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More congruences for numerical data of an embedded resolution

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Abstract. To an arbitrary intersection of exceptional varieties of an embedded resolution we associate a finite number of congruences between naturally occurring multiplicities. This theory generalizes previous results concerning just one exceptional variety. Moreover we describe precise equalities which imply the congruences and we give some applications on the poles of Igusa's local zeta function.

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Introduction

(0.1). Let k be an algebraically closed field of characteristic zero and $f \in k[x_1, \ldots, x_n]$. Let $h: X \to \mathbb{A}^n$ be an embedded resolution of singularities of $f^{-1}\{0\}$, considered as an algebraic set in affine space $\mathbb{A}^n$. We suppose that this resolution (X, h) is constructed by means of consecutive blowing-ups, according to Hironaka's Theorem [H].

We denote by $E_i, i \in S$, the irreducible components of $h^{-1}(f^{-1}\{0\})$ and by N_i the multiplicity of E_i in the divisor of $f \circ h$.

(0.2). Fix one exceptional variety E_j. When $n = 2$ the following congruence is now well known. Say E_j intersects k times other components $E_1, \ldots, E_k$. Then

$$\sum_{i=1}^{k} N_i \equiv 0 \mod N_j \qquad (*)$$

and more precisely $\Sigma_{i=1}^{k} N_i = N_j(1 + \rho)$, where ρ is the number of times that a point of E_j occurs as centre of some blowing–up during the resolution process. The original proof for analytically irreducible $f(x_1, x_2)$ was derived by consecutive work of Strauss [S], Meuser [M] and Igusa [I2], and for general $f(x_1, x_2)$ by Loeser [L].

(0.3). When n is arbitrary we developed in [V2] a general theory of congruences,

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extending $(*)$. The essential feature of dimension $n \geqslant 3$ is that E_j is subject to a 'historical evolution' during the resolution process. Let $E_j \subset X$ be the strict transform in X of the variety E_j^0, created at some stage of the resolution process h as exceptional variety of a blowing-up. Then in general E_j is *not* isomorphic to E_j^0; more precisely E_j itself is obtained from E_j^0 by a sequence of blowing-ups. (When $n = 2$ this phenomenon does not occur for then $E_j^0 \cong E_j \cong \mathbb{P}^1$.)

In fact we associated a finite number of congruences mod N_j to E_j; there are *Basic Congruences* associated to its creation as E_j^0 in the resolution process, generalizing $(*)$, and an *Additional Congruence* associated to each blowing-up of the sequence that produces E_j out of E_j^0.

(0.4). In this paper we will generalize this theory further to congruences 'in arbitrary codimension'. We first give an example.

When $n = 3$ let E_{j_1} and E_{j_2} be two intersecting exceptional surfaces and suppose that the curve $D := E_{j_1} \cap E_{j_2}$ is irreducible and projective. Say D intersects k times other components $E_1, \ldots, E_k$. Then

$$\sum_{i=1}^{k} N_i \equiv 0 \mod \gcd(N_{j_1}, N_{j_2}),$$

where gcd denotes the greatest common divisor. This 'codimension 2'-congruence cannot be derived as a consequence of the ordinary 'codimension 1'–congruences of [V2]. In fact there is an explicit equality $\Sigma_{i=1}^{k} N_i + \kappa_2 N_{j_1} + \kappa_1 N_{j_2} = 0$, where κ_ℓ is the self-intersection number of D on E_{j_ℓ}.

(0.5). We will associate to each irreducible component D of a nonempty intersection of exceptional varieties $\cap_{j \in J} E_j$ a finite number of congruences mod $\gcd_{j \in J} N_j$, and moreover we will describe equalities from which they can be obtained. We want to remark here that the congruences can be proved directly in an elegant way without reference to the equalities. (For $|J| = 1$ this was not mentioned explicitly in [V2].)

We now state these congruences more precisely. In general the variety D goes through a historical evolution during the resolution process: it is obtained by a finite succession of blowing-ups

$$D^0 \xleftarrow{\pi_1} D^1 \xleftarrow{\pi_2} \cdots D^{i-1} \xleftarrow{\pi_i} D^i \cdots \xleftarrow{\pi_{m-1}} D^{m-1} \xleftarrow{\pi_m} D^m = D,$$

with irreducible nonsingular centre $Z_{i-1} \subset D^{i-1}$ and exceptional variety $C_i \subset D^i$ for $i = 1, \ldots, m$. The variety D^0 is created at some step of the global resolution process (in fact at the creation of the 'last' of the $E_j, j \in J$).

There are two kinds of intersections of D with components $E_\ell, \ell \notin J$. We have the strict transforms in D of the exceptional varieties $C_1, \ldots, C_m$; and also the strict transforms in D of certain varieties $C_k, k \in T^0$, (of codimension one) in D^0. We have moreover that the strict transform of each $C_i, i \in T^0 \cup \{1, \ldots, m\}$, is (an irreducible component of) the intersection of D with exactly one component of $h^{-1}(f^{-1}\{0\})$; slightly abusing notation let this component have multiplicity N_i in the divisor of $f \circ h$.

THEOREM. *Set* $N_J := \gcd_{j\in J} N_j$. *Using the notation above we have for* $i = 0, \ldots, m-1$ *that*

(Congruence A)

$$N_{i+1} \equiv \sum_{k\in T^0\cup\{1,\ldots,i\}} \mu_k N_k \mod N_J,$$

where μ_k *is the multiplicity of the generic point of* Z_i *on* (*the strict transform in* D^i *of*) C_k. *We have also*

(Congruence B)

$$\sum_{k\in T^0} N_k C_k = 0 \quad \textit{in} \quad \frac{\mathrm{Pic}\, D^0}{N_J\, \mathrm{Pic}\, D^0}.$$

Whenever D^0 is complete, Congruence B induces a finite number of ordinary congruences mod N_J.

(0.6). For $i \in S$ let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $h^*(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n)$ on X. Classically the $(N_i, \nu_i), i \in S$, are called the *numerical data* of the resolution (X, h). The numbers $-(\nu_i/N_i), i \in S$, form a complete list of candidate poles for Igusa's local zeta function of f (when f is defined over a p-adic field). We will mention a straightforward generalization to arbitrary codimension of our 'codimension one'-theory of relations between numerical data [V1], which enables us to give some applications of the congruences of this paper concerning the poles of Igusa's local zeta function.

(0.7). The plan of the exposition is as follows. In Section 1 we recall briefly the important aspects of an embedded resolution and in Section 2 we prove the Congruences A and B. Their underlying equalities are studied separately in Section 3; this part is a bit technical and is not needed for the applications concerning Igusa's local zeta function. After developing the more general relations between numerical data in Section 4, those applications are treated in Section 5.

1. Embedded resolution

(1.1). Let k be an algebraically closed field of characteristic zero and let $f \in k[x_1, \ldots, x_n]$. Let $Y = f^{-1}\{0\}$ denote the zero set of f in affine space $\mathbb{A}^n$. We exclude the trivial case $f \in k$, so Y is a hypersurface in $\mathbb{A}^n$.

DEFINITION. An embedded resolution (X, h) for $Y \subset \mathbb{A}^n$ consists of a nonsingular variety X and a proper birational morphism $h\colon X \to \mathbb{A}^n$ such that the restriction

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$h|_{X\setminus h^{-1}Y}$ is an isomorphism and $h^{-1}Y$ has normal crossings in X. In particular the irreducible components of $h^{-1}Y$ are nonsingular hypersurfaces.

