, \tag{1.13}$$

is independent of $r(x)$ for the same reasons. We can integrate the numerator and the denominator separately over $r(x)$ by multiplying them with

$$1 = N(\xi)\int \mathcal{D}r\;\exp\left\{-i\int d^4x\,\frac{r^2(x)}{2\xi}\right\}, \tag{1.14}$$

where $N(\xi)$ is a normalization function defined by Eq. (1.14) and ξ is an arbitrary number. Multiplying both the numerator and the denominator of Eq. (1.13) by Eq. (1.14), canceling $N(\xi)$, and performing the r-integrals with the help of the delta functions, we obtain

$$\langle\mathcal{O}\rangle = \frac{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{O}\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\;\exp\left\{i\int d^4x\,\left(\mathcal{L}_{QCD}-\frac{1}{2\xi}\,[G(a)]^2\right)\right\}}{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\;\exp\left\{i\int d^4x\,\left(\mathcal{L}_{QCD}-\frac{1}{2\xi}\,[G(a)]^2\right)\right\}}. \tag{1.15}$$

Finally, in order to remove the determinants of Eq. (1.15) into the exponents one introduces the (unphysical) *Faddeev–Popov ghost field* $c^a(x)$, whose values are complex Grassmann numbers (Faddeev and Popov 1967, Feynman 1963, DeWitt 1967). The ghost field is a Lorentz scalar in the adjoint representation of SU(3). With the help of the Faddeev–Popov ghost field we write

$$\det\left(\frac{\delta G\,(A^\alpha)}{\delta\alpha}\right) = \int \mathcal{D}c\,\mathcal{D}c^*\;\exp\left\{-i\int d^4x\,c^*\,\frac{\delta G\,(A^\alpha)}{\delta\alpha}\,c\right\} \tag{1.16}$$

with c^* the complex conjugate of the c field. Using Eq. (1.16) in Eq. (1.15) we obtain

$$\langle\mathcal{O}\rangle = \frac{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}c\,\mathcal{D}c^*\,\mathcal{O}\;\exp\left\{i\int d^4x\,\mathcal{L}(A,q,\bar{q},c,c^*)\right\}}{\int \mathcal{D}A\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}c\,\mathcal{D}c^*\;\exp\left\{i\int d^4x\,\mathcal{L}(A,q,\bar{q},c,c^*)\right\}}, \tag{1.17}$$

where we have defined an effective Lagrangian

$$\mathcal{L}(A,q,\bar{q},c,c^*) \equiv \mathcal{L}_{QCD} - \frac{1}{2\xi}\,[G(A)]^2 - c^*\,\frac{\delta G(A^\alpha)}{\delta\alpha}\,c. \tag{1.18}$$

Now we are ready to derive the Feynman rules for QCD.

In this book we will employ two main gauge choices. One is the Lorenz gauge, defined by the gauge condition

$$\partial_\mu A^{a\,\mu} = 0. \tag{1.19}$$

Inserting $G(A) = \partial_\mu A^{a\,\mu}$ into Eq. (1.18), after some straightforward algebra (see e.g. Peskin and Schroeder (1995)) we end up with

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \left(\partial^\mu c^{a\,*}\right)\left(\delta^{ac}\,\partial^\mu + g f^{abc} A^b_\mu\right)c^c. \tag{1.20}$$

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Using Eq. (1.20) we can derive the Feynman rules for QCD by substituting the Lagrangian (1.20) into Eq. (1.7) in place of $\mathcal{L}_{QCD}$.

The other gauge choice that we will be using frequently throughout the book is the light cone gauge, defined by

$$\eta \cdot A^a = \eta^\mu A^a_\mu = 0, \tag{1.21}$$

with η^μ a constant four-vector that is light-like, so that $\eta^2 = \eta_\mu \eta^\mu = 0$. One can show that, in the light cone gauge, $\det[\delta G(A^\alpha)/\delta\alpha]$ does not depend on A^μ when we take the limit $\xi \to 0$. From Eq. (1.18) one can see that in this case the ghost field would not couple to the gluon field and so can be integrated out in the functional integrals of Eq. (1.17). Hence there is no ghost field in the light cone gauge. The effective Lagrangian (1.18) in the light cone gauge becomes

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{1}{2\xi}\left(\eta^\mu A^a_\mu\right)^2 \tag{1.22}$$

(with an implied $\xi \to 0$ limit).

Below we list the Feynman rules for QCD, in the Lorenz and light cone gauges, which follow from the Lagrangians in Eqs. (1.20) and (1.22). We use the standard notation for a product of two four-vectors $u \cdot v = u_\mu v^\mu$ and for the square of a single four-vector $v_\mu v^\mu = v^2$. The Dirac gamma matrices in the standard Dirac representation, which we will use here, are defined by

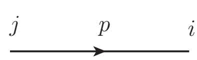
$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.23}$$

where $\mathbf{1}$ is a unit 2×2 matrix, $i = 1, 2, 3$, and σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.24}$$

As usual, we will write $\not{v} = \gamma^\mu v_\mu$. Arrows on the quark and ghost propagators (see below) indicate the flow of the particle number and, in the cases of the quark propagator and the ghost–gluon vertex, they also indicate the momentum flow. As ghost fields do not exist in the light cone gauge, the Feynman rules for ghosts listed below apply only in the Lorenz gauge.

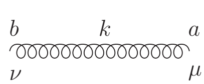
1.2.1 QCD Feynman rules

Quark propagator: 

$$= \frac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon}\,\delta^{ij}, \tag{1.25}$$

Ghost propagator: 

$$= \frac{i}{k^2 + i\epsilon}\,\delta^{ab}, \tag{1.26}$$

Gluon propagator: 

$$= \frac{-i D_{\mu\nu}(k)}{k^2 + i\epsilon}\,\delta^{ab}, \tag{1.27}$$

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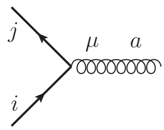
where in the Lorenz gauge ($\partial \cdot A^a = 0$)

$$D_{\mu\nu}(k) = g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}; \tag{1.28}$$

the choice $\xi = 0$ is referred to as the Landau gauge and the choice $\xi = 1$ is called the Feynman gauge. In the light cone gauge $\eta \cdot A^a = 0$ with $\xi \to 0$ one has

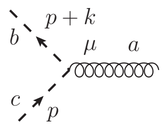
$$D_{\mu\nu}(k) = g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k}. \tag{1.29}$$

Quark–gluon vertex:



