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ON PRIME RIGHT ALTERNATIVE RINGS WITH COMMUTATORS IN THE LEFT NUCLEUS

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A ring is called s-prime if the 2-sided annihilator of a nonzero ideal must be zero. In particular, any simple ring or prime $(-1,1)$ ring is s-prime. Also, a nonzero s-prime right alternative ring, with characteristic $\neq 2$, cannot be right nilpotent. Let R be a right alternative ring with commutators in the left nucleus. Then R is associative in the following cases: (1) R is prime, with characteristic $\neq 2$, and has an idempotent $e \neq 1$ such that $(e,e,R) = 0$. (2) R is an algebra over a commutative-associative ring with $1/6$, and R is either s-prime, or R is prime and locally $(-1,1)$.

1. INTRODUCTION

A ring is right alternative if it satisfies the identity

$$(y,x,x) = 0, \tag{1}$$

where by definition the associator $(x,y,z) = (xy)z - x(yz)$. A right alternative ring which also satisfies the identity $S(x,y,z) = 0$, where $S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y)$, is called $(-1,1)$; and one which satisfies the weaker identity $S(xy,x,y) = 0$ is said to be locally $(-1,1)$.

In Section 2 we define a ring to be s-prime if the 2-sided annihilator of any nonzero ideal is zero. Any simple ring or prime $(-1,1)$ ring is s-prime. Also, if the attached Jordan ring $R^{(+)}$ of a right alternative ring R is prime, then R is s-prime, although the converse need not be true. We shall show that a nonzero s-prime right alternative ring, with characteristic $\neq 2$, cannot be right nilpotent. In particular, there does not exist a nonzero s-prime right alternative nil algebra, over a commutative-associative ring with $1/2$, which satisfies the minimum condition on right ideals.

In any ring R the left nucleus is the subring $N = \{n \in R \mid (n,R,R) = 0\}$. In Section 3 we shall consider a right alternative ring such that the linear span of all commutators $[x,y] = xy - yx$ is contained in N. Such rings were first considered by

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Paul [8], who showed that if R is a semiprime $(-1,1)$ ring with characteristic $\neq 2,3$, then $[R,R] \subseteq N$ implies R is associative. Likewise, the authors [3] have shown that a simple right alternative ring R with characteristic $\neq 2$ is associative if $[R,R] \subseteq N$. In Section 3, assuming $[R,R] \subseteq N$, we shall extend these results in the following ways. First, if a prime right alternative ring R with characteristic $\neq 2$ has an idempotent $e \neq 1$ such that $(e,e,R) = 0$, then R is associative. Next, assuming R is a right alternative algebra over a commutative-associative ring with $1/6$, then R is associative if R is s-prime, or if R is prime and locally $(-1,1)$.

Finally, we note that in our proofs we shall need to make use of the following identities:

(T) $$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z,$$

(S) $$S(x,y,z) = [xy,z] + [yz,x] + [zx,y],$$

(C) $$[xy,z] = x[y,z] + [x,z]y + (x,y,z) - (x,z,y) + (z,x,y),$$

(1′) $$(y,x,z) + (y,z,x) = 0,$$

(2) $$2S(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y],$$

(3) $$[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = 0,$$

(4) $$(w,x,yz) + (w,y,xz) = (w,x,z)y + (w,y,z)x,$$

(5) $$(xy,z,w) + (x,y,[z,w]) = x(y,z,w) + (x,z,w)y,$$

(6) $$([w,x],y,z) - [w,(x,y,z)] + [x,(w,y,z)] = (x,w,[y,z]) - (w,x,[y,z]),$$

(7) $$z(x,x,y) = (zx,x,y) + (z,yx,x),$$

(8) $$(x,x,y)^4 = 0,$$

(9) $$[y,(x,x,y)] = 0,$$

where as usual $x \circ y = xy + yx$. Straightforward verifications show that (T), (S), and (C) hold in any ring. Identity (1′) is just the linearised form of (1), and (2)–(8) are known to hold in any right alternative ring with characteristic $\neq 2$. For example, (2), (5), and (6) can be found directly in [14], and (4) is just the linearised form of an identity there. Also, in any right alternative ring $S(x,y,z) + S(x,z,y) = 0$, so (S) gives (3). Identity (7) can be found in [10], and (8) was established in [4]. That (9) holds in a locally $(-1,1)$ ring with characteristic $\neq 2$ also follows from [14].