Remember that a reduced hypersurface E of X has *normal crossings* if for all $x \in X$ there exists a regular system of parameters $t_1, \dots, t_n$ in the local ring $\mathcal{O}_{X,x}$ of X at x such that the ideal in $\mathcal{O}_{X,x}$ of each irreducible component of E containing x is generated by one of the t_i. (Analytically one can think of E being locally a union of coordinate hyperplanes.) Also E is said to have *normal crossings with* a (necessarily smooth) subvariety D of X if for all $x \in D$ the ideal of D in $\mathcal{O}_{X,x}$ is generated by some of the t_i.

(1.2). Hironaka [H] constructed an embedded resolution as a finite composition of *blowing-ups*. Recall that if $g: \tilde{Z} \to Z$ is the *blowing-up* of the variety Z with centre the closed subset B of Z, then the *exceptional divisor* $E = g^{-1}B$ is everywhere of codimension one on $\tilde{Z}$, and the restriction $g|_{\tilde{Z}\setminus E}$ is an isomorphism. For any subvariety V of Z with $V \not\subset B$ the closure of $g^{-1}(V\setminus B)$ in $\tilde{Z}$ is called the *strict transform* of V by g. If Z and B are nonsingular varieties, then the same is true for $\tilde{Z}$ and E.

More precisely Hironaka constructed a resolution (X, h) as a suitable composition of blowing-ups

$$\mathbb{A}^n = X_0 \xleftarrow{g_1} X_1 \xleftarrow{g_2} \cdots X_i \xleftarrow{g_{i+1}} X_{i+1} \cdots \xleftarrow{g_{r-1}} X_{r-1} \xleftarrow{g_r} X_r = X,$$

with irreducible nonsingular centre $B_i \subset X_i, 0 \leqslant i < r$, such that $\operatorname{codim}(B_i, X_i) \geqslant 2$. Moreover each B_i is contained in the (repeated) strict transform of Y in X_i, and the reduced hypersurface, consisting of the (repeated) strict transforms in X_i of the exceptional varieties of $g_1, \dots, g_i$, has normal crossings with B_i.

Finally $h^{-1}Y = (g_r \circ \cdots \circ g_1)^{-1}(Y)$ has thus normal crossings in X; its irreducible components are the strict transforms of the irreducible components of Y and the *exceptional varieties of* h, being the strict transforms in X of the exceptional varieties of $g_1, \dots, g_r$.

(1.3). Let now $D \subset X_i$ be any variety which intersects B_i transversely everywhere (and is not contained in B_i) and $\tilde{D} \subset X_{i+1}$ its strict transform by g_{i+1}. We have the following important fact (see e.g. [GH, page 605] for the first claim; the second is not difficult to verify).

PROPOSITION, The restriction $g_{i+1}|_{\tilde{D}}: \tilde{D} \to D$ is the blowing-up of D with (nonsingular) centre $B_i \cap D$. Moreover the exceptional divisor of $g_{i+1}|_{\tilde{D}}$ is the intersection of $\tilde{D}$ with the exceptional divisor of g_{i+1}.

Note that $B_i \cap D$ can eventually be reducible. The total blow-up of D with centre $B_i \cap D$ can then be considered as the result of consecutive blowing-ups of D with centres the irreducible components of $B_i \cap D$, which are necessarily disjoint.

We will use this proposition intensively for D a nonempty intersection of exceptional varieties of h; because of the normal crossings property the transversality condition is indeed satisfied.

(1.4). From now on we will denote the irreducible components of $h^{-1}Y$ by $E_i, i \in S$, and their multiplicity in the divisor of $f \circ h$ by N_i; alternatively $(f \circ h) = \Sigma_{i \in S} N_i E_i$. We also set $E_{\mathrm{I}} := \cap_{i \in \mathrm{I}} E_i$ for $\mathrm{I} \subset S$. While working with the resolution process h we will in general use the same notation for E_i, when created as exceptional variety, and for its strict transforms in any X_k.

2. Self–intersection divisors and congruences

(2.1). From now on we fix intersecting exceptional varieties $E_j, j \in J$, and an irreducible component D of E_J. Remark that because of the normal crossings property D is nonsingular and disjoint from eventual other components of E_J, and that $\mathrm{codim}(D, X) = |J|$.

Let $C_i, i \in T$, denote all the irreducible components of the intersections $D \cap E_\ell, \ell \notin J$. Set also $N_J := \gcd_{j \in J} N_j$. Our starting point is the following observation.

PROPOSITION 2.2. *For* $j \in J$ *we denote by* $\mathfrak{D}_{\langle j \rangle}$ *the self-intersection divisor of* D *on* $E_{J \setminus \{j\}}$, *considered as an element of* Pic D. *Then*

$$\sum_{j \in J} N_j \mathfrak{D}_{\langle j \rangle} + \sum_{i \in T} N_i C_i = 0 \quad \textit{in } \mathrm{Pic}\, D.$$

Proof. Denote by $\delta\colon D \hookrightarrow X$ the natural embedding and consider for each $j \in J$ the decomposition $D \overset{\alpha_j}{\hookrightarrow} E_{J \setminus \{j\}} \overset{\beta_j}{\hookrightarrow} X$ of δ. Since $\Sigma_{i \in S} N_i E_i = 0$ in Pic X we have that

$$\sum_{i \in S} N_i \delta^* E_i = 0 \quad \text{in } \mathrm{Pic}\, D. \tag{1}$$

Now $\delta^*(\Sigma_{i \notin J} N_i E_i) = \Sigma_{i \in T} N_i C_i$ and for each $j \in J$ we have that $\delta^* E_j = \alpha_j^*(\beta_j^* E_j) = \alpha_j^*(E_J) = \alpha_j^*(D) = \mathfrak{D}_{\langle j \rangle}$. Substituting all this in (1) yields the stated expression. □

COROLLARY 2.3. $\Sigma_{i \in T} N_i C_i = 0$ *in* Pic D / N_J Pic D.

COROLLARY 2.4. *Let* D *be a projective curve. Then* (*taking degrees*)

$$\sum_{i \in T} N_i \equiv 0 \mod N_J,$$

and more precisely, if κ_j *denotes the self-intersection number of* D *on* $E_{J \setminus \{j\}}$, *then* $\Sigma_{j \in J} \kappa_j N_j + \Sigma_{i \in T} N_i = 0$.

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Remark 2.5. Proposition 2.2 and its corollaries are in fact valid in a more general context.

(i) They are true for *any* embedded resolution of Y, i.e. not necessarily obtained à la Hironaka.

(ii) The $E_j, j \in J$, can be arbitrary (intersecting) components of $h^{-1}Y$. Now for example when f is irreducible and the strict transform of Y is one of the $E_j, j \in J$, then $N_J = 1$ and the congruences are meaningless. See (2.8) for an application of the equalities.

(2.6). We now fix the notation for our general congruences and equalities. Using Proposition 1.3 the following is not difficult to verify.

(i) The variety D is the strict transform in X of a nonsingular variety D^0, created at some step of the global resolution process. (In fact D^0 appears in this process at the creation of the 'last' of the $E_j, j \in J$, as exceptional variety of a blowing-up of h; and more precisely D^0 is a component of the intersection of this variety with the other $E_j, j \in J$, at that stage of h.)