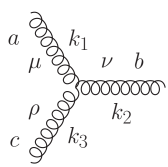
$$= i g \gamma^\mu \,(t^a)_{ji}, \tag{1.30}$$

Ghost–gluon vertex
(Lorenz gauge only):



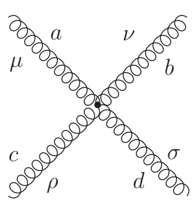
$$= g(p+k)^\mu f^{abc} \tag{1.31}$$

Three-gluon vertex
(all momenta flow
into the vertex):



$$= \begin{aligned} &-g f^{abc}\,[(k_1-k_3)^\nu g^{\mu\rho} \\ &\quad + (k_2-k_1)^\rho g^{\mu\nu} + (k_3-k_2)^\mu g^{\nu\rho}] \end{aligned} \tag{1.32}$$

Four-gluon vertex:



$$= \begin{aligned} &-ig^2\left[f^{abe}\, f^{cde}\,(g^{\mu\rho}\, g^{\nu\sigma} - g^{\mu\sigma}\, g^{\nu\rho})\right. \\ &\quad + f^{ace}\, f^{bde}\,(g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\sigma}\, g^{\nu\rho}) \\ &\quad \left. + f^{ade}\, f^{bce}\,(g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\rho}\, g^{\nu\sigma})\right] \end{aligned} \tag{1.33}$$

The Feynman rules that are standard for all field theories, such as the conservation of four-momentum in the vertices and the inclusion of a factor -1 for each fermion loop or of proper symmetry factors, apply to QCD as well and will not be explicitly spelled out here.

1.3 Rules of light cone perturbation theory

Many calculations in this book will not be performed using the Feynman rules. Instead we will use light cone perturbation theory (LCPT), following the rules introduced by Lepage and Brodsky (1980) (see Brodsky and Lepage (1989) and Brodsky, Pauli, and Pinsky (1998) for a detailed derivation of the LCPT rules). We begin by introducing the light cone notation.

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For any four-vector v^μ we define

$$v^+ = v^0 + v^3, \quad v^- = v^0 - v^3. \tag{1.34}$$

With this notation we see immediately that

$$v^2 = v^+ v^- - \vec{v}_\perp^{\,2}, \tag{1.35}$$

where we have defined a vector of transverse components $\vec{v}_\perp = (v^1, v^2)$. A product of two four-vectors v^μ and u^μ in light cone notation is

$$u \cdot v = \frac{1}{2} u^+ v^- + \frac{1}{2} u^- v^+ - \vec{u}_\perp \cdot \vec{v}_\perp. \tag{1.36}$$

The metric has nonzero components $g_{+-} = g_{-+} = 1/2$, $g_{11} = g_{22} = -1$. This gives

$$v_- = \frac{v^0 + v^3}{2} = \frac{v^+}{2}, \quad v_+ = \frac{v^0 - v^3}{2} = \frac{v^-}{2}. \tag{1.37}$$

Note also that $\partial_+ = (1/2)\,\partial^-$ and $\partial_- = (1/2)\,\partial^+$.

Light cone perturbation theory is similar to time-ordered perturbation theory, except that the light cone x^+-direction plays the role of time. (For a good presentation of time-ordered perturbation theory see Sterman (1993).) Our discussion of LCPT here will closely follow Lepage and Brodsky (1980) and Brodsky and Lepage (1989). We will work in the particular light cone gauge

$$A^+ = 0, \tag{1.38}$$

which can be obtained from Eq. (1.21) by choosing $\eta^\mu = (0, 2, \vec{0}_\perp)$, in the $(+, -, \perp)$ notation. Of the remaining A^- and $A_\perp^i$ components of the gluon field ($i = 1, 2$), only the transverse components $A_\perp^i$ are independent. The component A^- can be expressed in terms of the $A_\perp^i$ using the equations of motion for the QCD Lagrangian (1.1). The quark field, which we will denote by $q(x)$, dropping the flavor label, is separated into two spinor components q_+ and q_- defined by

$$q_\pm(x) = \Lambda_\pm\, q(x), \tag{1.39}$$

where the projection operators $\Lambda_\pm$ are given by

$$\Lambda_\pm = \frac{1}{2} \gamma^0 \gamma^\pm \tag{1.40}$$

and the Dirac matrix $\gamma^\pm = \gamma^0 \pm \gamma^3$. Note that, just like any other projection operators, $\Lambda_\pm$ obey the following relations: $\Lambda_+ \Lambda_- = 0$, $\Lambda_\pm^2 = \Lambda_\pm$, and $\Lambda_+ + \Lambda_- = 1$. The two projections q_+ and q_- are not independent and can also be related using the constraint part of the equations of motion. The dependent field operators A^- and q_- are expressed in terms of $A_\perp^i$ and q_+ as (see Lepage and Brodsky (1980))[2]

$$A^- = -\frac{2}{\partial^+}\,\partial_{\perp j}\cdot A_\perp^j + \frac{2g}{(\partial^+)^2}\left\{\left[i\partial^+ A_\perp^j, A_\perp^j\right] + 2q_+^\dagger t^a q_+ t^a\right\}, \tag{1.41}$$

$$q_- = \frac{1}{i\partial^+}\,\gamma^0\left(-i\,\gamma_\perp^j D_{\perp j} + m\right) q_+ \tag{1.42}$$

where $j = 1, 2$. Next one defines free gluon and quark fields $\tilde{A}^\mu$ and $\tilde{q}$ by

$$\tilde{A}^\mu = (0, \tilde{A}^-, \vec{A}_\perp), \tag{1.43}$$

in the $(+, -, \perp)$ notation, with

$$\tilde{A}^- \equiv -\frac{2}{\partial^+}\,\partial_{\perp j}\cdot A_\perp^j \tag{1.44}$$

and

$$\tilde{q} \equiv q_+ + \frac{1}{i\partial^+}\,\gamma^0\left(-i\gamma_\perp^j\partial_{\perp j} + m\right) q_+. \tag{1.45}$$