2. s-PRIME RINGS

Let A and B be ideals of a ring R. If $AB = 0$ implies either $A = 0$ or $B = 0$, then R is said to be prime; and if $AB = 0 = BA$ implies $A = 0$ or $B = 0$, then R is said to be weakly-prime. Also, if $A^2 = 0$ implies $A = 0$, then R is semiprime. It is clear that any prime ring is weakly-prime, and that any weakly-prime ring is semiprime.

PROPOSITION 1. *A ring R is prime if and only if R is weakly-prime.*

PROOF: Let A and B be ideals of a weakly-prime ring R with $AB = 0$. Then $(A \cap B)^2 \subseteq AB = 0$ implies $A \cap B = 0$ since R is semiprime, and so also $BA \subseteq A \cap B = 0$. Since R is weakly-prime, this means either $A = 0$ or $B = 0$, that is, R is prime. ∎

We now define a ring R to be s-prime if $Ann(A) = \{x \in R \mid xA = 0 = Ax\} = 0$ for any nonzero ideal A of R. It is easy to see that any simple ring is s-prime, any s-prime ring is weakly-prime, and also that an s-prime ring is without nonzero nilpotent ideals. We note that Miheev [5] has constructed a finite-dimensional, prime, right alternative nil algebra with nilpotent heart. Thus a prime right alternative ring need not be s-prime. However, Sterling [13] has shown that in a $(-1,1)$ ring the 2-sided annihilator of an ideal is itself an ideal. Thus from Proposition 1 it follows that a $(-1,1)$ ring is prime if and only if it is s-prime. We also note that Pchelincev [9] has constructed a prime $(-1,1)$ ring that is not alternative, that is, does not also satisfy the identity $(x,x,y) = 0$. Thus even an s-prime $(-1,1)$ ring need not be alternative.

It is well-known that if R is a right alternative ring, then redefining multiplication by $x \circ y = xy + yx$ gives a ring $R^{(+)}$ which is Jordan, that is, satisfies $[x,y] = 0 = (x^2, y, x)$. This ring $R^{(+)}$ is referred to as the attached Jordan ring.

PROPOSITION 2. *Let R be a right alternative ring. If the attached Jordan ring $R^{(+)}$ is prime, then R is s-prime.*

PROOF: Let R be a right alternative ring such that $R^{(+)}$ is prime, and let A be a nonzero ideal of R. Clearly A is also an ideal of $R^{(+)}$. Now suppose $x \in Ann(A)$. Then by (1′) $A(x \circ R) \subseteq (Ax)R + (AR)x \subseteq Ax = 0$, and $(x \circ R)A \subseteq (xR)A + (Rx)A \subseteq (xA)R + x(R \circ A) + (RA)x + R(x \circ A) \subseteq xA + Ax = 0$. Thus $Ann(A)$ is also an ideal of $R^{(+)}$. But $A \circ Ann(A) = 0$, so $R^{(+)}$ prime implies $Ann(A) = 0$, that is, R is s-prime. ∎

We note that the converse of Proposition 2 is not true. In particular, Miheev [6] has constructed a simple right alternative nil ring that is not alternative, and in his example the subspace spanned by the set DB_t is a trivial Jordan ideal. Thus R an s-prime right alternative ring does not even imply $R^{(+)}$ is semiprime.

We next define an element x of a ring R to be anticommutative if $x \circ R = 0$. We note that Kleinfeld [2] has shown that a semiprime alternative ring can have no nonzero anticommutative elements. However, this is not so for prime right alternative rings in general. In the finite-dimensional, prime, right alternative nil algebra constructed by Miheev [5], the basis element e_{10} is anticommutative.

PROPOSITION 3. *Let R be an s-prime right alternative ring with characteristic $\neq 2$. If $t \in R$ is anticommutative, then $t = 0$.*

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PROOF: Since by assumption t is anticommutative, from (3) we have $0 = [x \circ y, t] + [y \circ t, x] + [t \circ x, y] = [x \circ y, t]$. But also $(x \circ y) \circ t = 0$, so characteristic $\neq 2$ gives

$$(x \circ y)t = 0 = t(x \circ y). \tag{10}$$

We now let $V = \{v \in R \mid tv = 0\}$. Then by (10) and (1′) we have $0 = t(x \circ v) = (tx)v + (tv)x = (tx)v$, and so also $(xt)v = 0$ since t anticommutes. Now using these calculations and (1′), we see $t(xv) = -(xv)t = (xt)v - x(t \circ v) = 0$, and then $t(vx) = -t(xv) + (tv)x + (tx)v = 0$. Hence V is an ideal of R with $tV = 0 = Vt$. Since R is s-prime, this means either $t = 0$ or $V = 0$. But $0 = t \circ t = 2t^2$ and characteristic $\neq 2$ imply $t^2 = 0$. Thus $t \in V$, and so in either case we arrive at $t = 0$. □

Now let R be any ring. We set $R^{[1]} = R$ and then define inductively $R^{[k]} = R^{[k-1]}R$. If $R^{[n]} = 0$ for some integer $n \geqslant 1$, then R is said to be right nilpotent.