(ii) So D itself is obtained from D^0 by a finite succession of blowing-ups

$$D^0 \xleftarrow{\pi_1} D^1 \xleftarrow{\pi_2} \cdots D^{i-1} \xleftarrow{\pi_i} D^i \cdots \xleftarrow{\pi_{m-1}} D^{m-1} \xleftarrow{\pi_m} D^m = D,$$

with irreducible nonsingular centre $Z_{i-1} \subset D^{i-1}$ and exceptional variety $C_i \subset D^i$ for $i = 1, \ldots, m$. In fact C_i is (a component of) the intersection of D^i with some global exceptional variety E_ℓ at the stage where E_ℓ is created.

(iii) For $i = 1, \ldots, m$ and for any variety $V \subset D^j, 0 \leqslant j < i$, let the strict transform of V in D^i (by $\pi_i \circ \cdots \circ \pi_{j+1}$) be denoted by $V^{(i)}$.

Let $C_i, i \in T$, be the intersections of D with components $E_\ell, \ell \notin J$. They consist of the strict transforms $C_1^{(m)}, \ldots, C_m^{(m)}$ in D of the exceptional varieties $C_1, \ldots, C_m$ and of the strict transforms $C_i^{(m)}$ in D of varieties $C_i, i \in T^0$, (of codimension one) in D^0. Those last varieties are the intersections of D^0 with components $E_\ell, \ell \notin J$, at the stage of h where D^0 is created. (So $T = T^0 \cup \{1, \ldots, m\}$.)

(iv) Since $\cup_{\ell \in S} E_\ell$ has normal crossings in X we have for each $i \in T$ that $C_i^{(m)}$ is (a component of) the intersection of D with exactly one component of $h^{-1}Y$, different from the $E_j, j \in J$. For simplicity of notation let this component be E_i.

THEOREM 2.7. *For* $i = 0, \ldots, m-1$ *we have that*

(Congruence A)

$$N_{i+1} \equiv \sum_{k \in T^0 \cup \{1,\ldots,i\}} \mu_k N_k \mod N_J,$$

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where μ_k is the multiplicity of the generic point of Z_i on $C_k^{(i)}$. We have also that

(Congruence B)

$$\sum_{k\in T^0} N_k C_k = 0 \quad \textit{in } \frac{\text{Pic } D^0}{N_J \text{ Pic } D^0}.$$

Remark. (i) Of course these congruences are only meaningful when $N_J > 1$.

(ii) In general Congruence B thus has divisors as 'coefficients'. When Pic $D^0 \cong \mathbb{Z}$, for example if D^0 is some projective space, Congruence B becomes an ordinary congruence.

(iii) More generally, whenever D^0 is complete, Congruence B induces a finite number of numerical congruences. For then we can consider it in $\text{Num } D^0/N_J$ $\text{Num } D^0$, where $\text{Num } D^0$ is the group of divisors on D^0 modulo numerical equivalence, which is a quotient of Pic D^0. Since $\text{Num } D^0$ is a finitely generated free Abelian group we get rank $(\text{Num } D^0)$ congruences.

Proof. Consider for a fixed $i \in \{0, \dots, m-1\}$ the blowing-up $\pi_{i+1}: D^{i+1} \to D^i$. It is not difficult to verify that the classical isomorphism $\text{Pic } D^{i+1} \cong \pi_{i+1}^*$ $\text{Pic } D^i \oplus \mathbb{Z}C_{i+1}$ (with injective π_{i+1}^*) induces

$$\frac{\text{Pic } D^{i+1}}{N_J \text{ Pic } D^{i+1}} \cong \pi_{i+1}^* \frac{\text{Pic } D^i}{N_J \text{ Pic } D^i} \oplus \frac{\mathbb{Z}}{N_J\mathbb{Z}} C_{i+1}, \tag{2}$$

where π_{i+1}^* is still injective.

Suppose now that $\Sigma_{\ell\in T^0\cup\{1,\dots,i+1\}} N_\ell C_\ell^{(i+1)} = 0$ in $\text{Pic } D^{i+1}/N_J \text{ Pic } D^{i+1}$. This is equivalent to $\Sigma_{k\in T^0\cup\{1,\dots,i\}} N_k(\pi_{i+1}^* C_k^{(i)} - \mu_k C_{i+1}) + N_{i+1}C_{i+1} = 0$, and using (2) we obtain

$$\sum_{k\in T^0\cup\{1,\dots,i\}} N_k C_k^{(i)} = 0 \quad \text{in } \frac{\text{Pic } D^i}{N_J \text{ Pic } D^i}$$

and

$$N_{i+1} = \sum_{k\in T^0\cup\{1,\dots,i\}} \mu_k N_k \quad \text{in } \frac{\mathbb{Z}}{N_J\mathbb{Z}}.$$

Starting from the fact that $\Sigma_{i\in T} N_i C_i^{(m)} = 0$ in $\text{Pic } D/N_J \text{ Pic } D$ (Corollary 2.3) we use the arguments above consecutively for $i = m-1, \dots, 0$ to obtain the Congruences A and finally end up with Congruence B. □

For concrete varieties D^0 one can make Congruence B more explicit. When D^0 is a projective space bundle over some base variety this has been done in [V2]. (When $|J| = 1$ then D^0 is always such a bundle.)

(2.8). Take $n = 3$ and suppose that (X, h) is the embedded resolution of an isolated singularity P of the irreducible surface Y. The equalities of Corollary 2.4 are useful to determine the self-intersection numbers in the resolution graph of $P \in Y$, when an explicit embedded resolution of $Y \subset \mathbb{A}^3$ is given.

Indeed let D be (a component of) the intersection of the strict transform of Y with some exceptional surface E_j, and let D intersect k times other exceptional surfaces, say $E_1, \ldots, E_k$. We want to know the self-intersection number κ of D on the strict transform of Y (which is a resolution of $P \in Y$). Now Corollary 2.4 says that

$$\kappa_j + N_j \kappa + \sum_{i=1}^{k} N_i = 0,$$

where κ_j is the self-intersection number of D on E_j, which can very easily be computed.

3. Precise equalities

(3.1). We keep using the notation of (2.6). The congruences in the preceding section were completely determined by the configuration of the $C_i, i \in T$, on D; or equivalently of the $C_i, i \in T^0$, on D^0. For example we did not need any information concerning how D is embedded in X or in the $E_{J'}, J' \subset J$, or analogously for D^0. In my opinion precisely this feature makes these congruences attracting and useful for applications. See Section 5.

The underlying equalities for these congruences however depend rather intensively on knowledge about the global resolution process. We will derive them from the key Lemma 3.3, for which we now introduce the data.