The light cone Hamiltonian H is defined as the minus component of the four-momentum vector, P^-. It can be written as the sum of free and interaction terms:

$$H = P^- = H_0 + H_{int}, \tag{1.46}$$

where (Lepage and Brodsky 1980, Brodsky and Lepage 1989, Brodsky, Pauli, and Pinsky 1998)

$$H_0 = \frac{1}{2}\int dx^-\, d^2x_\perp\left(\bar{\tilde{q}}\,\gamma^+\,\frac{m^2-\nabla_\perp^2}{i\partial^+}\,\tilde{q} - \tilde{A}_\mu^a\,\nabla_\perp^2\tilde{A}^{a\,\mu}\right) \tag{1.47}$$

is the free part of the Hamiltonian, while the interaction part is given by

$$\begin{aligned} H_{int} = \int dx^- d^2x_\perp \Bigg[& -2g\,\mathrm{tr}\left(i\partial^\mu\tilde{A}^\nu[\tilde{A}_\mu, \tilde{A}_\nu]\right) - \frac{g^2}{2}\,\mathrm{tr}\left([\tilde{A}^\mu, \tilde{A}^\nu][\tilde{A}_\mu, \tilde{A}_\nu]\right) \\ & - g\bar{\tilde{q}}\gamma^\mu A_\mu\tilde{q} + g^2\,\mathrm{tr}\left([i\partial^+\tilde{A}^\mu, \tilde{A}_\mu]\,\frac{1}{(i\partial^+)^2}\,[i\partial^+\,\tilde{A}^\nu, \tilde{A}_\nu]\right) \\ & + g^2\bar{\tilde{q}}\gamma^\mu A_\mu\gamma^+\frac{1}{2i\partial^+}\gamma^\nu A_\nu\tilde{q} - g^2\bar{\tilde{q}}\gamma^+\left(\frac{1}{(i\partial^+)^2}\,[i\partial^+\tilde{A}^\mu, \tilde{A}_\mu]\right)\tilde{q} \\ & + \frac{g^2}{2}\bar{q}\gamma^+t^a q\frac{1}{(i\partial^+)^2}\bar{q}\gamma^+t^a q\Bigg]. \end{aligned} \tag{1.48}$$

Quantizing the theory by expanding $A_\perp^i$ and q_+ in terms of creation and annihilation operators while treating the x^+ light cone direction as time, one can construct light cone time-ordered perturbation theory with the help of the light cone Hamiltonian H. The rules of LCPT for the calculation of scattering amplitudes are given in the following subsection (Lepage and Brodsky 1980, Brodsky and Lepage 1989, Zhang and Harindranath 1993, Brodsky, Pauli, and Pinsky 1998).

[2] Our notation in Eqs. (1.1), (1.2), and (1.4), and therefore throughout the book, can be obtained from that of Lepage and Brodsky (1980) and Brodsky and Lepage (1989) by making the replacement $g \to -g$.

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1.3.1 QCD LCPT rules

1. Draw all diagrams for a given process at the desired order in the coupling constant, including all possible orderings of the interaction vertices in the light cone time x^+. Assign a four-momentum k^μ to each line such that it is on mass shell, so that $k^2 = m^2$ with m the mass of the particle. Each vertex conserves only the k^+ and $\vec{k}_\perp$ components of the four-momentum. Hence for each line the four-momentum has components as follows:

$$k^\mu = \left(k^+, \frac{\vec{k}_\perp^2 + m^2}{k^+}, \vec{k}_\perp^2\right). \tag{1.49}$$

2. With quarks associate on-mass-shell spinors in the Lepage and Brodsky (1980) convention:

$$u_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ + m\gamma^0 + \gamma^0\,\vec{\gamma}_\perp \cdot \vec{p}_\perp\right)\, \chi(\sigma), \tag{1.50}$$

$$v_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ - m\gamma^0 + \gamma^0\,\vec{\gamma}_\perp \cdot \vec{p}_\perp\right)\, \chi(-\sigma), \tag{1.51}$$

with

$$\chi(+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{1.52}$$

Gluon lines come with a polarization vector $\epsilon_\lambda^\mu(k)$. In the $A^+ = 0$ gauge this vector is given by

$$\epsilon_\lambda^\mu(k) = \left(0, \frac{2\,\vec{\epsilon}_\perp^{\,\lambda} \cdot \vec{k}_\perp}{k^+}, \vec{\epsilon}_\perp^{\,\lambda}\right) \tag{1.53}$$

with transverse polarization vector

$$\vec{\epsilon}_\perp^{\,\lambda} = -\frac{1}{\sqrt{2}}\,(\lambda, i)\,, \tag{1.54}$$

where $\lambda = \pm 1$. Equation (1.53) follows from requiring that $\epsilon_\lambda^+ = 0$ and $\epsilon_\lambda(k) \cdot k = 0$.

3. For each intermediate state there is a factor equal to the light cone energy denominator

$$\frac{1}{\sum\limits_{inc} k^- - \sum\limits_{interm} k^- + i\,\epsilon} \tag{1.55}$$

where the sums run respectively over all incoming particles present in the initial state in the diagram (“inc”) and over all the particles in the intermediate state at hand (“interm”). According to rule 1 above, for each particle we have $k^- = (\vec{k}_\perp^2 + m^2)/k^+$. Since the k^- momentum component is not conserved at the vertices the intermediate states are not on the “energy shell” and the light cone denominator in (1.55) is nonzero. Note that the light

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cone energy is conserved for the whole scattering process: $\sum_{inc} k^-$ is equal to $\sum_{out} k^-$, where “out” stands for all outgoing particles.[3]

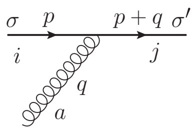
4. Include a factor

$$\frac{\theta(k^+)}{k^+} \tag{1.56}$$

for each internal line, where k^+ flows in the future light cone time direction.

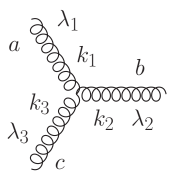
5. For vertices include factors as follows (we assume that the light cone time flows from left to right).