THEOREM 1. *A nonzero s-prime right alternative ring with characteristic $\neq 2$ cannot be right nilpotent.*

PROOF: Suppose R is an s-prime right alternative ring with characteristic $\neq 2$, and that R is right nilpotent. Then by Skosyrskiĭ [11] $R^{(+)}$ is nilpotent, say of index n. Now if $n > 1$, then for $n-1$ factors the elements $(((R \circ R) \circ R) \dots) \circ R$ are anticommutative, and so must be zero by Proposition 3. But this contradicts that n is the index of nilpotency for $R^{(+)}$, and so it must be that $n = 1$, that is, $R = 0$. □

COROLLARY. *There does not exist a nonzero s-prime right alternative nil algebra R, over a commutative-associative ring with $1/2$, such that R satisfies the minimum condition on right ideals.*

PROOF: By Skosyrskiĭ[12] such an R would be right nilpotent, and so by Theorem 1 it must be zero. □

As noted earlier, Miheev [6] has constructed a simple right alternative nil ring that is not alternative. His example is also without proper left ideals. Thus, in contrast to the preceding Corollary, there do exist nonzero s-prime right alternative nil rings with minimum condition on left ideals.

3. COMMUTATORS IN THE LEFT NUCLEUS

In this section we shall be considering a right alternative ring R, with characteristic $\neq 2$, such that $[R, R] \subseteq N$. We first observe that such an R satisfies the following identities:

$$-[w, (x, y, z)] + [x, (w, y, z)] = (x, w, [y, z]) - (w, x, [y, z]), \tag{11}$$

$$[[x, y], (a, b, c)] = (a, [b, c], [x, y]) = -[[b, c], (a, x, y)]. \tag{12}$$

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Identity (11) is just (6) and the assumption $[w,x] \in N$. Then $[x,y] \in N$, (11), and (1′) give $[[x,y],(a,b,c)] = [[x,y],(a,b,c)] - [a,([x,y],b,c)] = -(a,[x,y],[b,c]) + ([x,y],a,[b,c]) = -(a,[x,y],[b,c]) = (a,[b,c],[x,y])$. From this and (1′) we also have $[[b,c],(a,x,y)] = (a,[x,y],[b,c]) = -(a,[b,c],[x,y])$, which establishes (12).

For R any ring, we next let $A = (R,R,R) + (R,R,R)R$. Using (T) it follows directly that A is an ideal of R, which is commonly referred to as the associator ideal of R. Also, suppose B is an ideal of R contained in the left nucleus N of R. Then by (T) we have $B(R,R,R) \subseteq (BR,R,R) + (B,R^2,R) + (B,R,R^2) + (B,R,R)R \subseteq (B,R,R) = 0$, and so also $B((R,R,R)R) = (B(R,R,R))R = 0$. Thus we have shown

(13) $$BA = 0 \text{ if } B \text{ is an ideal contained in } N.$$

Theorem 2. *Let R be a prime right alternative ring, with characteristic $\neq 2$, such that $[R,R] \subseteq N$. If R has an idempotent $e \neq 1$ such that $(e,e,R) = 0$, then R is associative.*

Proof: Since $(e,e,R) = 0$, the right alternative ring R permits a Peirce decomposition with respect to e. Thus $R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$ (module direct sum), where $R_{ij} = \{x \in R \mid ex = ix,\ xe = jx\}$ for $i,j = 0,1$. Also, by Humm [1] these submodules R_{ij} have the following multiplication table:

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	$R_{11}+R_{01}$	R_{10}	R_{10}	0
R_{10}	0	$R_{11}+R_{01}$	R_{11}	R_{10}
R_{01}	R_{01}	R_{00}	$R_{10}+R_{00}$	0
R_{00}	0	R_{01}	R_{01}	$R_{10}+R_{00}$

Now for $i \neq j$ we have $(i-j)R_{ij} = [e,R_{ij}] \subseteq N$, so $R_{ij} \subseteq N$. Then $0 = (R_{ij},e,R_{ij}) = (j-i)R_{ij}^2$ implies $R_{ij}^2 = 0$. Thus in our case this multiplication table becomes