(3.2). We fix some blowing-up g with centre B of the global resolution process h, occurring *after* the creation of D^0 (and thus of all the $E_j, j \in J$). We denote the strict transform of D^0 right before and after g respectively by $D^\dagger$ and $D^\ddagger$ and the restriction of g to $D^\ddagger$ by π. So $\pi\colon D^\ddagger \to D^\dagger$ itself is a blowing-up with (eventually reducible) centre $B \cap D^\dagger$. (We may suppose that $B \cap D^\dagger \neq \emptyset$, otherwise nothing relevant happens.) Analogously we denote for each $J' \subset J$ the 'ancesters' of $E_{J'} \subset X$ before and after g respectively by $E^\dagger_{J'} \subset X^\dagger$ and $E^\ddagger_{J'} \subset X^\ddagger$. For each $j \in J$ we thus have the following diagram

$$\begin{tikzcd}
D^\ddagger \arrow[r] \arrow[d, "\pi"'] & E^\ddagger_{J\setminus\{j\}} \arrow[r] \arrow[d] & X^\ddagger \arrow[d, "g"] \\
D^\dagger \arrow[r] & E^\dagger_{J\setminus\{j\}} \arrow[r] & X^\dagger.
\end{tikzcd} \tag{3}$$

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Let also $\mathfrak{D}^{\dagger}_{\langle j\rangle}$ denote the self-intersection divisor of $D^{\dagger}$ on $E^{\dagger}_{J\setminus\{j\}}$, considered as an element of Pic $D^{\dagger}$, and $\mathfrak{D}^{\ddagger}_{\langle j\rangle}$ the analogous element of Pic $D^{\ddagger}$.

LEMMA 3.3. *We use the notation of* (3.2). *Let* $C^{\ddagger}_e, e \in \mathcal{E}$, *be the* (*necessarily disjoint*) *irreducible components of the exceptional divisor of* π, *and* $C^{\ddagger}_i, i \in \mathrm{I}$, *all other irreducible components of the intersection of* $D^{\ddagger}$ *with components* $E_\ell, \ell \notin J$, *in* $X^{\ddagger}$.

First case: $\operatorname{codim}(B \cap D^{\dagger}, D^{\dagger}) \geqslant 2$.

So for each $e \in \mathcal{E}$ *we have that* $Z_e := \pi(C^{\ddagger}_e)$ *is of codimension at least* 2 *in* $D^{\dagger}$ *and the irreducible components of the intersections of* $D^{\dagger}$ *with the* $E_\ell, \ell \notin J$, *in* $X^{\dagger}$ *are precisely the* $C^{\dagger}_i := \pi(C^{\ddagger}_i), i \in \mathrm{I}$.

If $\Sigma_{j\in J} a_j \mathfrak{D}^{\ddagger}_{\langle j\rangle} + \Sigma_{\ell\in \mathrm{I}\cup\mathcal{E}} a_\ell C^{\ddagger}_\ell = 0$ *in* Pic $D^{\ddagger}$, *then*

(i) $$\sum_{j\in J} a_j \mathfrak{D}^{\dagger}_j + \sum_{i\in \mathrm{I}} a_i C^{\dagger}_i = 0 \quad \textit{in } \operatorname{Pic} D^{\dagger}, \quad \textit{and}$$

(ii) $$a_e = \sum_{i\in\mathrm{I}} \mu^{(e)}_i a_i + \sum_{j\in J} \delta_j a_j \quad \textit{for all } e \in \mathcal{E},$$

where $\mu^{(e)}_i$ is the multiplicity of the generic point of Z_e on $C^{\dagger}_i$, and $\delta_j = 1$ if $B \subset E_j$ and $\delta_j = 0$ if $B \not\subset E_j$.

Second case: $\operatorname{codim}(B \cap D^{\dagger}, D^{\dagger}) = 1$.

So $\pi\colon D^{\ddagger} \to D^{\dagger}$ *is an isomorphism and the irreducible components of the intersections of* $D^{\dagger}$ *with the* $E_\ell, \ell \notin J$, *in* $X^{\dagger}$ *are precisely the* $C^{\dagger}_\ell := \pi(C^{\dagger}_\ell), \ell \in \mathrm{I} \cup \mathcal{E}$.

If $\Sigma_{j\in J} a_j \mathfrak{D}^{\ddagger}_{\langle j\rangle} + \Sigma_{\ell\in \mathrm{I}\cup\mathcal{E}} a_\ell C^{\ddagger}_\ell = 0$ *in* Pic $D^{\ddagger}$, *then*

(iii) $$\sum_{j\in J} a_j \mathfrak{D}^{\dagger}_{\langle j\rangle} + \sum_{i\in\mathrm{I}} a_i C^{\dagger}_i + \sum_{e\in\mathcal{E}} \left(a_e - \sum_{j\in J} \delta_j a_j \right) C^{\dagger}_e = 0 \quad \textit{in } \operatorname{Pic} D^{\dagger}.$$

Proof. We first show for each $j \in J$ that in Pic $D^{\ddagger}$

$$\mathfrak{D}^{\ddagger}_{\langle j\rangle} = \pi^* \mathfrak{D}^{\dagger}_{\langle j\rangle} - \delta_j \sum_{e\in\mathcal{E}} C^{\ddagger}_e. \tag{4}$$

Consider the natural embeddings $\alpha_\dagger\colon D^{\dagger} \mapsto X^{\dagger}$ and $\alpha_\ddagger\colon D^{\ddagger} \mapsto X^{\ddagger}$ in the diagram (3) of (3.2) and let $E^{\ddagger}$ be the exceptional divisor of g in $X^{\ddagger}$. Then as in the proof of Proposition 2.2 we have that $\mathfrak{D}^{\dagger}_{\langle j\rangle} = \alpha^*_\dagger E^{\dagger}_j$ and $\mathfrak{D}^{\ddagger}_{\langle j\rangle} = \alpha^*_\ddagger E^{\ddagger}_j$. So

$$\mathfrak{D}^{\ddagger}_{\langle j\rangle} = \alpha^*_\ddagger (g^* E^{\dagger}_j - \delta_j E^{\ddagger}) = \pi^* \alpha^*_\dagger E^{\dagger}_j - \delta_j \alpha^*_\ddagger E^{\ddagger} = \pi^* \mathfrak{D}^{\dagger}_{\langle j\rangle} - \delta_j \sum_{e\in\mathcal{E}} C^{\ddagger}_e.$$

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First case. Substituting (4) and the identities $\pi^* C_i^\dagger = C_i^\ddagger + \Sigma_{e\in\mathcal{E}} \mu_i^{(e)} C_e^\ddagger, i \in \mathrm{I}$, in the given expression yields

$$\sum_{j\in J} a_j \left(\pi^* \mathfrak{D}^\dagger_{\langle j\rangle} - \delta_j \sum_{e\in\mathcal{E}} C_e^\ddagger\right)$$
$$+ \sum_{i\in\mathrm{I}} a_i \left(\pi^* C_i^\dagger - \sum_{e\in\mathcal{E}} \mu_i^{(e)} C_e^\ddagger\right) + \sum_{e\in\mathcal{E}} a_e C_e^\ddagger = 0 \quad \text{in Pic } D^\ddagger,$$

which is equivalent to

$$\pi^* \left(\sum_{j\in J} a_j \mathfrak{D}^\dagger_{\langle j\rangle} + \sum_{i\in\mathrm{I}} a_i C_i^\dagger\right) + \sum_{e\in\mathcal{E}} \left(a_e - \sum_{i\in\mathrm{I}} \mu_i^{(e)} a_i - \sum_{j\in J} \delta_j a_j\right) C_e^\ddagger = 0.$$

Now since Pic $D^\ddagger \cong \pi^*$ Pic $D^\dagger \oplus (\oplus_{e\in\mathcal{E}} \mathbb{Z} C_e^\ddagger)$, where π^* is injective, we obtain the stated results.

