Canad. Math. Bull. Vol. 32 (2), 1989

MONTEL SUBSPACES IN THE COUNTABLE PROJECTIVE LIMITS OF $L^p(\mu)$-SPACES

BY
J. C. DÍAZ

ABSTRACT. Let us suppose one of the following conditions: (a) $p \geqq 2$ and F is a closed subspace of a projective limit $\varprojlim(L^p(\mu_n), I_{nm})$; (b) $p = 1$ and F is a complemented subspace of an echelon Köthe space of order 1, $\Lambda(X, \beta, \mu, g_k)$; and (c) $1 < p < 2$ and F is a quotient of a countable product of $L^p(\mu_n)$ spaces. Then, F is Montel if and only if no infinite dimensional subspace of F is normable.

It is clear that if F is a Fréchet Montel space then, no infinite dimensional subspace of F is normable. We are concerned then with the reciprocal question: (*) Let F be a Fréchet space such that no infinite dimensional subspace of F is normable. Then, is F a Montel space?

The answer to (*) is not always positive since in [4] the author gives an example of a Fréchet space, not Montel, without infinite dimensional normable subspaces. However, it is known that the answer is positive if F is an echelon sequence space ([5]), an X-Köthe sequence space ([2]), or an echelon space of order 0 ([4]).

In this paper, we study (*) on the closed subspaces of the countable projective limits of $L^p(\mu)$-spaces, obtaining a positive answer if $p \geqq 2$, and some partial results if $1 \leqq p < 2$.

The vector spaces we use are defined on the field R of reals. Given a topological vector space E, we denote E' and E'' the dual and bidual of E. If $\langle E, F\rangle$ is a dual pair, it will be denoted by $\sigma(E, F)$ the weak topology on E, and by $\langle x, y\rangle$ the canonical bilinear form on $E \times F$. If $S \subset E$, $\overline{\langle S\rangle}$ denotes the (closed linear) subspace spanned by the elements of S.

Notations and preliminary results. Let F be a Fréchet space and let us fix a fundamental system of seminorms $(\|\cdot\|_k)_k$. Then, F_k denotes the local Banach space generated by the seminorm $\|\cdot\|_k$, i.e. F_k is the completion of $(F/\|\cdot\|_k^{-1}(0), \|\cdot\|_k)$. I_k will denote the canonical mapping from F to F_k, for each $k \in N$.

A sequence $(x_n)_n$ in a Fréchet space E has been called a basis if for every $x \in E$ there exists a unique sequence of scalars $(a_n)_n \subset R$, such that $\sum_{n=1}^{\infty} a_n x_n$. A sequence $(x_n)_n$ in E is said to be a basic sequence if it is a basis of the closed subspace $\overline{\langle\{x_n; n \in N\}\rangle}$ of E. Checking the following fact is left to the reader ([14], and [2]).

Received by the editors June 10, 1987.
AMS Subject Classification (1980): 46A14.

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press

LEMMA 1. *Let $(x_n)_n$ be a sequence in a Fréchet space E. If $(I_1(x_n))_n$ is a basic sequence in E_1, and if for each k, there is an n_k so that $(I_k(x_n))_{n \geqq n_k}$ is also a basic sequence in E_k, then $(x_n)_n$ is a basic sequence in E.*

Two bases, $(x_n)_n$ in E, and $(y_n)_n$ in F, are called equivalent provided that a series $\Sigma_n a_n x_n$ converges if and only if (iff) $\Sigma_n a_n y_n$ converges. It follows (E and F being Fréchet spaces) that $(x_n)_n$ is equivalent to $(y_n)_n$ iff there is a (linear) isomorphism T from E onto F for which $Tx_n = y_n$, for all $n \in N$.

It is known that if $(x_n)_n$ is a sequence of vectors in a Banach space X, weakly convergent to 0 and such that $\lim_n \inf \|x_n\| > 0$, then $(x_n)_n$ has a subsequence which is a basic sequence ([8], pg. 5). Thus, the next result (which is essentially known) is straightforward.

LEMMA 2. *Let $(x_n)_n$ be a sequence in $L^2(\mu)$ satisfying the following conditions:*

(1) *$(x_n)_n$ weakly converges to 0,*

(2) $\inf\{\|x_n\|; n \in N\} > 0$.

Then, there exists a subsequence $(x_{\sigma(n)})_n$ which is a basic sequence equivalent to the unit vector basis in l^2.

Note. It is a well known fact that if G is a separable subset of the Lebesgue space $L^p(\mu)\,(1 \leqq p < \infty)$, then, there exists a complemented sublattice, isomorphic to l^p or to $L^p([0,1])\,(L^p)$, containing G. From this remark and some results of Kadec and Pelczynski ([6], Corollaries 1 and 2) Lemma A and Lemma B follow.