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	$R_{11}+R_{01}$	R_{10}	R_{10}	0
R_{10}	0	0	R_{11}	R_{10}
R_{01}	R_{01}	R_{00}	0	0
R_{00}	0	R_{01}	R_{01}	$R_{10}+R_{00}$

At this point, using the last table and (1′), straightforward and standard calculations show $H = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$ is an ideal of R. Also, since by (T) N is a subring and we know $R_{ij} \subseteq N$ for $i \neq j$, this ideal H is contained in the left nucleus

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N. Thus by (13) $HA = 0$. Now since by assumption R is prime, either $H = 0$ or $A = 0$. But if $H = 0$, then $R_{10} = 0 = R_{01}$ implies R_{11} and R_{00} are orthogonal ideals. Since R is prime and $e \neq 0$ is in R_{11}, this means $R = R_{11}$. Hence $e = 1$, which is a contradiction. Therefore we must have the associator ideal $A = 0$, which proves that R is associative. □

Now let R be any ring with $[R,R] \subseteq N$, and set $\overline{N} = \{n \in N \mid nA = 0\}$.

LEMMA 1. *For R a ring with $[R,R] \subseteq N$, the following are equivalent for $n \in N$:*

(a) $n \in \overline{N}$,
(b) $nR \subseteq N$,
(c) $Rn \subseteq N$.

PROOF: From (T) we have $n(x,y,z) = (nx,y,z)$. Thus $nR \subseteq N \Leftrightarrow n(R,R,R) = 0 \Leftrightarrow nA = 0$, that is, $(a) \Leftrightarrow (b)$.

Since $nx = [n,x] + xn$ and $[R,R] \subseteq N$, $nR \subseteq N \Leftrightarrow RN \subseteq N$, that is, $(b) \Leftrightarrow (c)$. □

COROLLARY. *If R is a ring with $[R,R] \subseteq N$, then $\overline{N}R \subseteq \overline{N}$.*

PROOF: By Lemma 1(b) $\overline{N}R \subseteq N$, and $(\overline{N}R)A = \overline{N}(RA) \subseteq \overline{N}A = 0$. Thus $\overline{N}R \subseteq \overline{N}$. □

LEMMA 2. *If R is a ring with $[R,R] \subseteq N$, then $[N,N] \subseteq \overline{N}$.*

PROOF: Let $n,m \in N$. Then using (C), $[R,R] \subseteq N$, and that N is a subring, we have $[n,m]y = [ny,m] - n[y,m] - (n,y,m) + (n,m,y) - (m,n,y) = [ny,m] - n[y,m] \in N$. Thus $[N,N] \subseteq \overline{N}$ by Lemma 1. □

LEMMA 3. *If R is a ring with $[R,R] \subseteq N$, then $(R,N,N) \subseteq N$*

PROOF: $S(x,y,z) \in N$ from (S) and $[R,R] \subseteq N$. Thus for $n,m \in N$ we have $(x,n,m) = (x,n,m) + (n,m,x) + (m,x,n) = S(x,n,m) \in N$. □

LEMMA 4. *Let R be a right alternative algebra over a commutative-associative ring with $1/2$. If $[[R,R],R] \subseteq N$, then the ideal generated in R by $[[R,R],R]$ is $\langle [[R,R],R] \rangle = [[R,R],R] + [[R,R],R]R$.*

PROOF: First $R[[R,R],R] \subseteq [R,[[R,R],R]] + [[R,R],R]R$, and $([[R,R],R]R)R = [[R,R],R]R^2 \subseteq [[R,R],R]R$ using $[[R,R],R] \subseteq N$. Then $R([[R,R],R]R) \subseteq (R,[[R,R],R],R) + (R[[R,R],R])R \subseteq (R,R,[[R,R],R]) + [[R,R],R]R$ by $(1')$ and the preceding. But by (C), $(1')$, and $[[R,R],R] \subseteq N$, we have $2(x,y,[[a,b],c]) = [xy,[[a,b],c]] - x[y,[[a,b],c]] - [x,[[a,b],c]]y - ([[a,b],c],x,y) = [xy,[[a,b],c]] - x[y,[[a,b],c]] - [x,[[a,b],c]]y$. Thus also $(R,R,[[R,R],R]) \subseteq [R,[[R,R],R]] + R[R,[[R,R],R]] + [R,[[R,R],R]]R \subseteq [[R,R],R] + [[R,R],R]R$ by the preceding. Hence $R([[R,R],R]R) \subseteq [[R,R],R] + [[R,R],R]R$, which proves the lemma. □