Second case. Now substituting (4) and $\pi^* C_e^\dagger = C_e^\ddagger, \ell \in \mathrm{I} \cup \mathcal{E}$, in the given expression yields

$$\sum_{j\in J} a_j \left(\pi^* \mathfrak{D}^\dagger_{\langle j\rangle} - \delta_j \sum_{e\in\mathcal{E}} \pi^* C_e^\dagger\right) + \sum_{i\in\mathrm{I}} a_i \pi^* C_i^\dagger + \sum_{e\in\mathcal{E}} a_e \pi^* C_e^\dagger = 0 \quad \text{in Pic } D^\ddagger,$$

which clearly implies the stated expression in Pic $D^\dagger$. □

THEOREM 3.4. *We use the notation of* (2.6).

(A) *Fix $i \in \{0, \ldots, m-1\}$. Let B denote the centre of the global blowing-up in the resolution process h by which C_{i+1} is created* (*as irreducible component of the intersection with the global exceptional variety*) *and set $\delta_j = 1$ if $B \subset E_j$ and $\delta_j = 0$ if $B \not\subset E_j$ for $j \in J$. For all $k \in T^0 \cup \{1, \ldots, i+1\}$ and $j \in J$ let $m_k^{\langle j\rangle}$ be the number of centres B_ℓ of global blowing-ups in the subsequent stages of the resolution process that satisfy $C_k^{(\geqslant i+1)} \subset B_\ell \subset E_j$. Then*

$$N_{i+1} = \sum_{k\in T^0\cup\{1,\ldots,i\}} \mu_k N_k + \sum_{j\in J} \left(m_{i+1}^{\langle j\rangle} - \sum_{k\in T^0\cup\{1,\ldots,i\}} \mu_k m_k^{\langle j\rangle} + \delta_j\right) N_j,$$

where μ_k is the multiplicity of the generic point of Z_i on $C_k^{(i)}$.

(B) *For all $k \in T^0$ and $j \in J$ let $m_k^{\langle j\rangle >}$ be the number of centres B_ℓ of global blowing-ups in stages of the resolution process h after the creation of D^0 that satisfy $C_k^{(\geqslant 0)} \subset B_\ell \subset E_j$. Then*

$$\sum_{k\in T^0} N_k C_k = \sum_{j\in J} N_j \left(\sum_{k\in T^0} m_k^{\langle j\rangle} C_k - \mathfrak{D}^0_{\langle j\rangle}\right) \quad \text{in Pic } D^0,$$

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where $\mathfrak{D}^0_{\langle j\rangle} \in \operatorname{Pic} D^0$ *denotes the self-intersection divisor of* D^0 *on the intersection of the* $E_\ell, \ell \in J\backslash\{j\}$, *at the stage where* D^0 *is created.*

Proof. (A) For simplicity of notation we will suppose that C^{i+1} is exactly the intersection of the global exceptional variety associated to B with the strict transform of D^0; so this strict transform may be identified with D^{i+1}. The general case is entirely similar.

For $i = 0, \ldots, m$ we denote by $\mathfrak{D}^{(i)}_{\langle j\rangle} \in \operatorname{Pic} D^i$ the self-intersection divisor of D^i on the intersection of the $E_\ell, \ell \in J\backslash\{j\}$, at the appropriate stage of the global resolution process. Starting from the equality

$$\sum_{k\in T} N_k C_k + \sum_{j\in J} N_j \mathfrak{D}^{(m)}_{\langle j\rangle} = 0 \quad \text{in Pic}\, D,$$

(Proposition 2.2), consecutive applications of Lemma 3.3(i, iii) yield

$$\sum_{k\in T^0\cup\{1,\ldots,i+1\}} \left(N_k - \sum_{j\in J} m_k^{\langle j\rangle} N_j\right) C_k + \sum_{j\in J} N_j \mathfrak{D}^{(i+1)}_{\langle j\rangle} = 0 \quad \text{in Pic}\, D^{i+1}.$$

Then by Lemma 3.3(ii) we have that

$$N_{i+1} - \sum_{j\in J} m_{i+1}^{\langle j\rangle} N_j$$

$$= \sum_{k\in T^0\cup\{1,\ldots,i\}} \mu_k \left(N_k - \sum_{j\in J} m_k^{\langle j\rangle} N_j\right) + \sum_{j\in J} \delta_j N_j,$$

which is equivalent to the stated expression for N_{i+1}.

(B) Further applications of Lemma 3.3(i, iii) finally yield

$$\sum_{k\in T^0} \left(N_k - \sum_{j\in J} m_k^{\langle j\rangle} N_j\right) C_k + \sum_{j\in J} N_j \mathfrak{D}^{(0)}_{\langle j\rangle} = 0 \quad \text{in Pic}\, D^0. \qquad \square$$

Caution. The $m_k^{\langle j\rangle}$ in (A) depend on the chosen $i \in \{0, \ldots, m-1\}$.

(3.5). We can extend all previously obtained results to the following situation. Instead of the polynomial function f on $\mathbb{A}^n$ we can consider in (1.1) any nonsingular variety A and any rational function f on A. Let now Y denote the support of the divisor of f, the map $h: X \to A$ an embedded resolution of $Y \subset A$, and again $E_i, i \in S$, the irreducible components of $h^{-1}Y$ with multiplicity N_i in the divisor of $f \circ h$.

The essential difference is now that the $N_i, i \in S$, can also be negative and eventually even zero. This however does not cause any trouble. Of course when $N_J = 0$ a congruence mod N_J becomes an equality.

4. Relations between numerical data in any codimension

(4.1). We now introduce besides the $N_i, i \in S$, other invariants of the embedded resolution (X, h) for $Y \subset \mathbb{A}^n$. For $i \in S$ let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$ on X; alternatively the canonical divisor on X is $\Sigma_{i \in S}(\nu_i - 1)E_i$. Remark that $\nu_i = 1$ for any irreducible component of the strict transform of Y. Classically the $(N_i, \nu_i), i \in S$, are called the *numerical data* of the resolution (X, h).

The rational numbers $-(\nu_i/N_i), i \in S$, are important because they form an exhaustive list of candidate poles for certain zeta functions associated to f, see Section 5. In particular a nonempty intersection E_{I} for which all $\nu_i/N_i, i \in \mathrm{I}$, are equal induces in general a candidate pole of order $|\mathrm{I}|$.

(4.2). We now fix a nonempty intersection E_J of exceptional varieties such that $s_0 = -(\nu_j/N_j)$ for all $j \in J$, and an irreducible component D of E_J. Let $C_i, i \in T$, still denote all the irreducible components of the intersections $D \cap E_\ell, \ell \notin J$. Remember that each $C_i, i \in T$, is the intersection with D of exactly one component of $h^{-1}Y$; slightly abusing notation we let this component have numerical data (N_i, ν_i) and we denote $\alpha_i := \nu_i + s_0 N_i$.

These numbers α_i occur naturally in the expression for the 'residue' of the candidate pole s_0 for the zeta functions mentioned above; see Section 5. When $|J| = 1$ we developed in [V1] a general theory of linear relations between the $\alpha_i, i \in T$. We now present shortly a straightforward generalization when E_J is of arbitrary codimension.

In the sequel for a nonsingular variety V we denote by K_V its canonical divisor.

PROPOSITION 4.3. $K_D = \Sigma_{i \in T}(\alpha_i - 1)C_i$ *in* Pic $D \otimes \mathbb{Q}$.