LEMMA A. *Let $p > 2$ and let $(x_n)_n$ be a sequence in $L^p(\mu)$ so that*

(1) *$(x_n)_n$ weakly converges to 0.*

(2) $\lim\sup(\|x_n\|)_n > 0$.

Then, there exists a subsequence of $(x_n)_n$ whcih is a basic sequence equivalent either to the unit vector basis in l^p or to the unit vector basis in l^2.

LEMMA B. *Let $p > 2$ and let S be an infinite-dimensional separable and closed subspace of $L^p(\mu)$. It follows that*

a) *If S is isomorphic to l^2, then it is complemented in $L^p(\mu)$.*

b) *If S is isomorphic to l^p, it contains a subspace isomorphic to l^p and complemented in $L^p(\mu)$.*

Likewise, if $p = 2$, we know, from the Hilbert spaces theory, that an infinite dimensional, separable, and closed subspace of $L^2(\mu)$ is isomorphic to l^2 and complemented in $L^2(\mu)$.

Main results.

Case $p \geqq 2$. Our main theorem, in this case, follows from the following, more general, result.

PROPOSITION 1. *Let $p \geqq 2$ and let F be a Fréchet space admitting a fundamental system of seminorms $(\|\cdot\|_k)_k$ such that, for each $k \in N$, F_k is isomorphic to a subspace*

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press

of some $L^p(\mu_k)$. Then, the following are equivalent:

(a) *F is not a Montel space.*

(b) *There exists a subspace of F which is isomorphic either to l^p or to l^2.*

(c) *F contains a complemented subspace isomorphic either to l^p or to l^2.*

PROOF. a $\longrightarrow$ b) Let us denote by T the topology of F. F is reflexive from the hypotheses; thus since it is not Montel we can find a sequence $(y_n)_n \subset F$, weakly convergent to 0, and such that no subsequence of $(y_n)_n$ T-converges to 0. So, there exists $r_k \in R$, for all $k \in N$, so that

$$\|y_n\|_k \leqq r_k, \quad \forall n, k \in n \tag{1}$$

and there are $k_0 \in N, \xi \in R^+$, $n_0 \in N$ such that, for $n \geqq n_0$

$$\|y_n\|_{k_0} \geqq \xi. \tag{2}$$

We can assume, without loss of generality, that $k_0 = n_0 = 1$. Consider now the sequence $(I_1(y_n))_n \subset F_1 \subset L^p(\mu_1)$. I_1 is a continuous mapping, hence weakly continuous. Thus, $(I_1(y_n))_n$ is weakly convergent to 0; besides, we have

$$\limsup \|I_1(y_n)\|_1 = \limsup \|y_n\|_1 \geqq \xi > 0.$$

Then, from Lemma A if $p > 2$, and from Lemma 2 if $p = 2$, there exists a subsequence of $(y_n)_n$, say $(y_n^1)_n$, such that $(I_1(y_n^1)_n$ is a basic sequence equivalent to the usual basis of $l^{\alpha(1)}$ ($\alpha(1)$ is either p or 2). We proceed inductively in the same way, choosing sequences $(y_n^k)_n$ for all $k \in N$, so that $(y_n^{k+1})_n$ is a subsequence of $(y_n^k)_n$ and $(I_k(y_n^k))_n$ is a basic sequence equivalent to the usual basis of $l^{\alpha(k)}$ ($\alpha(k) = p$ or 2).

To end the proof we take the diagonal sequence $(y_n^n)_n$ which is a basic sequence in F from Lemma 1. Since $\alpha(k)$ can change only from p to 2, it is easy to check that $\overline{\langle\{y_n^n\}\rangle}$ is isomorphic to the echelon sequence space $\lambda^\alpha(a_n^k)$ (where $\alpha = p$ or 2, and $a_n^k = \|y_n^n\|_k, n, k \in N$) which is normable from (1) and (2), hence isomorphic to $l^{p \text{ or } 2}$.

b $\longrightarrow$ c) It is easy to check that, if S is a normable subspace of F, there is an index $k \in N$ such that S is isomorphic to a subspace of F_k. The result follows now from the hypotheses and Lemma B.

c $\longrightarrow$ a) Trivial. □

THEOREM 1. *Let $p \geqq 2$ and let F be a closed subspace of a projective limit $\varprojlim(L^p(\mu_n), I_{nm})$. Then, F is Montel iff it contains no complemented subspace isomorphic to $l^{p \text{ or } 2}$.*