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Lemma 5. *Let R be a right alternative ring with characteristic $\neq 2$. If $[R,R] \subseteq N$ and $[N,N]=0$, then $[[R,R],(R,R,R)]=0$.*

Proof: First by (2), $[R,R] \subseteq N$, and $[N,N]=0$, we have $2(R,N,N) = 2S(R,N,N) \subseteq [[R,N],N]+[[N,N],R]+[[N,R],N] = 0$, so characteristic $\neq 2$ implies $(R,N,N)=0$. Thus (12) and $[R,R] \subseteq N$ give $[[R,R],(R,R,R)] \subseteq (R,[R,R],[R,R]) \subseteq (R,N,N)=0$. ∎

Lemma 6. *Let R be a semiprime right alternative algebra over a commutative-associative ring with $1/2$. If $[R,R] \subseteq N$ and $[N,N]=0$, then $[[R,R],R]=0$.*

Proof: We shall show that $\langle [[R,R],R] \rangle$ is a trivial ideal. First, using $[[a,b],[[y,z],x]] \in [N,N]=0$, (C) and (1′), we have $[[a,b],[y,z]x+[x,z]y] = [[a,b],x[y,z]+[x,z]y] = [[a,b],[xy,z]-2(x,y,z)-(z,x,y)] \in [N,N]+[[R,R],(R,R,R)] = 0$ by Lemma 5. This then gives $[[a,b],[[s,t],z]x] = -[[a,b],[x,z][s,t]] \in [N,N]=0$, that is, $[[R,R],[[R,R],R]R]=0$. This, (C) and (1′), $[R,R] \subseteq N$, and $[N,N]=0$, then imply $[[R,R],R]^2 \subseteq [[R,R],[[R,R],R]R] + [[R,R],[[R,R],R]]R + 2([[R,R],R],R,[R,R]) + ([R,R],[[R,R],R],R) = [[R,R],[[R,R],R]]R \subseteq [N,N]R = 0$. Hence also $[[R,R],R]([[R,R],R]R) = [[R,R],R]^2R = 0$, so by Lemma 4 we have $[[R,R],R] \langle [[R,R],R] \rangle = 0$. Then likewise $([[R,R],R]R)\langle [[R,R],R] \rangle = [[R,R],R](R\langle [[R,R],R] \rangle) \subseteq [[R,R],R]\langle [[R,R],R] \rangle = 0$. Thus it follows $\langle [[R,R],R] \rangle^2 = 0$, and so R semiprime gives $[[R,R],R]=0$. ∎

Corollary. *Let R be a semiprime right alternative algebra over a commutative-associative ring with $1/6$. If $[R,R] \subseteq N$ and $[N,N]=0$, then R is associative.*

Proof: By Lemma 6 $[[R,R],R]=0$, and so R is associative by [8]. ∎

Lemma 7. *Let R be a semiprime right alternative ring with $[R,R] \subseteq N$. If $L \subseteq A \cap \overline{N}$ is a left ideal, then $L=0$.*

Proof: First $RL \subseteq L$, $(LR)R = LR^2 \subseteq LR$, and using (1′) $R(LR) \subseteq (R,L,R) + (RL)R \subseteq (R,R,L)+LR \subseteq L+LR$. Thus the ideal generated by L in R is $\langle L \rangle = L + LR$. Now by the Corollary to Lemma 1 $\langle L \rangle = L+LR \subseteq A \cap \overline{N}$. Hence $\langle L \rangle^2 \subseteq \overline{N}A = 0$, so R semiprime gives $0 = \langle L \rangle = L$. ∎

At this point we let $I = \{\overline{n} \in \overline{N} \mid A\overline{n} = 0\} = N \cap Ann(A)$.

Lemma 8. *For R a semiprime right alternative ring with $[R,R] \subseteq N$, the following are equivalent for $\overline{n} \in \overline{N}$:*

(a) $\overline{n} \in I$,

(b) $(R,A,\overline{n}) = 0$,

(c) $R\overline{n} \subseteq \overline{N}$.

Proof: If $\overline{n} \in I$, then $(R,A,\overline{n}) \subseteq (RA)\overline{n} + R(A\overline{n}) \subseteq A\overline{n} = 0$. Thus (a) ⇒ (b). If $(R,A,\overline{n}) = 0$, then using (1′) we have $(R\overline{n})A \subseteq (R,\overline{n},A) + R(\overline{n}A) \subseteq (R,A,\overline{n}) = 0$.