Proof. By definition of the numerical data we have that $K_X = \Sigma_{\ell \in S}(\nu_\ell - 1)E_\ell$ and $\Sigma_{\ell \in S} N_\ell E_\ell = 0$ in Pic X, and consequently

$$K_X = \sum_{\ell \in S}(\nu_\ell - 1)E_\ell + s_0 \sum_{\ell \in S} N_\ell E_\ell = -\sum_{j \in J} E_j + \sum_{\ell \notin J}(\nu_\ell + s_0 N_\ell - 1)E_\ell$$

in Pic $X \otimes \mathbb{Q}$. This implies the stated expression after applying $|J|$ times the adjunction formula, or at once by [F, Example 3.2.12]. $\square$

Example. When D is a projective curve of genus g we obtain the relation $2g - 2 = \Sigma_{i \in T}(\alpha_i - 1)$.

Remark. Proposition 4.3 is valid for *any* embedded resolution, not necessarily à la Hironaka.

(4.4). When dim $D \geqslant 2$ we obtain a finite number of relations by analyzing as before the historical evolution of D.

THEOREM. *We use the notation of* (2.6). *For* $i = 0, \ldots, m-1$ *we have that*

(Relation A)

$$\alpha_{i+1} = \sum_{k \in T^0 \cup \{1,\ldots,i\}} \mu_k(\alpha_k - 1) + r_i,$$

where μ_k *is the multiplicity of the generic point of* Z_i *on* $C_k^{(i)}$, *and* $r_i = \operatorname{codim}(Z_i, D^i)$. *We have also*

(Relation B)

$$K_{D^0} = \sum_{k \in T^0} (\alpha_k - 1) C_k \quad \textit{in } \operatorname{Pic} D^0 \otimes \mathbb{Q}.$$

Idea of the proof. It is quite analogous to the proof of Theorem 2.7 starting now from Proposition 4.3. Investigating 'backwards' the evolution of the canonical divisors K_{D^i} and using the identities $K_{D^{i+1}} = \pi_{i+1}^* K_{D^i} + (r_i - 1)C_{i+1}$ we derive for $i = m-1, \ldots, 0$ that

$$K_{D^i} = \sum_{k \in T^0 \cup \{1,\ldots,i\}} (\alpha_k - 1) C_k \quad \text{in } \operatorname{Pic} D^i \otimes \mathbb{Q},$$

and as a bonus we obtain the Relations A. See [V1] for the complete proof of the case $|J| = 1$, which is in fact also valid in the general case. □

(4.5). For concrete varieties D^0 we can make Relation B more explicit. For example when $D^0 \cong \mathbb{P}^m$ then it becomes

$$\sum_{k \in T} d_k(\alpha_k - 1) + m + 1 = 0,$$

where d_k is the degree of the hypersurface C_k. See [V1] when D^0 is an arbitrary projective space bundle.

5. Poles of zeta functions

(5.1). Let K be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, R the valuation ring of K, P the maximal ideal of R, and $\bar{K} = R/P$ the residue field with cardinality q. For $z \in K$ we denote by $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ its valuation, $|z| = q^{-\operatorname{ord} z}$ its absolute value, and $\operatorname{ac}(z) = z\pi^{-\operatorname{ord} z}$ its angular component, where π is a fixed uniformizing parameter for R.

Let $f(x) \in K[x] = K[x_1, \ldots, x_n]$ and $\varkappa\colon R^\times \to \mathbb{C}^\times$ a character of $R^\times$, the group of units of R. (We formally put $\varkappa(0) = 0$.) To these data one associates *Igusa's local zeta function*

$$Z(s) = Z(s, f, \varkappa) := \int_{R^n} \varkappa(\operatorname{ac} f(x)) |f(x)|^s |\mathrm{d}x|,$$

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for $s \in \mathbb{C}$ with $\Re(s) > 0$. Here $|\mathrm{d}x|$ denotes the Haar measure on K^n, normalized such that R^n has measure 1. Igusa [I1] showed that it is a rational function of q^{-s}, so it extends to a meromorphic function on $\mathbb{C}$.

For more information and references on Igusa's local zeta function, see for example the overview paper [D3].

(5.2). From now on we suppose that $\varkappa$ is trivial on $1 + P$, i.e. it is induced by a character of $\bar{K}$; this is the relevant case (see [D3, Thm 3.3]). Let also d denote the order of $\varkappa$.

We choose an embedded resolution $h: X \to \mathbb{A}^n$ of $f^{-1}\{0\}$, constructed entirely over K (this in possible by [H]), for which we use the notation of (1.1), where now the $E_i, i \in S$, are the K-irreducible components of $h^{-1}(f^{-1}\{0\})$. We also set $\overset{\circ}{E}_{\mathrm{I}} := E_{\mathrm{I}} \setminus \cup_{\ell \notin \mathrm{I}} E_\ell$ for $\mathrm{I} \subset S$. Igusa's proof of the rationality of $Z(s)$ yields the following: *All real poles of $Z(s)$ are among the values $-(\nu_j/N_j)$, where $j \in S$ and $d|N_j$.*

Moreover the following formula gives a closed expression for $Z(s)$ in terms of the resolution (X, h). In the sequel we denote reduction mod P by $(\cdot)_{\bar{K}}$.

THEOREM 5.3 [D3, Sec. 3]. *Suppose that the resolution (X, h) has good reduction* mod P (*see* [D3, (3.2)]). *Then*

$$Z(s) = q^{-n} \sum_{\mathrm{I} \subset S} c_{\mathrm{I}}^{\varkappa} \prod_{i \in \mathrm{I}} \frac{q-1}{q^{\nu_i + sN_i} - 1},$$

with

$$c_{\mathrm{I}}^{\varkappa} = \sum_k (-1)^k \operatorname{Tr}[\operatorname{Frob}, H_c^k((\overset{\circ}{E}_{\mathrm{I}})_{\bar{K}}, \mathcal{L}_\varkappa)].$$

Here $\mathcal{L}_\varkappa$ is a certain ℓ-adic sheaf on $X_{\bar{K}}$ associated to $\varkappa$, Tr *denotes the trace, and* Frob *is the geometric Frobenius of $\bar{K}$.* (*Remark that $c_{\mathrm{I}}^{\varkappa} = 0$ when $E_{\mathrm{I}} = \emptyset$.*)

Remark 5.4. (i) 'Good reduction mod P' is a technical condition. When f and (X, h) are defined over a number field F, then we have good reduction for all but a finite number of completions K of F.

(ii) The sheaf $\mathcal{L}_\varkappa$ is in fact zero on $\cup_{d \nmid N_i} (E_i)_{\bar{K}}$ and locally constant of rank one elsewhere; we can thus restrict the summation above to subsets I for which $d|N_i$ for all $i \in \mathrm{I}$.

(iii) When $\varkappa$ is the trivial character the sheaf $\mathcal{L}_\varkappa$ is constant on $\bar{X}$ and so $c_{\mathrm{I}}^{\varkappa}$ is just the number of $\bar{K}$-rational points on $(\overset{\circ}{E}_{\mathrm{I}})_{\bar{K}}$.

(5.5). Let now E_J be a nonempty intersection for which $s_0 = -(\nu_j/N_j)$ for all $j \in J$, and $s_0 \neq -(\nu_i/N_i)$ for other components E_i of $h^{-1}Y$ that intersect E_J.