COROLLARY 1. *Let $p \geqq 2$ and let F be a closed subspace of a projective limit $\varprojlim(L^p(\mu_n), I_{nm})$. Then, F is Montel iff no infinite dimensional subspace of F is normable.*

REMARK. Theorem 1 and Proposition 1 can fail if the assumption on p is removed. Indeed, it is known that for $1 \leqq p < r \leqq 2$, L^r can be isomorphically embedded into

L^p, hence, l^r is a closed, not Montel, subspace of L^p, containing no copy of $l^{p \text{ or } 2}$. Even more, let us take $p \geqq 1$, and put

$$\lambda_p = \bigcap_{q>p} l^q$$

λ_p is a Fréchet space, not Montel, without infinite dimensional normable subspaces ([4]; furthermore, let us also note that λ_p has no infinite dimensional Banach quotient since l^q and l^r are totally coincomparable spaces if $q \neq r$); however, if $p < 2$, then λ_p fulfils the hypotheses in Proposition 1. We don't know whether the same can be said on Corollary 1. However, we have some partial results if $1 \leqq p < 2$.

Case $p = 1$. The next lemma is a generalization of a result of Rosenthal in the Banach space theory ([12]).

LEMMA 3. *Let $(x_n)_n$ be a bounded sequence in a Fréchet space F. Then $(x_n)_n$ has a subsequence $(x_{\sigma(n)})_n$ satisfying one of the following two mutually exclusive alternatives.*
(1) $(x_{\sigma(n)})_n$ is a weak-Cauchy sequence.
(2) $(x_{\sigma(n)})_n$ is equivalent to the usual l^1-basis.

PROOF. Let us fix a sequence of seminorms $(\|\cdot\|_k)_k$ defining the topology of F. Then, $(I_1(x_n))_n$ is a bounded sequence in the local Banach space F_1. So, from the results in [12] there exists a subsequence $(x_n^1)_n$ satisfying either 1) $(I_1(x_n^1))_n$ is a $\sigma(F_1, F_1')$-Cauchy sequence, or 2) $(I_1(x_n^1)_n$ is equivalent to the usual l^1-basis.

We consider now the sequence $(I_2(x_n^1)_n$ and the procedure continues inductively. If there is $r \in N$ so that $(I_r(x_n^r))_n$ is equivalent to the usual l^1-basis, then it is easy to check that $(x_n^r)_n$ is equivalent to the usual l^1-basis. In the other case, the diagonal sequence $(x_n^n)_n$ will provide a $\sigma(F, F')$-Cauchy subsequence of $(x_n)_n$. □

The next corollary follows as for Banach spaces.

COROLLARY 2. *Let F be a Fréchet space $\sigma(F, F')$-sequentially complete. Then, F is reflexive iff none of its subspaces is isomorphic to l^1.*

We shall need the following result; its proof is left to the reader.

LEMMA 4. *Let F be the projective limit of a projective sequence (F_n, I_{nm}) such that F_n is a $\sigma(F_n, F_n')$ sequentially complete locally convex space for each $n \in N$. Then, F is $\sigma(F, F')$ sequentially complete.*

COROLLARY 3. *Let F be the projective limit of a projective sequence $(L^1(\mu_n), I_{nm})$, and let S be a closed subspace of F. Then S is reflexive iff it contains no copy of l^1.*

PROOF. In fact, the spaces $L^1(\mu_n)$ are weakly sequentially complete and $\sigma(F, F')$ induces $\sigma(S, S')$ on S, so the result follows from Lemma 4 and Corollary 2. □

Our main result in this section will be stated in the framework of the echelon Köthe spaces of order 1. So we shall need a definition.

DEFINITION ([9]). *Given a measure space (X, β, μ) and a sequence of β-measurable functions $g_k : X \longrightarrow R$, so that $0 \leqq g_k(x) \leqq g_{k+1}(x)$ μ-a.e., and $\mu(\{x \in X; g_k(x) = 0, \forall k \in N\}) = 0$, we define the echelon Köthe space of order 1, $\Lambda(X, \beta, \mu, g_k)$, as the space of all (equivalence classes of) functions $f : X \longrightarrow R$ such that*

$$\|f\|_k = \int_X |f| g_k d\mu < \infty, \quad \forall k \in N.$$

We shall also denote it by Λ and we consider on Λ the Fréchet space topology, say T, generated by the seminorms $(\|\cdot\|_k)_k$. Obviously Λ is isomorphic to the projective limit of the projective sequence $(L^1(g_n d\mu), I_{nm})$, where I_{nm} is the restriction mapping, $n \leqq m$. (Note.– The author has recently proved that a reduced projective limit of a projective sequence $(L_{1,\lambda_n}, I_{nm})$, admitting a Fréchet lattice structure is isomorphic to some echelon Köthe space of order 1.)

