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A HELLY TYPE THEOREM FOR CONVEX SETS

BY

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Abstract. A ray in Euclidean n-dimensional space R^n is a set of the form $\{a+\lambda b: \lambda \geq 0\}$ where a and b are fixed points in R^n and $b \neq 0$.

The subject of this paper is a Helly type theorem for convex sets in R^n.

If $\mathcal{A}$ is a finite family of at least $2n$ convex sets in R^n and if the intersection of any $2n$ members of $\mathcal{A}$ contains a ray then $\cap\mathcal{A}$ contains a ray.

1. **Introduction.** The subject of this paper is a Helly type theorem for convex sets in Euclidean n-dimensional space R^n.

For related results consult [1] and for standard notation and terminology see [4].

For a set S in R^n, conv S will denote the convex hull of S, aff S the affine hull of S and dim S the dimension of aff S. A ray with apex a in R^n is a set of the form $\{a+\lambda b: \lambda \geq 0\}$ where a and b are fixed points in R^n and $b \neq 0$.

The following two theorems are known.

Theorem A. *If $\mathcal{A}$ is a family of at least $2n$ convex sets in R^n and if the intersection of each $2n$ members of $\mathcal{A}$ is at least 1-dimensional then the intersection $\cap\mathcal{A}$ is at least 1-dimensional.*

Theorem B. *If $\mathcal{A}$ is a family of at least n convex sets in R^n and if the intersection of each n members of $\mathcal{A}$ contains a line then $\cap\mathcal{A}$ contains a line.*

For a short proof of Theorem A consult [3] or [5] (the values of $h(k, n)$ in [3] are wrong for $1<k<n$, see [5] for the correct values). For a proof of Theorem B consult [2].

The gap between Theorem A and Theorem B is filled by

Theorem C. *If $\mathcal{A}$ is a finite family of at least $2n$ convex sets in R^n and if the intersection of each $2n$ members of $\mathcal{A}$ contains a ray then $\cap\mathcal{A}$ contains a ray.*

By constructing a suitable family of half spaces with the origin on their boundary it is possible to show that $2n$ in Theorem C cannot be replaced by a smaller number.

2. **Proof of Theorem C.** The proof is by induction on n. For $n=1$ the theorem is obvious so assume that $n>1$. By a standard argument it is sufficient to prove the theorem for $|\mathcal{A}|=2n+1$.

Let $\mathcal{A}=\{A_1, \ldots, A_{2n+1}\}$, let $s=\dim \cap\mathcal{A}$ and let $R^s=\text{aff } \cap\mathcal{A}$.

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By Theorem A dim $\cap \mathcal{A} = s \geq 1$. Assume, without loss of generality that $0 \in$ relint $\cap \mathcal{A}$. This implies that R^s is an s-dimensional subspace of R^n.

There are two cases to consider.

CASE 1. $1 \leq s < n$.

Let I be the set of indexes i such that

$$R^s \cap \cap (\mathcal{A} \backslash \{A_i\}) \text{ contains a ray.}$$

If $|I| \geq 2s+1$ then the family of convex sets

$$\mathcal{B} = \left\{ R^s \cap A_i \cap \bigcap_{j \notin I} A_j : i \in I \right\}$$

satisfies the assumptions of Theorem C with s replacing n and R^s replacing R^n. By the induction hypothesis $\cap \mathcal{B}$ contains a ray and since $\cap \mathcal{A} = \cap \mathcal{B}$ the intersection $\cap \mathcal{A}$ contains a ray.

Suppose that $|I| \leq 2s$. It will be shown that this assumption leads to a contradiction.

Let $H = H^{n-s}$ be a subspace of R^n which is complementary to R^s. Let

$$\mathcal{C} = \left\{ H \cap A_j \cap \bigcap_{i \in I} A_i : j \notin I \right\}$$

be a family of convex sets in H. The family $\mathcal{C}$ is of cardinality $|\mathcal{A}| = 2n+1-|I| \geq 2n+1-2s = 2(n-s)+1 = 2 \dim H + 1$.

Since $R^s = \operatorname{aff} \cap \mathcal{A}$ and $0 \in \operatorname{relint} \cap \mathcal{A}$ and from the definition of I it follows that the intersection of any $2(n-s)$ members of $\mathcal{A}$ with H contains a point different from 0, and is therefore of dimension at least 1.

By Theorem A with $\mathcal{C}$ replacing $\mathcal{A}$ and H replacing R^n, $\cap \mathcal{C}$ contains a point different from 0. Since $\cap \mathcal{C} \subset \cap \mathcal{A}$, dim $\cap \mathcal{A} \geq s+1$, a contradiction.

CASE 2. $s = n$.

The following observation will be used.

If a convex set S contains a ray C with apex p and if $q \in$ relint S then S contains the ray $C-p+q$ with apex q.

Since $0 \in \operatorname{relint} \cap \mathcal{A} = \operatorname{int} \cap \mathcal{A}$, for each $1 \geq i \leq n$ there is a ray C_i with apex 0 such that $C_i \subset \cap (\mathcal{A} \backslash \{A_i\})$.

Define for each $1 \leq i \leq 2n+1$

$$D_i = \operatorname{conv} \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{2n+1} C_j \right).$$

Then for each $1 \leq i \leq 2n+1$, D_i is a convex cone with apex 0, $D_i \subset A_i$, and $\bigcap_{j \neq 1} D_j \supset C_i$ so that dim $\bigcup_{j \neq i} D_j \geq 1$. By Theorem A applied to the family $\{D_i : 1 \leq i \leq 2n+1\}$ the convex cone $\bigcap_{i=1}^{2n+1} D_i$ is of dimension at least 1 and therefore contains a ray. It follows that $\cap \mathcal{A}$ contains a ray. This completes the proof of Theorem C.

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