Thus this and Lemma 1(c) imply $R\overline{n} \subseteq \overline{N}$, that is, (b) $\Rightarrow$ (c). If $R\overline{n} \subseteq \overline{N}$, then using (1′) we see $R(A\overline{n}) \subseteq (R, A, \overline{n}) + (RA)\overline{n} \subseteq (R, \overline{n}, A) + A\overline{n} \subseteq \overline{N}A + A\overline{n} \subseteq A\overline{n}$. Thus $A\overline{n} \subseteq A \cap \overline{N}$ is a left ideal, so by Lemma 7 $A\overline{n} = 0$. Therefore (c) $\Rightarrow$ (a), which proves the lemma. □

Suppose now that R is a right alternative ring with characteristic $\neq 2$, and let $M = \{m \in R \mid (m, R, R) = 0 = (R, R, m)\}$ be the nucleus of R. If R is prime and not associative, then by [7] M coincides with the centre of R. Thus under the indicated assumptions we have

$$[M, R] = 0. \tag{14}$$

Lemma 9. *Let R be a prime right alternative ring with characteristic $\neq 2$ and $[R, R] \subseteq N$. If $\overline{n} \in \overline{N}$, then $\overline{n} \in I$ if and only if $(R, [R, R], \overline{n}) = 0$.*

Proof: First let $\overline{n} \in I$. Then using (1′), (5), $[R, R] \subseteq N$, and (T), we have $(R, [R, R], \overline{n}) = -(R, \overline{n}, [R, R]) \subseteq (R\overline{n}, R, R) - R(\overline{n}, R, R) - (R, R, R)\overline{n} = (\overline{n}R, R, R) = \overline{n}(R, R, R) = 0$.

Conversely, suppose that $(R, [R, R], \overline{n}) = 0$. We may assume that R is not associative, since otherwise $A = 0$ clearly implies $\overline{N} = R = I$. Now by (5), $[R, R] \subseteq N$, (1′), and (T), we see $(R, R, R)\overline{n} \subseteq (R\overline{n}, R, R) + (R, \overline{n}, [R, R]) - R(\overline{n}, R, R) = (\overline{n}R, R, R) - (R, [R, R], \overline{n}) = \overline{n}(R, R, R) = 0$. Thus

$$(R, R, R)\overline{n} = 0. \tag{15}$$

Then since (7) shows $R(x, x, R) \subseteq (R, R, R)$, by (15) we have

$$(R(x, x, R))\overline{n} \subseteq (R, R, R)\overline{n} = 0. \tag{16}$$

Next, using (5) and $(R, [R, R], \overline{n}) = 0$, we see $(R, R, [[R, R], \overline{n}]) \subseteq -(R^2, [R, R], \overline{n}) + R(R, [R, R], \overline{n}) + (R, [R, R], \overline{n})R = 0$. Since $[R, R] \subseteq N$, this shows $[\overline{n}, [R, R]] \subseteq M$; and so by (14) $[[\overline{n}, [R, R]], R] = 0$. It now follows directly that the ideal generated in R by $[\overline{n}, [R, R]]$ is $\langle [\overline{n}, [R, R]] \rangle = [\overline{n}, [R, R]] + [\overline{n}, [R, R]]R$, which by Lemma 2 and the Corollary to Lemma 1 is contained in $\overline{N}$. Therefore $\langle [\overline{n}, [R, R]] \rangle A = 0$ and R prime but not associative give

$$[\overline{n}, [R, R]] = 0. \tag{17}$$

Then by (17) and $\overline{n}[(R, R, R), R] \subseteq \overline{n}A = 0$, we have

$$[(R, R, R), R]\overline{n} = 0. \tag{18}$$

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From (18), (T), and (15), we thus see $(w(x,y,z))\overline{n} = ((x,y,z)w)\overline{n} = \{(xy,z,w) - (x,yz,w) + (x,y,zw) - x(y,z,w)\}\overline{n} = -(x(y,z,w))\overline{n} = -(x(w,y,z))\overline{n}$ using $(1')$ and linearised (16). This, linearised (16), and $(1')$ now show

(19) $[w(x,\ y,\ z)]\,\overline{n}$ is an alternating function in $w,\ x,\ y,\ z$.

We shall now show that $(R(R,R,R))\overline{n} = (R,(R,R,R),\overline{n}) = 0$. Since by (T) we also have $A = (R,R,R) + R(R,R,R)$, this with (15) will prove $A\overline{n} = 0$, and so prove the lemma.