Quark–gluon vertex (i and j are quark color indices):



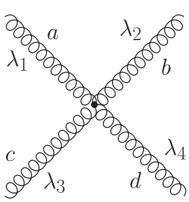
$$= -g\bar{u}_{\sigma' j}(p+q)\, \not\epsilon_\lambda(q)\, (t^a)_{ji}\, u_{\sigma\, i}(p). \tag{1.57}$$

Three-gluon vertex (all momenta flow into the vertex; asterisks denote complex conjugation):



$$= \begin{aligned} -igf^{abc}\,[&(k_1 - k_3)\cdot \epsilon^*_{\lambda_2}(k_2)\, \epsilon_{\lambda_1}(k_1)\cdot \epsilon_{\lambda_3}(k_3) \\ &+ (k_2 - k_1)\cdot \epsilon_{\lambda_3}(k_3)\, \epsilon_{\lambda_1}(k_1)\cdot \epsilon^*_{\lambda_2}(k_2) \\ &+ (k_3 - k_2)\cdot \epsilon_{\lambda_1}(k_1)\, \epsilon_{\lambda_3}(k_3)\cdot \epsilon^*_{\lambda_2}(k_2)]. \end{aligned} \tag{1.58}$$

Four-gluon vertex:



$$= \begin{aligned} g^2\Big[&f^{abe} f^{cde}\, (\epsilon_{\lambda_1}\cdot \epsilon_{\lambda_3}\, \epsilon^*_{\lambda_2}\cdot \epsilon^*_{\lambda_4} - \epsilon_{\lambda_1}\cdot \epsilon^*_{\lambda_4}\, \epsilon_{\lambda_3}\cdot \epsilon^*_{\lambda_2}) \\ &+ f^{ace} f^{bde} (\epsilon_{\lambda_1}\cdot \epsilon^*_{\lambda_2}\, \epsilon_{\lambda_3}\cdot \epsilon^*_{\lambda_4} - \epsilon_{\lambda_1}\cdot \epsilon^*_{\lambda_4}\, \epsilon_{\lambda_3}\cdot \epsilon^*_{\lambda_2}) \\ &+ f^{ade} f^{bce} (\epsilon_{\lambda_1}\cdot \epsilon^*_{\lambda_2}\, \epsilon_{\lambda_3}\cdot \epsilon^*_{\lambda_4} - \epsilon_{\lambda_1}\cdot \epsilon_{\lambda_3}\, \epsilon^*_{\lambda_2}\cdot \epsilon^*_{\lambda_4})\Big]. \end{aligned} \tag{1.59}$$

In addition to the above vertices, which are (up to some trivial factors due to a different convention) identical to the same vertices in the Feynman rules, there are instantaneous terms in the light cone Hamiltonian giving the four vertices below. Again, light cone time flows to the right while the momentum flow direction is indicated by arrows. Instantaneous quark and gluon lines are denoted by regular quark and gluon lines with a short

[3] This light cone energy conservation condition does not apply to light cone wave functions, to be discussed shortly, as they represent only part of the scattering process.

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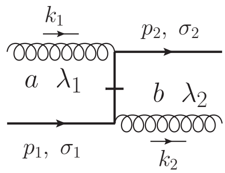
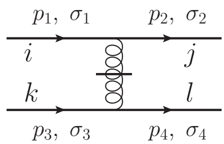
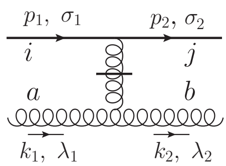
line crossing them.



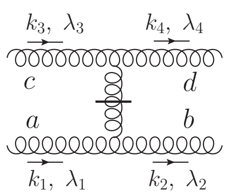
$$= g^2\, \bar{u}_{\sigma_2 j}(p_2)\, \not\epsilon_{\lambda_1}(k_1)\, \frac{\gamma^+}{2(p_1^+ - k_2^+)}\, \not\epsilon^*_{\lambda_2}(k_2) \\ \times (t^a\, t^b)_{ji}\, u_{\sigma_1 i}(p_1), \tag{1.60}$$



$$= g^2\, \bar{u}_{\sigma_2 j}(p_2)\, \gamma^+\, (t^a)_{ji}\, u_{\sigma_1 i}(p_1) \\ \times \bar{u}_{\sigma_4 l}(p_4)\, \gamma^+\, (t^a)_{lk}\, u_{\sigma_3 k}(p_3)\, \frac{1}{(p_1^+ - p_2^+)^2}, \tag{1.61}$$



$$= -\, g^2\, \bar{u}_{\sigma_2 j}(p_2)\, \gamma^+\, (t^c)_{ji}\, u_{\sigma_1 i}(p_1) \\ \times \frac{k_1^+ + k_2^+}{(k_1^+ - k_2^+)^2}\, i f^{abc} \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}, \tag{1.62}$$



$$= g^2\, f^{abe}\, f^{cde} \epsilon^*_{\lambda_2} \cdot \epsilon_{\lambda_1}\, \epsilon^*_{\lambda_4} \cdot \epsilon_{\lambda_3} \\ \times \frac{(k_1^+ + k_2^+)\,(k_3^+ + k_4^+)}{(k_1^+ - k_2^+)^2}. \tag{1.63}$$

6. For each independent momentum k^μ integrate with the measure

$$\int \frac{dk^+\, d^2k_\perp}{2(2\pi)^3}. \tag{1.64}$$

Sum over all internal quark and gluon polarizations and colors.

Again, standard parts of the rules, common to both LCPT and Feynman diagram calculations, such as symmetry factors and a factor -1 for fermion loops and for fermion lines beginning and ending at the initial state, are assumed implicitly.

The rules of LCPT are supplemented by tables of Dirac matrix elements in appendix section A.1. These tables are very useful in the evaluation of LCPT vertices.

1.3.2 Light cone wave function

An important quantity in LCPT, which is hard to construct in the standard Feynman diagram language, is the light cone wave function. Its definition is similar to that of the wave function

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in quantum mechanics. In our presentation of the light cone wave function we will follow Brodsky, Pauli, and Pinsky (1998). Imagine that we have a hadron state $|\Psi\rangle$. In general this is a superposition of different Fock states

$$\big|n_G, n_q\big\rangle \equiv \big|n_G, \{k_i^+, \vec{k}_{i\perp}, \lambda_i, a_i\}; n_q, \{p_j^+, \vec{p}_{j\perp}, \sigma_j, \alpha_j, f_j\}\big\rangle, \tag{1.65}$$

where a particular Fock state has n_G gluons and n_q quarks (and antiquarks). The gluon momenta are labeled k_i^+, $\vec{k}_{i\perp}$, with polarizations λ_i and gluon color indices a_i where $i = 1, \ldots, n_G$. (As usual in LCPT $k_i^- = \vec{k}_{i\perp}^2 / k_i^+$, as all particles are on mass shell.) The quark momenta are labeled p_j^+, $\vec{p}_{j\perp}$, with helicities σ_j, colors α_j, and flavors f_j where $j = 1, \ldots, n_q$.

