A NOTE ON THE HEAT KERNEL ON THE HEISENBERG GROUP

ADAM SIKORA AND JACEK ZIENKIEWICZ

We describe the analytic continuation of the heat kernel on the Heisenberg group $\mathbb{H}_n(\mathbb{R})$. As a consequence, we show that the convolution kernel corresponding to the Schrödinger operator e^{isL} is a smooth function on $\mathbb{H}_n(\mathbb{R}) \setminus S_s$, where $S_s = \left\{ (0,0,\pm sk) \in \mathbb{H}_n(\mathbb{R}) : k = n, n+2, n+4, \dots \right\}$. At every point of S_s the convolution kernel of e^{isL} has a singularity of Calderón–Zygmund type.

Let $\mathbb{H}_n(\mathbb{R})$ denote the $(2n+1)$-dimensional real Heisenberg group, that is, $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law

$$(x,y,t)(x',y',t') = \left(x + x', y + y', t + t' - \frac{1}{2} \operatorname{Im} \sum_{r=1}^{n} (x_r + iy_r)(x'_r - iy'_r) \right)$$

for all x, x', y, y' in $\mathbb{R}^n$ and t and t' in $\mathbb{R}$. For s in $\mathbb{R}$, we define the set S_s by

$$S_s = \left\{ (0,0,\pm sk) \in \mathbb{H}_n(\mathbb{R}) : k = n, n+2, n+4, \dots \right\}.$$

We define $\mathbb{H}_n(\mathbb{C})$ like $\mathbb{H}_n(\mathbb{R})$. For $(x,y,t) \in \mathbb{H}_n(\mathbb{C})$, we write x^2 for $\sum_{r=1}^{n} x_r^2$, and A and B for $it - (x^2 + y^2)/4$ and $-(x^2 + y^2)/2$ respectively.

The vector fields X_r, Y_r (where $r = 1, \dots, n$) and T, given by

$$X_r = \frac{\partial}{\partial x_r} - \frac{y_r}{2} \frac{\partial}{\partial t}, \quad Y_r = \frac{\partial}{\partial y_r} + \frac{x_r}{2} \frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t},$$

form a basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}_n(\mathbb{R})$. The Heisenberg Laplacian L is defined by

$$L = -\sum_{r=1}^{n} X_r^2 + Y_r^2.$$

The subelliptic operator L admits a spectral resolution

$$L = \int_0^{\infty} \lambda \, dE(\lambda),$$

and therefore when $\operatorname{Re} s \geqslant 0$, one can define the operator e^{-sL}, bounded on $L^2\big(\mathbb{H}_n(\mathbb{R})\big)$, by the spectral theorem:

$$e^{-sL} = \int_0^\infty e^{-s\lambda}\,dE(\lambda).$$

Let p_s be the convolution kernel of the operator e^{-sL} (see [5, (1.10), (1.11)]). When $s > 0$, e^{-sL} is the solution operator for the Heisenberg heat equation $\partial_s u = -Lu$ and p_s is called the heat kernel (see [6, (7.30), p.71].

The goal of this note is to study the analytic continuation of the heat kernel p_s. This is interesting from the point of view of the theory of analytic hypoellipticity (see [1, 2]). Another reason to study the analytic continuation of p_s is to investigate the operator L_α, equal to $L + i\alpha T$, where $\alpha \in \mathbb{C}$ (see Remark 2 below and [6, (7.53) p.73] (see also [3] for a detailed study of the operators L_α and an explanation of the significance of L_α).

However, we are also motivated by the possibility of explicitly computing the kernel of the Schrödinger propagator e^{isL}, where $s \in \mathbb{R}$, using the analytic continuation of the heat kernel. Indeed, e^{isL} is the solution operator for the Schrödinger equation

(1) $\partial_s u(x,y,t,s) = iLu(x,y,t,s).$

In [5, p.392–394], Strichartz noticed that "In principle we could attempt to solve (1) by analytic continuation from the solution of the heat equation". However, he abandoned this idea as "this analytic continuation is delicate, so we approach the problem directly". Strichartz proved that, when $s \in \mathbb{R}$, the convolution kernel of the Schrödinger operator p_{is} is a smooth function on the open set $\big\{(x,y,t) \in \mathbb{H}_n(\mathbb{R}) : |t| < n|s|\big\}$.