First, using (11) and (15), we have $[t,(x,(y,z,w),\overline{n})] = [x,(t,(y,z,w),\overline{n})] + (t,x,[(y,z,w),\overline{n}]) - (x,t,[(y,z,w),\overline{n}]) = [x,(t,(y,z,w),\overline{n})]$. This, (15), and (19), give $[t,(x,(y,z,w),\overline{n})] = [x,(t,(y,z,w),\overline{n})] = -[x,(y,(t,z,w),\overline{n})] = [x,(y,(z,t,w),\overline{n})] = -[x,(y,(z,w,t),\overline{n})]$. Then iteration of this last identity shows $[t,(x,(y,z,w),\overline{n})] = -[t,(x,(y,z,w),\overline{n})]$, so characteristic $\neq 2$ implies

$$[R,(R,(R,R,R),\overline{n})] = 0. \tag{20}$$

Now $(R,(R,R,R),\overline{n}) \subseteq N$ by (15) and Lemma 1. Thus using (C), $(1')$, and (20), we see $2(R,R,(R,(R,R,R),\overline{n})) \subseteq ((R,(R,R,R),\overline{n}),R,R) + [R^2,(R,(R,R,R),\overline{n})] + R[R,(R,(R,R,R),\overline{n})] + [R,(R,(R,R,R),\overline{n})]R = 0$. Since characteristic $\neq 2$, this means $(R,(R,R,R),\overline{n})$ is contained in the centre of R; and so it follows directly that the ideal generated in R by $(R,(R,R,R),\overline{n})$ is $\langle (R,(R,R,R),\overline{n})\rangle = (R,(R,R,R),\overline{n}) + (R,(R,R,R),\overline{n})R$.

Now let a and a' be associators, that is, elements of the form (x,y,z). Using (4), (15), and $(1')$, we have $(w,a,\overline{n})a' = -(w,a',\overline{n})a + (w,a,a'\overline{n}) + (w,a',a\overline{n}) = -(w,a',\overline{n})a = -(w,aa',\overline{n}) - (w,\overline{n}a',a) + (w,a',a)\overline{n} = -(w,aa',\overline{n}) = -(w,a'a,\overline{n})$, since $(R,[R,R],\overline{n}) = 0$. Thus $(w,a,\overline{n})a' = -(w,a'a,\overline{n}) = (w,\overline{n}a,a') - (w,a,\overline{n})a' - (w,a,a')\overline{n} = -(w,a,\overline{n})a'$, again by (4) $(1')$, and (15). Since characteristic $\neq 2$, we thus have $(R,(R,R,R),\overline{n})A = 0$; and so $\langle (R,(R,R,R),\overline{n})\rangle = (R,(R,R,R),\overline{n}) + (R,(R,R,R),\overline{n})R \subseteq \overline{N}$ by the Corollary to Lemma 1. But then $\langle (R,(R,R,R),\overline{n})\rangle^2 \subseteq \overline{N}A = 0$, so R semiprime implies $(R(R,R,R))\overline{n} = (R,(R,R,R),\overline{n}) = 0$ as claimed. □

Lemma 10. *Let R be a right alternative ring with characteristic $\neq 2$. If $[R,R] \subseteq N$, then $(R,N,[N,N])$, $R[N,[N,N]] \subseteq \overline{N}$.*

Proof: First, using (11) and Lemmas 3 and 2, we have $(R,N,[N,N]) \subseteq (N,R,[N,N]) + [N,(R,N,N)] + [R,(N,N,N)] = [N,(R,N,N)] \subseteq [N,N] \subseteq \overline{N}$. Next, using Lemmas 2 and 1, we see $R[N,[N,N]] \subseteq R[N,N] \subseteq R\overline{N} \subseteq N$. Now by (5) we have $(R,R,[N,[N,N]]) \subseteq (R^2,N,[N,N]) + R(R,N,[N,N]) + (R,N,[N,N])R$. Since we have already shown $(R,N,[N,N]) \subseteq \overline{N}$, it thus follows from Lemma 1 that $(R,R,[N,[N,N]]) \subseteq N$. Then this, $(1')$, Lemma 2, and Lemma 1 with its Corollary

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give $(R[N,[N,N]])R \subseteq (R,[N,[N,N]],R) + R([N,[N,N]]R) \subseteq (R,R,[N,[N,N]]) + R(\overline{N}R) \subseteq N + R\overline{N} \subseteq N$. But since $R[N,[N,N]] \subseteq R\overline{N} \subseteq N$ by Lemmas 2 and 1, this shows $R[N,[N,N]] \subseteq \overline{N}$ by Lemma 1. ∎