The Fock states form a complete basis such that

$$\sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, |n_G, n_q\rangle\langle n_G, n_q| = \mathbf{1}, \tag{1.66}$$

where the phase-space integral is defined by

$$\begin{aligned} \int d\Omega_{n_G+n_q} = \frac{2P^+ (2\pi)^3}{S_n} \int \prod_{i=1}^{n_G} \sum_{\lambda_i, a_i} \frac{dk_i^+ \, d^2 k_{i\perp}}{2k_i^+ (2\pi)^3} \prod_{j=1}^{n_q} \sum_{\sigma_j, \alpha_j, f_j} \frac{dp_j^+ \, d^2 p_{j\perp}}{2p_j^+ (2\pi)^3} \\ \times \delta\left(P^+ - \sum_{l_1=1}^{n_G} k_{l_1}^+ - \sum_{l_2=1}^{n_q} p_{l_2}^+\right) \delta^2\left(\vec{P}_\perp - \sum_{m_1=1}^{n_G} \vec{k}_{m_1\perp} - \sum_{m_2=1}^{n_q} \vec{p}_{m_2\perp}\right) \end{aligned} \tag{1.67}$$

with symmetry factor $S_n = n_G! \; n_Q! \; n_{\bar{Q}}!$. Here n_Q and $n_{\bar{Q}}$ are respectively the numbers of quarks and antiquarks in the wave function, so that $n_q = n_Q + n_{\bar{Q}}$. The delta functions in Eq. (1.67) represent the conservation of the "plus" and transverse components of the momenta, according to rule 1 of LCPT. The incoming hadron has longitudinal momentum P^+ and transverse momentum $\vec{P}_\perp$. We assume that each Fock state is normalized to 1, so that $\langle n_G, n_q | n_G, n_q\rangle = 1$.

Using Eq. (1.66) we can write

$$\big|\Psi\big\rangle = \sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, \big|n_G, n_q\big\rangle \big\langle n_G, n_q \big| \Psi\big\rangle. \tag{1.68}$$

The quantity

$$\Psi(n_G, n_q) = \big\langle n_G, n_q \big| \Psi\big\rangle \tag{1.69}$$

is called *the light cone wave function*. It is a multi-particle wave function, describing a Fock state in the hadron with n_G gluons and n_q quarks.

Note that requiring that the state $|\Psi\rangle$ is normalized to unity, $\langle\Psi|\Psi\rangle = 1$, and using Eq. (1.68) we can write

$$1 = \big\langle \Psi \big| \Psi \big\rangle = \sum_{n_G, n_q} \int d\Omega_{n_G+n_q} \, \big|\Psi(n_G, n_q)\big|^2. \tag{1.70}$$

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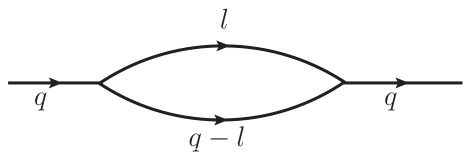

Fig. 1.1. A Feynman diagram in the ϕ^3-theory considered here. The arrows indicate the momentum flow.

We see that each light cone wave function $\Psi(n_G, n_q)$ is normalized to a number less than or equal to 1.

1.4 Sample LCPT calculations

While we expect that the reader has a fluent knowledge of Feynman rules, we realize that it is less likely that he or she is equally fluent with LCPT rules. Therefore, to help the reader become more familiar with LCPT, here we will perform two LCPT calculations. We will first "cross-check" LCPT by calculating a sample scattering amplitude using both the Feynman and LCPT rules and showing that we obtain the same result. We will then set up the rules for calculating light cone wave functions, by considering an example of a basic wave function containing $1 \to 2$ particle splitting.

1.4.1 LCPT "cross-check"

We begin by calculating a simple amplitude in a real scalar ϕ^3 field theory in two ways: using standard Feynman rules and using the rules of LCPT. We will show that the two ways give identical results. This demonstrates that LCPT is indeed equivalent to the standard Feynman diagram approach.

The process we consider is illustrated in Fig. 1.1. We consider a field theory for a real massive scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{3!}\,\phi^3. \tag{1.71}$$

The contribution of the diagram in Fig. 1.1 (henceforth labeled A) can be written down using the Feynman rules for the real scalar field theory having Lagrangian (1.71) (see e.g. Sterman (1993) on Peskin and Schroeder (1995)):

$$-i\,\Sigma = \frac{(-i\lambda)^2}{2!}\int \frac{d^4 l}{(2\pi)^4}\,\frac{i}{l^2 - m^2 + i\epsilon}\,\frac{i}{(q-l)^2 - m^2 + i\epsilon}. \tag{1.72}$$

Here $1/2!$ is a symmetry factor and m is the mass of the scalar particles.

Working in the light cone variables

$$q^\mu = (q^+, q^-, \vec{q}_\perp), \quad l^\mu = (l^+, l^-, \vec{l}_\perp), \tag{1.73}$$

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we write $l^2 = l^+ l^- - \vec{l}_\perp^2$ and $(q-l)^2 = (q^+ - l^+)(q^- - l^-) - (\vec{q}_\perp - \vec{l}_\perp)^2$. Equation (1.72) can now be rewritten as

$$-i\Sigma = \frac{\lambda^2}{4} \int \frac{dl^+ dl^- d^2 l_\perp}{(2\pi)^4} \frac{1}{l^+ l^- - \vec{l}_\perp^2 - m^2 + i\epsilon} \times \frac{1}{(q^+ - l^+)(q^- - l^-) - (\vec{q}_\perp - \vec{l}_\perp)^2 - m^2 + i\epsilon}. \tag{1.74}$$

Now we need to integrate over l^-. In the complex l^--plane the integrand in Eq. (1.74) has two poles,

$$l_1^- = \frac{\vec{l}_\perp^2 + m^2 - i\epsilon}{l^+} \quad \text{and} \quad l_2^- = q^- - \frac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2 - i\epsilon}{q^+ - l^+}. \tag{1.75}$$

The l^--integral is nonzero only if these two poles lie in different half-planes. This happens for either (i) $l^+ > 0$, $q^+ - l^+ > 0$ or (ii) $l^+ < 0$, $q^+ - l^+ < 0$. As the incoming particle with momentum q is physical we have $q^+ > 0$, which makes case (ii) impossible to achieve, as there one has $q^+ < l^+ < 0$. We are left with case (i). Closing the l^--integration contour in the lower half-plane we pick up the pole at l_1^-, obtaining

$$\begin{aligned} \Sigma &= \frac{\lambda^2}{2} \int \frac{dl^+ d^2 l_\perp}{2(2\pi)^3} \frac{\theta(l^+)\theta(q^+ - l^+)}{l^+ (q^+ - l^+)} \\ &\quad \times \frac{1}{q^- - \frac{\vec{l}_\perp^2 + m^2 - i\epsilon}{l^+} - \frac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2 - i\epsilon}{q^+ - l^+}} \\ &= \frac{\lambda^2}{2!} \int \frac{dl^+ d^2 l_\perp}{2(2\pi)^3} \frac{\theta(l^+)\theta(q^+ - l^+)}{l^+ (q^+ - l^+)} \\ &\quad \times \frac{1}{q^- - \frac{\vec{l}_\perp^2 + m^2}{l^+} - \frac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2}{q^+ - l^+} + i\epsilon}. \end{aligned} \tag{1.76}$$