In this note we propose a simple computation which allows us to handle the analytic continuation of the heat kernel in a straightforward manner. Then, using the analytic continuation of the heat kernel we show that p_{is} is smooth on $\mathbb{H}_n(\mathbb{R}) \setminus S_s$ when $s \in \mathbb{R}$, and at points in S_s, the kernel p_{is} has singularities of Calderón–Zygmund type. This is in contrast to the Euclidean case where, for the standard Laplace operator Δ, the kernel of the operator $e^{is\Delta}$ is bounded and smooth. In this context it is interesting to note that the convolution kernel of the operator $e^{is(L-T^2)}$ is smooth. The smoothness of the convolution kernel of the operator $e^{is(L-T^2)}$ is the last result proved in this note.

A comprehensive discussion of harmonic analysis on the Heisenberg group can be found in [5] or [6, Chapter 1].

The following theorem is the main result of this note.

THEOREM 1. *The function p_1 extends to an analytic function on $\mathbb{H}_n(\mathbb{C})\setminus S$, where $S = \big\{(x,y,t) \in \mathbb{H}_n(\mathbb{C}) : \pm it - n - (x^2+y^2/4) \in 2\mathbb{N}\big\}$. For all $(x,y,t) \in \mathbb{H}_n(\mathbb{C}) \setminus S$,*

$$p_1(x,y,t) = \frac{1}{2(4\pi)^{n+1}}\big(P(x,y,t) + P(x,y,-t)\big),$$

where

$$(2) \quad P(x,y,t) = 2^n n! \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} \binom{n+j+h-1}{h} \binom{n+j}{j} \frac{B^j}{\big(n+2(j+h)-A\big)^{n+j+1}}.$$

The sum (2) *is absolutely uniformly convergent on compact subsets of* $\mathbb{H}_n(\mathbb{C}) \setminus S$. *Moreover,*

$$(3) \quad p_s(x,y,t) = \frac{1}{s^{n+1}} p_1\left(\frac{x}{\sqrt{s}}, \frac{y}{\sqrt{s}}, \frac{t}{s}\right)$$

for all s *such that* $\operatorname{Re} s \geqslant 0$.

PROOF: By virtue of the well known formula for the heat kernel on the Heisenberg group (see [**6**, (7.36), p.71], or [**5**, (5.20)])

$$
\begin{aligned}
2p_1(x,y,t) &= \frac{2}{(4\pi)^{n+1}} \int_0^{\infty} \frac{\lambda^n \cos t\lambda}{(\sinh \lambda)^n} \exp\left(\frac{-\lambda(x^2+y^2)\coth\lambda}{4}\right) d\lambda \\
&= \frac{1}{(4\pi)^{n+1}} \int_0^{\infty} \frac{\lambda^n e^{it\lambda}}{(\sinh \lambda)^n} \exp\left(\frac{-\lambda(x^2+y^2)\coth\lambda}{4}\right) d\lambda \\
&\qquad + \frac{1}{(4\pi)^{n+1}} \int_0^{\infty} \frac{\lambda^n e^{-it\lambda}}{(\sinh \lambda)^n} \exp\left(\frac{-\lambda(x^2+y^2)\coth\lambda}{4}\right) d\lambda \\
&= \frac{1}{(4\pi)^{n+1}} \big(P(x,y,t) + P(x,y,-t)\big),
\end{aligned}
$$

say. We note that, if $\lambda > 0$ and $m \in \mathbb{N}$, then

$$(e^{\lambda} - e^{-\lambda})^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} e^{-(2k+m)\lambda},$$

where $\binom{k-1}{k} = 1$ if $k = 0$ and $\binom{k-1}{k} = 0$ otherwise. Now

(4)

$$
\begin{aligned}
2^{-n} &P(x,y,t) \\
&= \int_0^{\infty} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \lambda^n e^{it\lambda} e^{-(2k+n)\lambda} \exp\left(\frac{\lambda B \coth \lambda}{2}\right) d\lambda \\
&= \int_0^{\infty} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \lambda^n \exp\left(\lambda(A - 2k - n)\right) \exp\left(\frac{\lambda B(\coth \lambda - 1)}{2}\right) d\lambda \\
&= \int_0^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{n+k-1}{k} \lambda^n \exp\left(\lambda(A - 2k - n)\right) \frac{(\lambda B(\coth \lambda - 1))^j}{2^j j!} d\lambda \\
&= \int_0^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{n+k-1}{k} \lambda^n \exp\left(\lambda(A - 2k - j - n)\right) \frac{(\lambda B)^j}{j!(e^{\lambda} - e^{-\lambda})^j} d\lambda
\end{aligned}
$$

