A REMARK ON (π, n)-TYPE CW-COMPLEXES

KENICHI SHIRAIWA

§ **1.** Let X be a space whose i-th homotopy group $\pi_i(X)$ vanishes for every $i \geqq 0$ except $i = n \geqq 1$, and whose n-th homotopy group is isomorphic to a group π. Then it is well known that the polyhedral homotopy type of X is completely determined by π and n. We call such a space a (π, n)-type space. Also it is well known that the minimal complex of the singular complex of a (π, n)-type space is isomorphic to the complex $K(\pi, n)$ defined by S. Eilenberg and S. MacLane [1]. We know also that for any $n \geqq 1$ and any group π (abelian if $n > 1$) there exists a (π, n)-type space (See [6]).

The purpose of this paper is to shown that if π is a finitely generated abelian group and $n \geqq 2$, then there exists a (π, n)-type CW-complex whose number of cells is algebraically minimal to realize the integral homology group $H_*(\pi, n; Z)$ of $K(\pi, n)$. Since $H_*(\pi, n; Z)$ is finitely generated in each dimension under our assumption (Cf. [3]), the number of cells of such a complex is finite in each dimension.

The author wishes to express his sincere gratitude to Mr. Nobuo Yoneda for his valuable suggestions in preparing this paper.

§ **2.** Throughout this paper we assume π is a finitely generated abelian group, $n > 1$, and the coefficient group is always the group of integers Z.

We know that $H_*(\pi, n)$ is finitely generated in each dimension, so we can decompose $H_q(\pi, n)$ as a finite sum of cyclic groups.

Let

$$H_q(\pi, n) = F_1^q + \dots + F_{r_q}^q + T_1^q + \dots + T_{l_q}^q \tag{1}$$

be such a decomposition, where F_i^q is an infinite cyclic group and T_i^q is a cyclic group of order t_i^q.

To each F_i^q $(i = 1, \dots, r_q)$ we associate a q-cell e_i^q and also to each T_i^q $(i = 1, \dots, l_q)$ we associate a q-cell $'e_i^q$ and a $(q+1)$-cell $''e_i^{q+1}$.

Received January 25, 1957.

https://doi.org/10.1017/S0027763000021917 Published online by Cambridge University Press

THEOREM. *There exists a* (π, n)*-type CW-complex* X *such that*

i) $$X = \bigcup_{q=0}^{\infty} \left(\bigcup_{i=1}^{r_q} e_i^q \bigcup_{i=1}^{l_q} {}'e_i^q \bigcup_{i=1}^{l_q} {}''e_i^{q+1} \right),$$

ii) $$\partial e_i^q = \partial' e_i^q = 0, \qquad \partial'' e_i^{q+1} = t_i^q\, {}'e_i^q,$$

where ∂ *is the boundary operator of the chain complex* $C(X)$ *of* X.

We prove this theorem in the following manner. Namely we shall construct CW-complexes X_k $(k = 0, 1, 2, \ldots)$ which satisfy the following conditions 1)—5).

1) $X_{k-1} \subset X_k$,

2) $X_k - X_{k-1} = \bigcup_{i=1}^{r_k} e_i^k \bigcup_{i=1}^{l_k} {}'e_i^k \bigcup_{i=1}^{l_{k-1}} {}''e_i^k \qquad (X_{-1} = \phi)$,

3) $\partial e_i^q = \partial' e_i^q = 0, \qquad \partial'' e_i^q = t_i^{q-1}\, {}'e_i^{q-1} \qquad (q \leqq k)$,

4) $\pi_i(X_k) = 0, \quad i \neq n \quad \text{and} \quad i < k,$

$\pi_n(X_k) \approx \pi, \quad \text{if} \quad k > n.$

By 1) and 2) X_k^q (q-skeleton of X_k) $= X_q$ $(q \leqq k)$, and then by 3) $H_k(X_k)$ is a free abelian group generated by $\{e_i^k, {}'e_i^k\}$.