We observe that Eq. (1.76) is identical to what one would obtain for the diagram in Fig. 1.1 if one calculated it using the rules of LCPT from Sec. 1.3 (modified for a scalar particle), as illustrated in Fig. 1.2. Indeed Eq. (1.76) can be obtained by assigning

$$\frac{\theta(l^+)}{l^+} \quad \text{and} \quad \frac{\theta(q^+ - l^+)}{q^+ - l^+} \tag{1.77}$$

for each internal line (LCPT rule 4), including an energy denominator

$$\frac{1}{\sum\limits_{inc} k^- - \sum\limits_{interm} k^- + i\,\epsilon} = \frac{1}{q^- - \frac{\vec{l}_\perp^2 + m^2}{l^+} - \frac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2}{q^+ - l^+} + i\epsilon} \tag{1.78}$$

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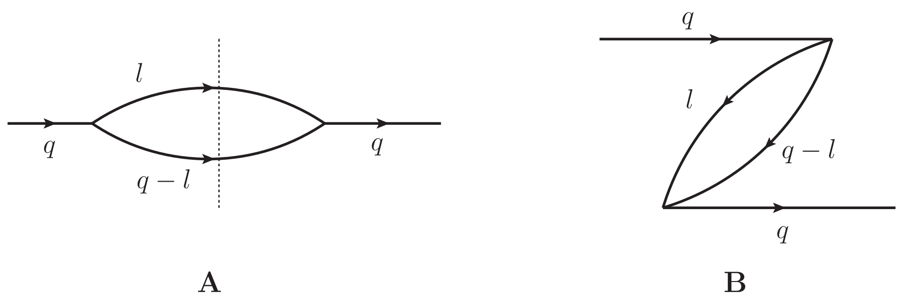

Fig. 1.2. Light cone perturbation theory diagrams in the ϕ^3-theory corresponding to the Feynman diagram in Fig. 1.1. Time flows to the right. The arrows indicate the momentum direction. The vertical dotted line indicates an intermediate state.

for the intermediate state (denoted by the dotted line in Fig. 1.2A), according to LCPT rule 3, and integrating over the internal momentum l with the integration measure

$$\int \frac{dl^+ \, d^2 l_\perp}{2(2\pi)^3}, \tag{1.79}$$

as prescribed by LCPT rule 6. In LCPT each vertex gives a factor λ (a modification of rule 5 for ϕ^3-theory) and one has to include the symmetry factor $1/2!$ as well. (Scalar particles obviously have no polarization. Neither do they have instantaneous terms.)

We have demonstrated that starting from the Feynman diagram amplitude expression (1.72) we can reduce it to the result that one would obtain by the rules of LCPT. Hence the two approaches in the end give identical expressions for the amplitudes, as expected.

A few words of caution are in order here. In principle the Feynman diagram in Fig. 1.1 corresponds to the two LCPT diagrams A and B shown in Fig. 1.2, which correspond to two different orderings of the vertices (see LCPT rule 1). The two graphs A and B in fact correspond to cases (i) and (ii) considered after Eq. (1.75). Our argument above was simplified by the fact that diagram B in Fig. 1.2 is zero as, according to the LCPT rules, it comes with a factor $\theta(-l^+)\,\theta(l^+ - q^+)$, which is zero for $q^+ > 0$. The physical meaning of this is quite clear: one cannot generate three particles with positive plus momenta out of nothing (see the lower vertex in Fig. 1.2B). Conversely, three particles with positive plus momenta cannot combine to give nothing (see the upper vertex in Fig. 1.2B). Because of this simplification, we have a one-to-one correspondence between the Feynman diagram in Fig. 1.1 and the LCPT diagram in Fig. 1.2A. In general, each Feynman diagram corresponds to a sum of all the LCPT diagrams with the same topology, including all possible time-orderings and instantaneous terms. A general derivation of an LCPT diagram starting from a Feynman diagram does not simply involve integration over the minus components of the internal momenta; one has to assign each vertex an x^+-coordinate and Fourier transform the diagram (by integrating over the minus momenta) into x^+ coordinate space. One then

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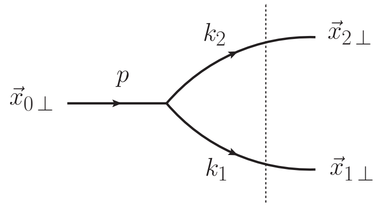

Fig. 1.3. Light cone wave function for a scalar particle splitting into two. The vertical dotted line denotes an intermediate state.

has to integrate over all the x^+-coordinates of the vertices, imposing different orderings: each ordering will lead to a different LCPT diagram.

1.4.2 A sample light cone wave function

Let us calculate, using the rules of LCPT, a sample light cone wave function. The calculation will be instructive, as the wave function we will calculate is similar to certain light cone wave functions that we will use throughout the book. In this calculation we will also illustrate in more detail what is actually meant by the light cone wave function definition (1.69) and will set up the rules for wave function calculations.

The sample wave function is depicted in Fig. 1.3. Again we are working in ϕ^3 real scalar field theory, with the Lagrangian (1.71). The wave function describes a single incoming particle splitting into two. For the scalar field theory only rules 1, 3, 4, and 6 from Sec. 1.3 apply. On top of these rules there is a factor equal to the coupling λ coming from the vertex. In calculating light cone wave functions one has to treat the "outgoing" state on the right of the diagram (the state denoted by the dotted line in Fig. 1.3) as an intermediate state. The reason is that, in describing a scattering process, the light cone wave function is thought of as a part of a larger diagram in which this "outgoing" state in fact undergoes subsequent interactions with other particles and therefore is truly an intermediate state. Our definition of the boost-invariant integration measure (1.67) dictates a slight modification of LCPT rule 4 as well, when calculating light cone wave functions: we treat the incoming lines (the external lines on the left, e.g. line p in Fig. 1.3) as "internal" and include a factor $1/p^+$ for them, while the outgoing lines (the lines on the right, e.g. lines k_1 and k_2 in Fig. 1.3) will be treated as "external" and so will not bring in such factors.