$$= \int_0^\infty \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \binom{n+k-1}{k}\binom{j+l-1}{l} \lambda^{n+j} \exp\left(\lambda\big(A-n-2(k+j+l)\big)\right) \frac{B^j}{j!}\, d\lambda$$

$$= n! \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \binom{n+k-1}{k}\binom{j+l-1}{l}\binom{n+j}{j} \frac{B^j}{(n+2(k+j+l)-A)^{n+j+1}}\ .$$

Note that, for any compact subset K of $\mathbb{H}_n(\mathbb{C}) \setminus S$, the sum in the last line of (4) is absolutely uniformly convergent on K and that for all $(x,y,t) \in \mathbb{H}_n(\mathbb{R})$ all expressions in the formula (4) are absolutely convergent. Indeed, there exists a constant C_K such that

$$\left| \big(n+2(k+j+l)-A\big)^{-n-j-1} \right| \leqslant C_K(k+j+l+1)^{-n-j-1} \quad \forall (x,y,t) \in K.$$

Next,

$$\binom{n+k-1}{k} \leqslant C_n k^{n-1} \quad \text{and} \quad \binom{j+l-1}{l} \leqslant (j+k+l+1)^{j-1}/j!.$$

Hence

$$\sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \binom{n+k-1}{k}\binom{j+l-1}{l}\binom{n+j}{j}\left| \frac{B^j}{\big(n+2(k+j+l)-A\big)^{n+j+1}} \right|$$

$$\leqslant C_n C_{n+1} C_K \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \frac{k^{n-1}j^n |B|^j}{j!(k+j+l+1)^{n+2}}$$

$$\leqslant C_n' C_K \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{k^{n-1}j^n |B|^j}{j!(k+j+1)^{n+1}}$$

$$\leqslant C_n' C_K \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{|B|^j j^n}{j!(k+j+1)^2}$$

$$\leqslant C_n'' C_K \sum_{j=0}^\infty \frac{|B|^j j^n}{(j+1)!}$$

$$\leqslant C_{K,n}\ .$$

Now we note that, for any $(x,y,t) \in \mathbb{H}_n(\mathbb{R})$,

$$\left| \big(n+2(k+j+l)-A\big) \right| \geqslant \left(n+2(k+j+l)-\frac{B}{2} \right),$$

and $\sup_{\lambda \in \mathbb{R}_+} \lambda(\coth \lambda - 1) = 1$, so

$$n! \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \binom{n+k-1}{k}\binom{j+l-1}{l}\binom{n+j}{j}\left| \frac{B^j}{(n+2(k+j+l)-A)^{n+j+1}} \right|$$

$$\leqslant n! \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \binom{n+k-1}{k}\binom{j+l-1}{l}\binom{n+j}{j} \frac{(-B)^j}{(n+2(k+j+l)-B/2)^{n+j+1}}$$

$$= \int_0^\infty \sum_{k=0}^\infty \binom{n+k-1}{k} \lambda^n \exp\left(-\lambda(2k+n-B/2)\right) \exp\left(\frac{-\lambda B(\coth \lambda - 1)}{2}\right) d\lambda$$

$$\leqslant 2^n \int_0^\infty \frac{\lambda^n}{(\sinh \lambda)^n} \exp\left(\frac{B(\lambda-1)}{2}\right) d\lambda.$$

The Lebesgue monotone convergence theorem proves the absolute convergence for all $(x,y,t) \in \mathbb{H}_n(\mathbb{R})$.