5) If $k > n$, there exists a homomorphism

$$\varphi_k : H_k(X_k) \longrightarrow H_k(\pi, n)$$

such that $\varphi_k e_i^k$, $\varphi_k{}'e_i^k$ generates F_i^k, T_i^k respectively and the following sequence

$$\pi_k(X_{k-1}) \xrightarrow{i_*} \pi_k(X_k) \xrightarrow{\eta} H_k(X_k) \xrightarrow{\varphi_k} H_k(\pi, n) \longrightarrow 0$$

is exact, where i is the injection map $X_{k-1} \to X_k$ and η is the Hurewicz homomorphism.

Obviously $X = \bigcup_k X_k$ will have the required property of our theorem.

§ **3.** We first construct X_k $(k \leqq n+1)$ as follows:

Let $X_{n+1} = e^0 \smile e_1^n \smile \ldots \smile e_{r_n}^n \smile {}'e_1^n \smile \ldots \smile {}'e_{l_n}^n \smile {}''e_1^{n+1} \smile \ldots \smile {}''e_{l_n}^{n+1}$ where e_i^n and ${}'e_i^n$ are n-cells attached to a 0-cell e^0 by constant mappings $\partial e_i^n \to e^0$, $\partial' e_i^n \to e^0$, and ${}''e_i^{n+1}$ is attached to ${}'e_i^n \smile e^0$ by a map $\partial'' e_i^{n+1} \to {}'e_i^n \smile e^0$ of degree t_i^n. Let X_k $(k \leqq n+1)$ be the k-skeleton X_{n+1}^k of X_{n+1}, then the conditions 1)—5) follows immediately from the fact that $H_{n+1}(\pi, n) = 0$ [2] and also that $i_* : \pi_{n+1}(X_n) \to \pi_{n+1}(X_{n+1})$ is onto [5].

Now assume we already have $X_0, \ldots, X_k$ $(k > n)$ with conditions 1)—5). The construction of X_{k+1} requires the following lemma.

https://doi.org/10.1017/S0027763000021917 Published online by Cambridge University Press

LEMMA. *Denoting by i the injection map $X_{k-1} \to X_k$ we have $H_{k+1}(\pi, n) \approx i_* \pi_k(X_{k-1})$ for $k \geqq n$.*

(Essentially the same lemma is proved in [4].)

Proof. Let Y be a (π, n)-type CW-complex obtained by killing the homotopy groups of X_k except for $\pi_n(X_k)$ in the usual way, and consider the commutative diagram

$$\begin{array}{ccccccc}
 & & & & \pi_k(X_k, X_{k-1}) & & \\
 & & & \overset{\partial_3}{\nearrow} & \uparrow \partial_1 & & \\
\pi_{k+2}(Y^{k+2}, Y^{k+1}) & \xrightarrow{\partial_2} & \pi_{k+1}(Y^{k+1}, X_k) & \xrightarrow{i_2} & \pi_{k+1}(Y^{k+2}, X_k) & \xrightarrow{j_2} & \pi_{k+1}(Y^{k+2}, Y^{k+1}) = 0 \\
 & & & & \uparrow j_1 & \overset{j_3}{\nearrow} & \\
 & & 0 = \pi_{k+1}(Y^{k+2}) & \longrightarrow & \pi_{k+1}(Y^{k+2}, X_{k-1}) & \xrightarrow{\partial_0} & \pi_k(X_{k-1}) \longrightarrow \pi_k(Y^{k+2}) = 0 \\
 & & & & \uparrow i_1 & \overset{\partial}{\nearrow} & \\
 & & & & \pi_{k+1}(X_k, X_{k-1}) & &
\end{array}$$

in which rows and columns are exact sequences of triples and a pair. Then, since $Y^k = X_k$ and $Y^{k-1} = X_{k-1}$, we have

$$H_{k+1}(\pi, n) \approx \operatorname{Ker} \partial_3 / \operatorname{Im} \partial_2 \approx \operatorname{Ker} \partial_1 \approx \operatorname{Coker} i_1 \approx \operatorname{Coker} \partial \approx i_* \pi_k(X_{k-1}).$$