In the sequel, we shall need the following unpublished result of López-Molina and López-Pellicer ([10]).

LEMMA C. *Let $\Lambda(X, \beta, \mu, g_k)$ be an echelon Köthe space. Let $(f_n)_n$ be a $\sigma(\Lambda, \Lambda')$ null sequence in Λ and let $(h_n)_n$ be a $\sigma(\Lambda', \Lambda'')$ null sequence in Λ'. Then*

$$(**) \qquad \lim_{n \to \infty} \langle f_n, h_n \rangle = 0.$$

We shall also use the following remarkable Orihuela's result ([11]).

LEMMA D. *If E is a DF-space then $(E, \sigma(E, E'))$ is an angelic space (i.e. every weaky relatively countably compact subset in E is weakly relatively sequentially compact).*

We can now state and prove the next theorem.

THEOREM 2. *Let F be a reflexive and complemented subspace of an echelon Köthe space Λ. Then F is Montel.*

PROOF. It is enough to show that all $\sigma(F, F')$-null sequences $(x_n)_n$, are convergent to 0. Indeed, let us assume, on the contrary, that there exists a sequence $(x_n)_n$, weakly convergent but not T-convergent to 0. We can then find $\xi > 0, k \in N$, and a subsequence of $(x_n)_n$ (denoted by $(x_n)_n$ again) such that

$$(1) \qquad \|x_n\|_k \geqq \xi, \quad \forall n \in N.$$

Let $U_k = \{x \in F; \|x\|_k \leqq 1\}$. Choose $f_j \in F' \cap U_k^0$, such that $f_j(x_j) = \|x_j\|_k, \quad j \in N$. From Lemma D, and since F is reflexive, we get that U_k^0 is $\sigma(F', F'')$ sequentially compact. Hence, there exists a subsequence of $(f_k)_k$ (let us denote it by $(f_n)_n$ again) which $\sigma(F', F'')$ converges to certain $f \in U_k^0$.

Let us now denote by P the continuous projection from Λ onto F. The adjoint map is continuous from $(F', \sigma(F', F''))$ into $(\Lambda', \sigma(\Lambda', \Lambda''))$, ([7]), so $(P'(f_n - f))$ is

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press

$\sigma(\Lambda', \Lambda'')$-convergent to 0. Then, from Lemma C, and because $(x_n)_n \subset F = P(F)$, we get

$$\langle f_n - f, x_n \rangle = \langle f_n - f, P(x_n) \rangle = \langle P'(f_n - f), x_n \rangle \overset{n\to\infty}{\longrightarrow} 0.$$

Likewise, $(\langle P'(f), x_n \rangle)_n$ is a null sequence, and so

$$\begin{aligned} \|x_n\|_k &= \langle f_n, x_n \rangle = \langle f_n, P(x_n) \rangle = \langle P'(f_n), x_n \rangle = \\ &= \langle P'(f_n - f), x_n \rangle + \langle P'(f), x_n \rangle \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

contradicting (1). This proves that $(x_n)_n$ converges to 0. □

Our main result follows.

COROLLARY 4. *Let Λ be an echelon Köthe space and let F be a complemented subspace of Λ. The F is Montel iff no infinite dimensional subspace of F is normable.*

PROOF. The condition is clearly necessary. Conversely, F is closed since it is complemented; by the assumptions, F contains no copy of l^1, hence it is reflexive from Corollary 3. The result follows from Theorem 2. □

Let us note that Theorem 2 (and so, Corollary 4) remains true replacing Λ by a projective limit $\underset{\longleftarrow}{\lim}(L^1(\mu_n), I_{nm})$ verifying the condition (**) in Lemma C (in particular, a reduced projective limit $F = \underset{\longleftarrow}{\lim}(L^1(\mu_n), I_{nm})$ such that $F'' \simeq \underset{\longleftarrow}{\lim}(L^1(\mu_n)'', I''_{nm})$).