To summarize, when calculating the light cone wave function using LCPT one should follow the rules stated in Sec. 1.3, with the following modifications.

(i) The outgoing state on the right of a diagram is treated as an internal state and brings in an energy denominator according to LCPT rule 3.
(ii) At the same time the outgoing external lines on the right of the diagram bring in only factors $\theta(k^+)$, in modification of LCPT rule 4. (As usual, light cone time flows to the right.)

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(iii) The incoming external lines on the left of a diagram bring in factors $1/p^+$, i.e., LCPT rule 4 is extended to apply to those lines. (We will drop $\theta(p^+ > 0)$ as incoming lines always have positive p^+ momentum.)

According to the above-stated rules, the light cone wave function depicted in Fig. 1.3 is

$$\Psi(k_1, k_2) = \frac{1}{p^+} \frac{\lambda}{p^- - k_1^- - k_2^-} = \frac{1}{p^+} \frac{\lambda}{\frac{\vec{p}_\perp^2 + m^2}{p^+} - \frac{\vec{k}_{1\perp}^2 + m^2}{k_1^+} - \frac{\vec{k}_{2\perp}^2 + m^2}{k_2^+}}, \tag{1.80}$$

where we have omitted the regulator $i\epsilon$ for simplicity (in fact we will not need it below). Before we simplify this expression, let us note that, as can be seen from Eq. (1.70), the probability of finding such a configuration in a general "dressed" state $|\Psi\rangle$ of the incoming particle is

$$\int d\Omega_2 \left|\Psi(k_1, k_2)\right|^2, \tag{1.81}$$

where, as follows from Eq. (1.67), the phase-space integral for two identical particles is given by

$$\int d\Omega_2 = \frac{2p^+ (2\pi)^3}{2!} \int \frac{dk_1^+ \, d^2k_{1\perp}}{2k_1^+ (2\pi)^3} \frac{dk_2^+ \, d^2k_{2\perp}}{2k_2^+ (2\pi)^3} \, \delta\big(p^+ - k_1^+ - k_2^+\big) \times \delta^2\Big(\vec{p}_\perp - \vec{k}_{1\perp} - \vec{k}_{2\perp}\Big) = \frac{1}{2!} \int \frac{dk_1^+ \, d^2k_{1\perp}}{2k_1^+ (2\pi)^3} \frac{p^+}{p^+ - k_1^+}. \tag{1.82}$$

We see that $k_2^+ = p^+ - k_1^+$ and $\vec{k}_{2\perp} = \vec{p}_\perp - \vec{k}_{1\perp}$. Using these to replace k_2^+ and $\vec{k}_{2\perp}$ in Eq. (1.80) and doing some algebra yields

$$\Psi(k_1, p - k_1) = -\frac{\lambda z_1 (1 - z_1)}{(\vec{k}_{1\perp} - z_1 \vec{p}_\perp)^2 + m^2 \, [1 - z_1(1 - z_1)]}, \tag{1.83}$$

where

$$z_1 = \frac{k_1^+}{p^+} \tag{1.84}$$

is the longitudinal fraction of the original particle's momentum p carried by the particle k_1, which will be identified as a Feynman-x variable in the next chapter. Equation (1.83) gives us the momentum-space two-particle light cone wave function at the lowest order in λ.

Substituting the wave function (1.83) into Eq. (1.81) and using Eq. (1.82) for the phase-space integration measure, one obtains the probability for one particle to fluctuate into two particles:

$$\frac{\lambda^2}{2!}\int \frac{dz_1\, d^2k_{1\perp}}{2(2\pi)^3}\, \frac{z_1(1-z_1)}{\left\{(\vec{k}_{1\perp}-z_1\vec{p}_\perp)^2+m^2\,[1-z_1(1-z_1)]\right\}^2} \sim \frac{\lambda^2}{m^2}. \tag{1.85}$$

Thus the probability of the configuration in Fig. 1.3 is proportional to the coupling constant squared. As the coupling in ϕ^3-theory has the dimension of the mass, the factor m^2 in the denominator of Eq. (1.85) makes the expression dimensionless. We note in passing that the effective dimensionless coupling constant for the perturbative expansion of ϕ^3-theory is λ/m.

It is also instructive to Fourier-transform the wave function (1.83) into transverse coordinate space. The transverse coordinates of the lines are shown in Fig. 1.3. The Fourier transform is accomplished by integrating over the independent transverse momenta, assigning a factor $e^{i\vec{k}_\perp\cdot\vec{x}_\perp}$ for each line, with k the net outgoing momentum carried by the line. For the two-particle wave function (1.83) we have

$$\begin{aligned}
&\Psi(\vec{x}_{1\perp},\vec{x}_{2\perp},\vec{x}_{0\perp},z_1)\\
&\quad=\int\frac{d^2k_{1\perp}\,d^2p_\perp}{(2\pi)^4}\, e^{i\vec{k}_{1\perp}\cdot\vec{x}_{1\perp}+i\vec{k}_{2\perp}\cdot\vec{x}_{2\perp}-i\vec{p}_\perp\cdot\vec{x}_{0\perp}}\,\Psi(k_1,p-k_1)\\
&\quad=\int\frac{d^2k_{1\perp}\,d^2p_\perp}{(2\pi)^4}\, e^{i\vec{k}_{1\perp}\cdot(\vec{x}_{1\perp}-\vec{x}_{2\perp})-i\vec{p}_\perp\cdot(\vec{x}_{0\perp}-\vec{x}_{2\perp})}\,\Psi(k_1,p-k_1).
\end{aligned} \tag{1.86}$$

Substituting Eq. (1.83) into Eq. (1.86) and integrating yields (see Eq. (A.11))

$$\begin{aligned}
\Psi(\vec{x}_{1\perp},\vec{x}_{2\perp},\vec{x}_{0\perp},z_1) = &-\frac{\lambda}{2\pi}\, z_1(1-z_1)\, K_0\Big(|\vec{x}_{12}|\, m\sqrt{1-z_1(1-z_1)}\Big)\\
&\times\,\delta^2(\vec{x}_{0\perp}-z_1\vec{x}_{1\perp}-(1-z_1)\vec{x}_{2\perp}),
\end{aligned} \tag{1.87}$$

where $\vec{x}_{ij}\equiv\vec{x}_{i\perp}-\vec{x}_{j\perp}$. Equation (1.87) gives us the $1\to 2$ splitting wave function shown in Fig. 1.3 in coordinate space. Even though this wave function has been obtained for the scalar ϕ^3-theory case it has a feature valid for theories with higher spin: it contains a delta function insuring that $\vec{x}_{0\perp}=z_1\vec{x}_{1\perp}+(1-z_1)\vec{x}_{2\perp}$. This means that the transverse coordinate positions of the two produced particles are indeed related to each other (Kopeliovich, Tarasov, and Schafer 1999): both the original particle and the two new particles lie on one straight line in transverse coordinate space, and $x_{02}:x_{01}=z_1:(1-z_1)$ where $x_{ij}=|\vec{x}_{ij}|$. The transverse coordinate space structure of the wave function (1.87) is illustrated in Fig. 1.4. The same constraint on the transverse plane locations of the produced particles as derived here for the ϕ^3-theory applies to the splittings of particles in quantum electrodynamics (QED) and in QCD.