For such an intersecting component E_i set $\alpha_i := \nu_i + s_0 N_i$. The contribution of E_J to the formula for $Z(s)$ above is

$$q^{-n}\frac{(q-1)^{|J|}}{\prod_{j\in J}(q^{\nu_j+sN_j}-1)}\sum_{\mathrm{I}\supset J} c_{\mathrm{I}}^{\varkappa}\prod_{i\in \mathrm{I}\setminus J}\frac{q-1}{q^{\nu_i+sN_i}-1}.$$

We are interested in the contribution of E_J to the problem whether s_0 is a pole of order $|J|$ of $Z(s)$, and thus in the nullity of

$$R_{s_0} := \sum_{\mathrm{I}\supset J} c_{\mathrm{I}}^{\varkappa}\prod_{i\in \mathrm{I}\setminus J}\frac{q-1}{q^{\alpha_i}-1}. \tag{$*$}$$

We may suppose that $d|N_j$ for all $j\in J$ since otherwise R_{s_0} is trivially zero.

Let $\chi(\cdot)$ denote the Euler–Poincaré characteristic with respect to singular cohomology. Inspired by Igusa's so–called Monodromy Conjecture [D3, Con. 2.3.2] and the formula of A'Campo [A, Thm 3] we expect the following. For a *generic* projective E_J with $\chi(\overset{\circ}{E}_J)=0$ we should have $R_{s_0}=0$.

(5.6). When $|J|=1$ then E_J is in fact an exceptional variety E_j and $s_0 = -(\nu_j/N_j)$. In the case of curves $(n=2)$ necessarily $E_j\cong \mathbb{P}^1$, and so the condition $\chi(\overset{\circ}{E}_j)=0$ is equivalent to E_j intersecting exactly twice other components, say E_1 and E_2. Then

$$R_{s_0} = c_{\{j\}}^{\varkappa} + c_{\{j,1\}}^{\varkappa}\frac{q-1}{q^{\alpha_1}-1} + c_{\{j,2\}}^{\varkappa}\frac{q-1}{q^{\alpha_2}-1}.$$

When $\varkappa$ is the trivial character we have $c_{\{j\}}^{\varkappa}=q-1$ and $c_{\{j,1\}}^{\varkappa}=c_{\{j,2\}}^{\varkappa}=1$ by Remark 5.4(iii) and consequently $R_{s_0}=0$ if we would have

$$\alpha_1+\alpha_2=0. \tag{5}$$

When $\varkappa$ is arbitrary using Remark 5.4(ii) it is not difficult to prove that $R_{s_0}=0$ if moreover we have $d|N_1 \Leftrightarrow d|N_2$. (See also Example 5.7.3.) This last equivalence is implied by the congruence

$$N_1+N_2\equiv 0 \mod N_j. \tag{6}$$

Now (6) and (5) are precisely Corollary 2.4 and the example after Proposition 4.3 for $|J|=1$! In fact the nullity of R_{s_0} was precisely the motivation for developing these relations and congruences for $n=2$ [S, M, I, L].

Using our theory of relations in codimension one we verified in [V3] that $R_{s_0}=0$ when expected for a lot of cases for surfaces $(n=3)$ and for some cases in arbitrary dimension n, assuming that $\varkappa$ is the trivial character. When $\varkappa$ is arbitrary we verified the nullity of R_{s_0} is some cases for surfaces using our theory of congruences (in codimension one); a couple of examples concerning the related topological zeta function (see (5.9)) appeared in [V2].

Here we should mention that when $n \geqslant 3$ there is a whole zoo of configurations satisfying $\chi(\overset{\circ}{E}_j) = 0$, and the vanishing of R_{s_0} seems a bit miraculous.

(5.7). Now when $|J|$ is arbitrary we can use the relations and congruences in arbitrary codimension of this paper to verify analogously the nullity of R_{s_0}. We give some examples, assuming that the resolution (X, h) has good reduction mod P, and for simplicity also that E_J is irreducible over an algebraic closure of K.

(5.7.1). If E_J is a projective curve then $\chi(\overset{\circ}{E}_J) = 0$ if and only if $E_J = \overset{\circ}{E}_J$ is an elliptic curve, or $E_J \cong \mathbb{P}^1$ and it intersects exactly twice other components. I doubt whether the first case can occur in an embedded resolution configuration. The second case certainly occurs and as above we have that $R_{s_0} = 0$, using Corollary 2.4 and the example after Proposition 4.3 (for arbitrary $|J|$).

(5.7.2). When $\varkappa$ is the trivial character all cases of [V3] where we verified for $|J| = 1$ that $R_{s_0} = 0$ can be extended to arbitrary codimension $|J|$.

(5.7.3). Let $E_J \cong \mathbb{P}^m (m \geqslant 2)$, and let the irreducible components of intersections of E_J with other $E_\ell, \ell \notin J$, be k hyperplanes in general position $(2 \leqslant k \leqslant m+1)$. One easily sees that $\chi(\overset{\circ}{E}_J) = 0$.

Let first $\varkappa$ be the trivial character. Then the numbers $c_{\mathrm{I}}^{\varkappa}$ in the expression $(*)$ are just the numbers of $\bar{K}$-rational points on the $(\overset{\circ}{E}_{\mathrm{I}})_{\bar{K}}$. When $|J| = 1$ we proved in [V3] that $R_{s_0} = 0$ (by induction on n and k); the same proof is valid for arbitrary $|J|$. Let now $\varkappa$ be arbitrary (of order d).

First case: $d | N_i$ for all $i = 1, \ldots, k$. By Remark 5.4(ii) we have that the sheaf $\mathcal{L}_\chi$ in the formula of 5.3 is locally constant on E_J and thus constant, since $E_J \cong \mathbb{P}^m$ is simply connected. Consequently the numbers $c_{\mathrm{I}}^{\varkappa}$ are just the numbers of $\bar{K}$-rational points on $(\overset{\circ}{E}_{\mathrm{I}})_{\bar{K}}$, and $R_{s_0} = 0$ arguing as above.

Second case: $d \nmid N_1$ and $d \nmid N_2$ (after permutation of the indices). We will show that all coefficients $c_{\mathrm{I}}^{\varkappa}$ in $(*)$ are zero, in fact more precisely that all the cohomology groups in the expression of 5.3 for $c_{\mathrm{I}}^{\varkappa}$ are zero, using Proposition 5.8 below. Indeed by an easy verification or by Proposition 5.8(ii) we have that $\chi(\overset{\circ}{E}_{\mathrm{I}}) = 0$ for any relevant $\mathrm{I} \supset J$, i.e. for I such that $J \subset \mathrm{I} \subset J \cup \{3, \ldots, k\}$ and $d | N_i$ for all $i \in \mathrm{I}$. Then Proposition 5.8(i) implies the nullity of all occurring cohomology groups.

Remark now that the eventual remaining case '$d \nmid N_1$ and $d | N_i$ for all $i = 2, \ldots, k$' is ruled out by Corollary 2.3. Indeed since Pic $\mathbb{P}^m \cong \mathbb{Z}$ this is equivalent to $\Sigma_{i=1}^k N_i \equiv 0 \bmod N_J$, which implies that $d | \Sigma_{i=1}^k N_i$. It is an exercise to check that in this hypothetical case we would in general have $R_{s_0} \neq 0$.