Case $1 < p < 2$. From [6] Corollary 2, and the note before Lemma A, we have

LEMMA 5. *Let $p > 2$ and let E be an infinite dimensional subspace of $L^p(\mu)$. Then, E contains a subspace isomorphic to $l^{p \text{ or } 2}$ and complemented in $L^p(\mu)$.*

We shall need the "dual version" of Lemma 5:

LEMMA 6. *Let $1 < p < 2$ and let E be an infinite dimensional quotient space of $L^p(\mu)$ (quotient map Q_1). Then E has a quotient space isomorphic to $l^{p \text{ or } 2}$ (quotient map Q_2) such that we have (with suitable T) the following commutative diagram:*

$$\begin{array}{ccc} L^p(\mu) & \xrightarrow{Q_1} & E \\ T \uparrow & & \downarrow Q_2 \\ 1^\alpha & \xrightarrow{id.} & 1^\alpha \end{array} \qquad \alpha = p \text{ or } 2$$

PROOF. Applying Lemma 5 to E' we obtain

$$\begin{array}{ccc} L^q(\mu) & \hookleftarrow & E' \\ \downarrow & & \uparrow \\ 1^\alpha & \xleftarrow{id} & 1^\alpha \end{array} \qquad \begin{array}{c} 1/p + 1/q = 1 \\ \\ \alpha = q \text{ or } 2 \end{array}$$

Then dualize this commutative diagram. □

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press

We can now state and prove our last result:

THEOREM 3. *Let $1 < p < 2$ and let F be a quotient of a countable product of $L^p(\mu_n)$-spaces. Then, F is Montel if it contains no complemented subspace isomorphic to $l^{p \text{ or } 2}$.*

PROOF. Denote by q the quotient map from $\Pi_{n\in N} L^p(\mu_n)$ onto F. F is a quojection since it is a quotient of a quojection; then, if F is not Montel, it must have an infinite dimensional normable quotient, say B, ([3]), denote by Q_0, the quotient map. It is easy to see that there exists $k \in N$ so that B is a quotient of $\Pi_{n=1}^k L^p(\mu_n)$. So, from Lemma 6, we get the following commutative diagram

$$\begin{array}{ccccc}
\prod_{n\in N} L^p(\mu_n) & \xrightarrow{q} & F & & \\
J_k \uparrow \; \downarrow I_k & & \downarrow Q_0 & & I_k \circ J_k = id \\
\prod_{n=1}^{k} L^p(\mu_n) & \xrightarrow{Q_1} & B & & \\
T \uparrow & & \downarrow Q_2 & & \\
l^\alpha & \xrightarrow{id} & l^\alpha & & \alpha = p \text{ or } 2
\end{array}$$

We put $A = q \circ J_k \circ T$, $B = Q_2 \circ Q_0$. Then we obtain

$$B \circ A = Q_2 \circ \underbrace{Q - 0 \circ q} J_k \circ T = Q_2 \circ Q_1 \circ \underbrace{I_k \circ J_k} \circ T = Q_2 \circ Q_1 \circ T = id$$

Hence A imbeds l^α as a complemented subspace into F. □

REFERENCES

1. B. Beauzamy, *Introduction to Banach spaces and their geometry*, Mathematics Studies, North-Holland, 1968.

2. S. F. Bellenot, *Basic sequences in non-Schwartz Fréchet spaces*, Trans. Amer. Math. Soc. **258** (1980), No. 1, 199–216.

3. ——— and E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients*, Trans. Amer. Math. Soc. **273**, 579–594 (1982).

4. J. C. Díaz, *An example of Fréchet space, not Montel, without infinite dimensional normable subspaces*, Proc. Amer. Math. Soc. **96, 4, 721** (1986).

5. J. Dieudonne and A. P. Gomes, *Sur certain espaces vectoriels topologiques*, C.R. Acad. Sci., Paris **230**, 1129–1130 (1957).

6. M. I. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L^p*, Studia Math. **21** (1962), 161–176.

7. G. Köthe, *Topological Vector Spaces I, II*, Springer Verlag, Berlin 1969.

8. J. Lindenstrauss and L. Tzafiri, *Classical Banach Spaces I.*, Surveys in Math. Springer Verlag, 1979.

9. J. A. Lopez-Molina, *The dual and bidual of an echelon Köthe space*, Collec. Math. **31, 2** (1980), 159–191.

10. ——— and Lopez-Pellicer, *Consequences of the Dunford-Pettis property in the echelon Köthe spaces*, Preprint.

11. J. Orihuela, *Pointwise compactness in spaces of continuous functions*, To appear in Journal London Math. Soc.

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press

12. H. P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. USA, **71**, No. 6, 2411–2413, June 1974.
13. H. H. Schaefer, *Topological vector spaces*, Springer Verlag, 1970.
14. M. Valdivia, *Topics in locally convex spaces*, North-Holland, 1982.

Cátedra de Matemáticas. E.T.S.I.A.
14004 Cordoba, Spain

https://doi.org/10.4153/CMB-1989-025-4 Published online by Cambridge University Press