Now by the condition 5) for X_k there exists $\alpha_i \in \pi_k(X_k)$ for each generator $t_i^k {}'e_i^k$ of $\operatorname{Ker} \varphi_k$, such that $\eta(\alpha_i) = t_i^k {}'e_i^k$. We attach new $(k+1)$-cells ${}''e_i^{k+1}$ $(i = 1, \ldots, l_k)$ to X_k each by a representative map $g_i'' : \partial {}''\bar{e}_i^{k+1} \to X_k$ of α_i. Let β_i $(i = 1, \ldots, r_{k+1})$, β_i' $(i = 1, \ldots, l_{k+1})$ be elements of $i_* \pi_k(X_{k-1})$ whose images under the isomorphism $H_{k+1}(\pi, n) \approx i_* \pi_k(X_{k-1})$ generate F_i^{k+1}, T_i^{k+1} respectively. We now attach $\bar{e}_i^{k+1}$ $(i = 1, \ldots, r_{k+1})$ and ${}'\bar{e}_i^{k+1}$ $(i = 1, \ldots, l_{k+1})$ by representative mappings $h_i : \partial \bar{e}_i^{k+1} \to X_{k-1}$ and $h' : \partial {}'\bar{e}_i^{k+1} \to X_{k-1}$ of β_i and β_i' respectively. Then the attached space

$$\overline{X_{k+1}} = X_k \overset{r_{k+1}}{\underset{i=1}{\cup}} \bar{e}_i^{k+1} \overset{l_{k+1}}{\underset{i=1}{\cup}} {}'\bar{e}_i^{k+1} \overset{l_k}{\underset{i=1}{\cup}} {}''\bar{e}_i^{k+1}$$

obviously satisfies conditions 1) and 2).

To see 3) is satisfied by X_{k+1}, we consider the following commutative diagram

$$\begin{array}{ccc}
\pi_{k+1}(X_{k+1}, X_k) & \xrightarrow{\partial_1} & \pi_k(X_k) \\
 & \overset{\partial_2}{\searrow} & \downarrow j \\
 & & \pi_k(X_k, X_{k-1})
\end{array}$$

https://doi.org/10.1017/S0027763000021917 Published online by Cambridge University Press

where ∂_1, ∂_2 are boundary homomorphisms. Since ∂_2 is equivalent to the homology boundary operator of the chain groups of X_{k+1}, and since ∂_1 makes each of the attached $(k+1)$-cells correspond to the attaching map, 3) follows directly by the construction of X_{k+1}.

To see 4) is satisfied, we only have to prove $\pi_k(\overline{X_{k+1}}) = 0$. In virtue of the exact sequence

$$0 \longrightarrow i_* \pi_k(X_{k-1}) \longrightarrow \pi_k(X_k) \xrightarrow{\eta} \mathrm{Im}\,\eta \longrightarrow 0$$

derived from condition 5) for X_k, α_i, β_i and β_i' generate $\pi_k(X_k)$, since β_i, β_i' generate $i_* \pi_k(X_{k-1})$ and $\eta(\alpha_i)$ generate $\mathrm{Im}\,\eta$. It follows then that in the exact sequence

$$\pi_{k+1}(\overline{X_{k+1}}, X_k) \xrightarrow{\partial_1} \pi_k(X_k) \longrightarrow \pi_k(\overline{X_{k+1}}) \longrightarrow 0$$

∂_1 is onto. Therefore we obtain $\pi_k(\overline{X_{k+1}}) = 0$.