1.5 Asymptotic freedom

A remarkable property of QCD, known as *asymptotic freedom*, is the fact that the running QCD coupling tends to be small at short distances (corresponding to large values of the

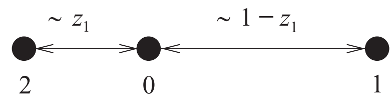

Fig. 1.4. The $1 \to 2$ splitting wave function pictured in transverse coordinate space. The circles represent particles and the numbers label these particles in agreement with the diagram in Fig. 1.3: 0 labels the original particle, while 1 and 2 label the produced particles.

relevant four-momentum squared, $q^2 = -Q^2$ with Q a real number). The running of the QCD coupling constant is given by (Gross and Wilczek 1973, Politzer 1973)[4]

$$\alpha_s(Q^2) \;=\; \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\,\beta_2\,\ln(Q^2/\mu^2)}, \tag{1.88}$$

where

$$\beta_2 = \frac{11N_c - 2N_f}{12\pi} \tag{1.89}$$

with $N_c = 3$ the number of colors and N_f the number of quark flavors. The QCD beta function is given by

$$\beta_{QCD}(\alpha) = -\beta_2\,\alpha^2 + O(\alpha^3). \tag{1.90}$$

While $N_f = 6$ in the Standard Model of particle physics, the effective number of flavors relevant for a given physical process depends on the momentum scale Q and may be smaller than six. One can clearly see from Eq. (1.88) that $\alpha_s(Q^2) \to 0$ as $Q^2 \to \infty$: the strong coupling is small at large momenta. Thus quarks and gluons interact *weakly* at asymptotically short distances; this is asymptotic freedom.

Such behavior is in striking contrast with the running of the coupling in quantum electrodynamics (QED), where β_2 is negative, making the QED coupling grow with Q^2 (Landau, Abrikosov, and Halatnikov 1956). The main difference between QED and QCD is in the non-Abelian interactions between the gluons. Owing to these interactions the gluon propagator receives corrections not only from quark loops (which are quite similar to the electron loops in QED) but also from gluon loops. The polarizations of virtual gluons in these loops can be either transverse or longitudinal. The transverse gluon and quark loops generate terms tending to make the QCD beta function positive (and $\beta_2 < 0$). Owing to a large contribution from the longitudinal gluon in the loop, however, the resulting QCD beta function is negative (and $\beta_2 > 0$), leading to asymptotic freedom (see Khriplovich (1969), Gribov (1978), and Dokshitzer and Kharzeev (2004) for more details).

The quantity μ in Eq. (1.88) is an arbitrary scale (known as the renormalization point): physical observables should not depend on its value. In fact Eq. (1.88) can be rewritten as

$$\alpha_s(Q^2) = \frac{1}{\beta_2\,\ln(Q^2/\Lambda_{QCD}^2)} \tag{1.91}$$

[4] The QCD beta function was also calculated by 't Hooft but the result was not included in t' Hooft (1972).

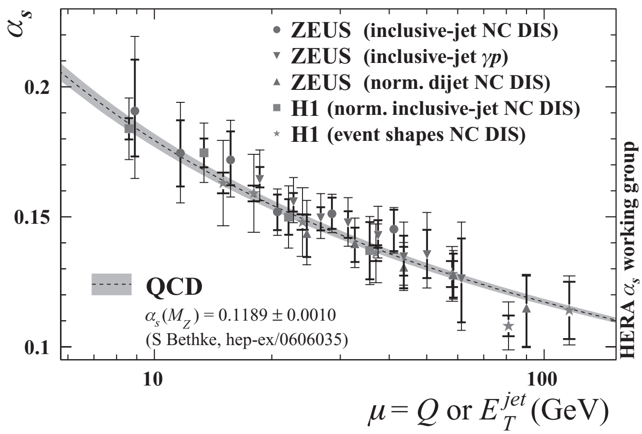

Fig. 1.5. The experimental data on the running QCD coupling from deep inelastic scattering (DIS) experiments at HERA. The dashed line with a band around it is the theoretical prediction for the strong coupling. (Reprinted with permission from H1 and ZEUS collaboration (2008). Copyright 2008 by IOP Publishing.) A color version of this figure is available online at www.cambridge.org/9780521112574.

where $\Lambda_{QCD} \approx 200-300$ MeV is the fundamental scale of QCD. (The exact value of Λ_{QCD} depends on the renormalization scheme used.) The strong coupling constant $\alpha_s(Q^2)$ becomes large near $Q \approx \Lambda_{QCD}$, leading to strong forces between the quarks and gluons. These strong forces presumably contribute to the confinement of quarks and gluons within hadrons.

For the purposes of this book the most important implication of Eq. (1.88) is that at short distances (large transverse momenta) the strong coupling is small. This small value of the dimensionless running QCD coupling gives the naturally small parameter needed to develop perturbation theory. Therefore the rules are simple: as we probe shorter and shorter distances inside the hadron perturbative QCD calculations become better justified, providing more theoretical control over the problem at hand.

Figure 1.5 shows a compilation of the data on the strong coupling constant determined from deep inelastic electron–proton scattering experiments at a single collider, the Hadron Electron Ring Accelerator (HERA) at the Deutsches Elektronen-Synchrotron (DESY) laboratory in Hamburg, Germany. The dashed line with a narrow band around it in Fig. 1.5 represents our theoretical knowledge of $\alpha_s(Q^2)$, which is based on Eq. (1.88) along with several higher-order corrections (up to three loops). The agreement between theory and data shown in Fig. 1.5 is quite remarkable and is a major triumph in our attempts to understand how QCD works.