(5.7.4). Using the notation of (2.6) we take $D^0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and let $D^1 = D = E_J$ be obtained from D^0 by the blowing-up π_1 at a point P. Let the $C_i, i \in T^0$, consist

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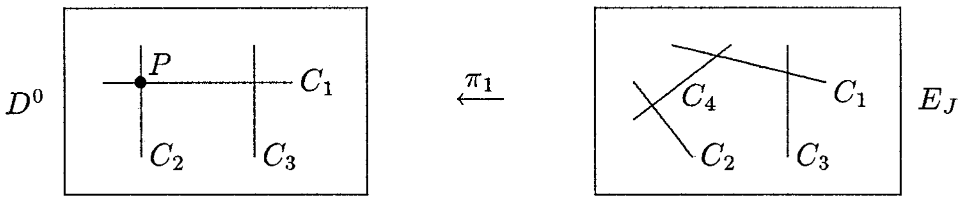
of a fibre C_1 of one projection $pr_1: D^0 \to \mathbb{P}^1$ and of two fibres C_2 and C_3 from the other projection pr_2, such that moreover $C_1 \cap C_2 = \{P\}$. Consequently the $C_i, i \in T$, consist of C_1, C_2, C_3, and the exceptional curve C_4 of π_1.



In this example Congruence B states that

$$N_1C_1 + N_2C_2 + N_3C_3 = 0 \quad \text{in } \frac{\operatorname{Pic} D^0}{N_J \operatorname{Pic} D^0}.$$

Since $\operatorname{Pic} D^0 \cong pr_1^* \operatorname{Pic} \mathbb{P}^1 \oplus pr_2^* \operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$ this is equivalent to

$$N_1 \equiv 0 \mod N_J \quad \text{and} \quad N_2 + N_3 \equiv 0 \mod N_J.$$

Furthermore Congruence A is

$$N_4 \equiv N_1 + N_2 \mod N_J.$$

One now verifies immediately that only the following two possibilities can occur:

(i) $d|N_i$ for $1 \leqslant i \leqslant 4$,
(ii) $d|N_1$ and $d \nmid N_i$ for $2 \leqslant i \leqslant 4$.

Case (i). As in the first case of (5.7.3) the numbers $c_{\mathrm{I}}^{\varkappa}$ are the numbers of $\bar{K}$-rational points on $(\overset{\circ}{E}_{\mathrm{I}})_{\bar{K}}$. Using the structure of $\operatorname{Pic} D^0$ and the fact that $K_{D^0} = pr_1^* K_{\mathbb{P}^1} + pr_2^* K_{\mathbb{P}^1}$, it is not difficult to verify that in this case the Relations B and A of Section 4 are $\{\alpha_1 = -1, \alpha_2 + \alpha_3 = 0\}$ and $\alpha_4 = \alpha_1 + \alpha_2$, respectively. Now it is an easy exercise to compute that $R_{s_0} = 0$.

Case (ii). In this case only $\overset{\circ}{E}_J$ and $\overset{\circ}{C}_1 := C_1 \backslash (C_3 \cup C_4)$ possibly contribute to R_{s_0}. Both contributions are however zero for we can show that, $\mathcal{L}_{\varkappa}$ being the sheaf of Theorem 5.3,

$$H_c^k((\overset{\circ}{C}_1)_{\bar{K}}, \mathcal{L}_{\varkappa}) = 0 \quad \text{for all } k, \text{ and} \tag{7}$$

$$H_c^k((\overset{\circ}{E}_J)_{\bar{K}}, \mathcal{L}_{\varkappa}) = 0 \quad \text{for all } k. \tag{8}$$

Indeed (7) is true because of Proposition 5.8 and the fact that $\chi(\overset{\circ}{C}_1) = 0$. We indicate a proof of (8), which gives the reader an idea of the arguments underlying

Proposition 5.8. First the exact sequence of cohomology with compact support for the inclusions $\overset{\circ}{E}_J \hookrightarrow \overset{\circ}{E}_J \cup \overset{\circ}{C}_1 \hookleftarrow \overset{\circ}{C}_1$, together with (7), yields

$$H_c^k((\overset{\circ}{E}_J)_{\bar{K}}, \mathcal{L}_\varkappa) \cong H_c^k((\overset{\circ}{E}_J \cup \overset{\circ}{C}_1)_{\bar{K}}, \mathcal{L}_\varkappa) \quad \text{for all } k.$$

Now since $\overset{\circ}{E}_J$ is affine these cohomology groups are zero for $k = 0, 1$. Using [SGA4$\frac{1}{2}$, Sommes Trig. 1.19.1] and Poincaré duality we have

$$\begin{aligned} H_c^k((\overset{\circ}{E}_J \cup \overset{\circ}{C}_1)_{\bar{K}}, \mathcal{L}_\varkappa) &\cong H^k((\overset{\circ}{E}_J \cup \overset{\circ}{C}_1)_{\bar{K}}, \mathcal{L}_\varkappa) \\ &\cong \check{H}_c^{4-k}((\overset{\circ}{E}_J \cup \overset{\circ}{C}_1)_{\bar{K}}, \check{\mathcal{L}}_\varkappa), \end{aligned}$$

for all k, where $\check{}$ denotes the dual. So $H_c^k((\overset{\circ}{E}_J)_{\bar{K}}, \mathcal{L}_\varkappa) = 0$ also for $k = 3, 4$ and consequently for $k = 2$ since $\chi(\overset{\circ}{E}_J) = 0$.

PROPOSITION 5.8. *Let $\mathcal{L}_\chi$ be the sheaf occurring in the formula of Theorem* 5.3. *Let E_J be a nonempty intersection of exceptional varieties with $d|N_j$ for all $j \in J$, and such that $E_J \setminus \cup_{d\nmid E_\ell} E_\ell$ is affine.*

(i) *For* $\mathrm{I} \supset J$ *such that* $d|N_i$ *for all* $i \in \mathrm{I}$ *we have that*

$$H_c^k((\overset{\circ}{E}_\mathrm{I})_{\bar{K}}, \mathcal{L}_\chi) = 0 \quad \text{for} \quad k \neq n - |\mathrm{I}| = \dim E_\mathrm{I}.$$

(ii) *If* $\chi(\overset{\circ}{E}_J) = 0$ *then for all* I *in* (i) *we have that* $\chi(\overset{\circ}{E}_\mathrm{I}) = 0$.

Proof. See [V4] when $|J| = 1$. The general case is analogous. □

(5.9) Finally we introduce the related topological zeta function. Taking heuristically the limit for $q \to 1$ in the formula in 5.3 yields

$$\sum_{\substack{\mathrm{I} \subset S \\ \forall i \in \mathrm{I}: d | N_i}} \chi(\overset{\circ}{E}_\mathrm{I}) \prod_{i \in \mathrm{I}} \frac{1}{\nu_i + sN_i}. \qquad (**)$$

Denef and Loeser [DL] define the *topological zeta function* $Z_{\text{top}}^{(d)}(s, f)$ associated to $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $d \in \mathbb{N}\setminus\{0\}$ as the rational function $(**)$ in the variable s. They prove that this defining formula does not depend on the chosen resolution (X, h) by expressing it in an exact way as a limit of Igusa's local zeta functions.

One can also state the Monodromy Conjecture for $Z_{\text{top}}^{(d)}(s, f)$, and our vanishing results about poles of Igusa's local zeta function are also valid for the topological zeta function, the latter results being easier then the first.

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