Now to get X_{k+1} satisfying 1)—5) we make some improvement on the cells $\overline{e_i^{k+1}}$, $\overline{'e_i^{k+1}}$. Namely we first imbed $\overline{X_{k+1}}$ in a (π, n)-type CW-complex Y in such a way that $\overline{X_{k+1}} = Y^{k+1}$. Then exactness holds in the following sequence

$$(2) \qquad \pi_{k+1}(X_k) \xrightarrow{\bar{i}_*} \pi_{k+1}(\overline{X_{k+1}}) \xrightarrow{\eta} H_{k+1}(\overline{X_{k+1}}) \xrightarrow{\bar{\varphi}_*} H_{k+1}(Y) \longrightarrow 0$$

where i, $\bar{\varphi}$ are injections. (This is essentially the same result as [1].) In fact, consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & \pi_{k+2}(Y^{k+2}, \overline{X_{k+1}}) & & \\
 & & & \swarrow{\scriptstyle \partial_1} & \downarrow{\scriptstyle \partial_2} & & \\
\pi_{k+1}(X_k) & \xrightarrow{\bar{i}_*} & \pi_{k+1}(\overline{X_{k+1}}) & \xrightarrow{j} & \pi_{k+1}(\overline{X_{k+1}}, X_k) & \longrightarrow & \pi_k(X_k) \\
 & & & & \downarrow{\scriptstyle \partial_3} & \swarrow & \\
 & & & & \pi_k(X_k, X_{k-1}) & &
\end{array}$$

where ∂_1 is onto and the row sequence is exact, and ∂_2, ∂_3 are equivalent to the boundary operators of the chain complex of Y. Thus (2) can be identified with the sequence

$$(2') \qquad \pi_{k+1}(X_k) \xrightarrow{\bar{i}_*} \pi_{k+1}(\overline{X_{k+1}}) \xrightarrow{j} \mathrm{Ker}\,\partial_3 \longrightarrow \mathrm{Ker}\,\partial_3/\mathrm{Im}\,\partial_2 \longrightarrow 0$$

which is obviously exact in virtue of the above diagram.

Now we identify $H_{k+1}(\pi, n)$ to $H_{k+1}(Y)$, then $\bar{\varphi}_*$ gives an onto homomorphism $\bar{\varphi}_{k+1} : H_{k+1}(\overline{X_{k+1}}) \to H_{k+1}(\pi, n)$. Since $H_{k+1}(\overline{X_{k+1}})$ is a free abelian group generated by e_i^{k+1} $(i=1, \ldots, r_{k-1})$ and $'e_i^{k+1}$ $(i=1, \ldots, l_{k-1})$, we can select another base $x_1, \ldots, x_{r_{k+1}}, x'_1, \ldots, x'_{l_{k+1}}$ of $H_{k+1}(X_{k+1})$ such that $\bar{\varphi}_{k+1}(x_i)$ and $\bar{\varphi}_{k+1}(x'_i)$ generate F_i^{k+1} and T_i^{k+1} respectively. The existence of such a base is readily verified by a quite elementary argument, and so the proof is omitted.

Let

$$x_i = \sum_j a_{ij} \overline{e_j^{k+1}} + \sum_j b_{ij} \overline{'e_j^{k+1}}$$

$$x'_i = \sum_j c_{ij} e_j^{k+1} + \sum_j d_{ij} {'e_j^{k+1}}$$

be the transformation of the bases. Then we attach new $(k+1)$-cells e_i^{k+1} $(i=1, \ldots, r_{k+1})$ to X_{k-1} each by a map representing $\sum_j a_{ij}\beta_j + \sum_j b_{ij}\beta'_j$ and $'e_i^{k+1}$ $(i=1, \ldots, l_{k+1})$ to X_{k-1} each by a map representing $\sum_j c_{ij}\beta_j + \sum_j d_{ij}\beta'_j$. Finally we attach $''e_i^{k+1}$ $(i=1, \ldots, l_k)$ to X_k each by a map representing α_i. Then the attached space

$$X_{k+1} = X_k \bigcup_{i=1}^{r_{k+1}} e_i^{k+1} \bigcup_{i=1}^{l_{k+1}} {'e_i^{k+1}} \bigcup_{i=1}^{l_k} {''e_i^{k+1}}$$

satisfies the required condition 1)—5). Infact, 1) and 2) are trivial and 3) is verified easily as in the case of X_{k+1}.