LEMMA 11. *Let R be a semiprime right alternative ring with characteristic $\neq 2$. If $[R,R] \subseteq N$, then $[N,[N,N]] \subseteq I$.*

PROOF: By Lemma 2 $[N,[N,N]] \subseteq [N,N] \subseteq \overline{N}$, and by Lemma 10 $R[N,[N,N]] \subseteq \overline{N}$. Thus by Lemma 8 $[N,[N,N]] \subseteq I$. ∎

LEMMA 12. *Let R be a prime right alternative ring with characteristic $\neq 2$. If $[R,R] \subseteq N$ and $I = 0$, then $[N,N] = 0$.*

PROOF: By Lemma 11 we have

$$[N,[N,N]] \subseteq I = 0. \tag{21}$$

Then (21), (5), and Lemma 10 show $R(A,[R,R],[N,N]) \subseteq (R,[R,R],[N,N])A + (RA,[R,R],[N,N]) + (R,A,[[R,R],[N,N]]) \subseteq (R,N,[N,N])A + (A,[R,R],[N,N]) + (R,A,[N,[N,N]]) \subseteq \overline{N}A + (A,[R,R],[N,N]) \subseteq (A,[R,R],[N,N])$. Thus $(A,[R,R],[N,N])$ is a left ideal. But $(A,[R,R],[N,N]) \subseteq (A,N,[N,N]) \subseteq \overline{N}$ by Lemma 10. Therefore we have this left ideal $(A,[R,R],[N,N]) \subseteq A \cap \overline{N}$, and so $(A,[R,R],[N,N]) = 0$ by Lemma 7. This, (5), and (21) then give $A(R,[R,R],[N,N]) \subseteq (AR,[R,R],[N,N]) + (A,R,[[R,R],[N,N]]) + (A,[R,R],[N,N])R \subseteq (A,[R,R],[N,N]) + (A,R,[N,[N,N]]) = 0$. Thus $(R,[R,R],[N,N]) \subseteq I = 0$, since $(R,[R,R],[N,N]) \subseteq (R,N,[N,N]) \subseteq \overline{N}$ by Lemma 10. But then by Lemmas 2 and 9 we have $[N,N] \subseteq I = 0$, which proves the lemma. ∎

COROLLARY. *Let R be a prime right alternative algebra over a commutative-associative ring with $1/6$. If $[R,R] \subseteq N$ and $I = 0$, then R is associative.*

PROOF: This follows directly from Lemma 12 and the Corollary to Lemma 6. ∎

THEOREM 3. *Let R be an s-prime right alternative algebra over a commutative-associative ring with $1/6$. If $[R,R] \subseteq N$, then R is associative.*

PROOF: Since R is s-prime, either $A = 0$ or $I = 0$. But if $I = 0$, then by the Corollary to Lemma 12 R is associative. Thus in either case we must have R associative. ∎

THEOREM 4. *Let R be a prime locally $(-1,1)$ algebra over a commutative-associative ring with $1/6$. If $[R,R] \subseteq N$, then R is associative.*

PROOF: Since R is locally $(-1,1)$ with characteristic $\neq 2$, R satisfies identity (9); and from the linearised form of this identity we obtain $[R,(x,x,I)] = -[I,(x,x,R)] = 0$. Now $(x,x,I) = -(x,I,x) \subseteq N$ by Lemma 8 and Lemma 1 with its Corollary. Thus, using (C) and (1′), analogous to earlier calculations it follows that $(R, R, (x, x, I)) = 0$; so (x, x, I) is contained in the centre of R. Now by (8), for $k \in I$ we have $(x, x, k)^4 = 0$. Thus the element $(x, x, k)^2$, which is contained in the centre of R, generates a trivial ideal. Since R is prime, this means $(x, x, k)^2 = 0$; and so the central element (x, x, k) likewise generates a trivial ideal. Thus we arrive at

$$(22) \qquad (x, x, I) = 0.$$

Then using (1′), linearised (22), and Lemma 8, we see $(A, I, R) = -(A, R, I) = (R, A, I) = 0$. Thus $A(IR) = (AI)R = 0$, and $A(RI) = (AR)I \subseteq AI = 0$. Since $IR \subseteq \overline{N}$ by the Corollary to Lemma 1, and $RI \subseteq \overline{N}$ by Lemma 8, this shows I is an ideal of R. But $AI = 0$, so R prime implies R is associative or $I = 0$. But if $I = 0$, then by the Corollary to Lemma 12 we also have R associative, which completes the proof of the theorem □

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