Let $g : C(X_{k+1}) \to C(\overline{X_{k+1}})$ be a chain map defined in the following way:

$$g : C_i(X_{k+1}) \to C_i(X_{k+1}), \qquad i \leqq k$$

is the identity map,

$$\bar{g} : C_{k+1}(X_{k+1}) \to C_{k+1}(X_{k+1})$$

is defined by

$$(3) \qquad \begin{aligned} g(e_i^{k+1}) &= \sum_j a_{ij} \overline{e_j^{k+1}} + \sum_j b_{ij} \overline{'e_j^{k+1}} = x_i, \\ g('e_i^{k+1}) &= \sum_j c_{ij} \overline{e_j^{k+1}} + \sum_j d_{ij} \overline{'e_j^{k+1}} = x'_i, \\ \bar{g}(''e_i^{k+1}) &= \overline{''e_i^{k+1}}. \end{aligned}$$

Let g' be the identity map of $X_{k+1}^k = X_k$ to $X_{k+1}^k = X_k$, then the following diagram is commutative.

$$\begin{array}{ccc} \pi_{k+1}(X_{k+1}, X_k) = C_{k+1}(X_{k+1}) & \xrightarrow{g} & C_{k+1}(X_{k+1}) = \pi_{k+1}(X_{k+1}, X_k) \\ \partial \downarrow & & \partial \downarrow \\ \pi_k(X_k) & \xrightarrow{g'_* = 1} & \pi_k(X_k) \end{array}$$

Therefore by a lemma of J. H. C. Whitehead [5], g' extends to a map $g : X_{k+1}$

https://doi.org/10.1017/S0027763000021917 Published online by Cambridge University Press

$\to X_{k+1}$ which realizes $\bar{g} : C(X_{k+1}) \to C(\overline{X_{k+1}})$. Therefore g induces an isomorphism of $H_*(X_{k+1}) \to H_*(\overline{X_{k+1}})$ and g is a homotopy equivalence (See [7]). This proves 4) for X_{k+1}.

Finally let us consider the following commutative diagram

$$\begin{array}{ccccccccc} \pi_{k+1}(X_k) & \xrightarrow{i_*} & \pi_{k+1}(X_{k+1}) & \xrightarrow{\eta} & H_{k+1}(X_{k+1}) & & & & \\ g_* \downarrow \wr & & g_* \downarrow \wr & & g_* \downarrow \wr & & & & \\ \pi_{k+1}(X_k) & \xrightarrow{\bar{i}_*} & \pi_{k+1}(\overline{X_{k+1}}) & \xrightarrow{\eta} & H_{k+1}(\overline{X_{k+1}}) & \xrightarrow{\bar{\varphi}_{k+1}} & H_{k+1}(\pi, n) & \longrightarrow & 0 \end{array}$$

Set $\varphi_{k+1} = \bar{\varphi}_{k+1} \circ g_*$. Then the condition 5) for X_{k+1} is now assured by (2) and (3), and this concludes the proof.

Bibliography

[1] S. Eilenberg and S. MacLane: Relations between homotopy and homology groups of spaces I, II, Ann. of Math., **46** (1945), pp. 480-509; **51** (1950), pp. 514-533.

[2] H. Cartan: Séminaire de topologie algébrique, 1954/55.

[3] J. P. Serre: Groupes d'homotopie et classes de groupes abéliens, Ann. of Math., **58** (1953), pp. 258-294.

[4] H. Toda: Generalized Whitehead products and homotopy groups of spheres, J. Inst. Polyt. Osaka City Univ., **3** (1952), pp. 43-82.

[5] J. H. C. Whitehead: On simply connected 4-dimensional polyhedra, Comm. Math. Helv., **22** (1949), pp. 48-92.

[6] J. H. C. Whitehead: On the realizability of homotopy groups, Ann. of Math., **50** (1949), pp. 261-263.

[7] J. H. C. Whitehead: On the homotopy type of ANR's, Bull. Amer. Math. Soc., **54** (1948), pp. 1133-1145.

Mathematical Institute
Nagoya University