We obtain (2) by virtue of the identity

$$\sum_{\substack{k \geqslant 0 \\ l \geqslant 0 \\ k+l=h}} \binom{n+k-1}{k}\binom{m+l-1}{l} = \binom{n+m+h-1}{h}$$

and (4). To conclude the proof of (3) we note that L is a homogeneous operator so (3) holds when $s > 0$ (see [6, Proposition 7.3, p.71]) and both sides of (3) are analytic as a function of s when $\operatorname{Re} s > 0$.   □

COROLLARY 2. *When $s \in \mathbb{R}$, the convolution kernel of the Schrödinger operator p_{is} is smooth on $\mathbb{H}_n(\mathbb{R}) \setminus S_s$, where $S_s = \{(0,0,\pm sk) \in \mathbb{H}_n(\mathbb{R}) : k = n, n+2, n+4, \dots\}$. At every point of S_s, the kernel p_{is} has a singularity of Calderón–Zygmund type.*

PROOF: Corollary 2 is a straightforward consequence of (3), and (4) or (2).   □

REMARK 1. There is an alternative proof of Theorem 1. One can use the formula [5, (2.28)] to prove that $P(x,y,t)$ is equal to

$$2^n \sum_{k=0}^\infty \left(\frac{(n+k)!}{k!}\frac{(B+(-A+n+2k))^k}{(-A+n+2k)^{n+1+k}}\right)\left(1 + \frac{k}{n+k}\frac{A-n-2k}{B+(-A+n+2k)}\right).$$

An argument similar to that in the proof of Theorem 1 shows that the above sum is uniformly absolutely convergent on any compact subset of $\mathbb{H}_n(\mathbb{C}) \setminus S$.

REMARK 2. In [6, (7.53), p.73], Taylor noticed that, when $s > 0$ and $|\operatorname{Re} \alpha| < n$,

$$(5) \qquad\qquad K_{e^{-sL_\alpha}}(x,y,t) = p_s(x,y,t-is\alpha),$$

where $K_{e^{-sL_\alpha}}$ is the convolution kernel of the operator e^{-sL_α} and $L_\alpha = L + i\alpha T$. By virtue of Theorem 1, (5) holds also when $|\operatorname{Re} \alpha| = n$. Note that, when $|\operatorname{Re} \alpha| = n$, the kernel $K_{e^{-sL_\alpha}}$, $s > 0$ is no longer smooth and it has a singularity of Calderón–Zygmund type at the point $(0,0,s\operatorname{Im}\alpha)$.

We would like to end with another observation, concerning a full Laplace operator on the Heisenberg group. We define this operator by the following formula

$$\widetilde{L} = -\sum_{r=1}^n X_r^2 + Y_r^2 - T^2 = L - T^2.$$

THEOREM 3. *For any $s \in \mathbb{R} \setminus \{0\}$, the convolution kernel of the operator $e^{-is\widetilde{L}}$ extends to an analytic function on $\mathbb{H}_n(\mathbb{C})$.*

PROOF: Write γ for $e^{i\pi/4}$. We denote the convolution kernels of the operators $e^{-is\widetilde{L}}$ and $e^{-s(\gamma L - T^2)}$ by $K_{e^{-is\widetilde{L}}}$ and $K_{e^{-s(\gamma L - T^2)}}$ respectively. Note that (see [6, (7.36)] and (3))

$$K_{e^{-s(\gamma L + T^2)}}(x, y, t) = \frac{1}{(4\pi\gamma)^{n+1}} \int_0^\infty \frac{\lambda^n \cos(t\lambda/(\gamma s))}{(\sinh \lambda)^n} \exp\left(\frac{-\lambda(x^2 + y^2)\coth \lambda}{4\gamma s}\right) e^{-s\lambda^2} \, d\lambda.$$

Hence $K_{e^{-s(\gamma L + T^2)}}$ is an analytic function on $\mathbb{H}_n(\mathbb{C})$. Now to finish the proof it is enough to note that

$$K_{e^{-is\widetilde{L}}}(x, y, t) = K_{e^{-s(\gamma \cdot \gamma L + \gamma^2 T^2)}}(x, y, t) = \frac{1}{\gamma^{n+1}} K_{e^{-s(\gamma L + T^2)}}\left(\frac{x}{\sqrt{\gamma}}, \frac{y}{\sqrt{\gamma}}, \frac{t}{\gamma}\right)$$

for all $s > 0$ and $(x, y, t) \in \mathbb{H}_n(\mathbb{R})$ (see [4, (18)]). $\qquad \qquad \square$

REMARK 3. When $\operatorname{Re} s > 0$, one can use the theory of analytic hypoellipticity to investigate the smoothness and analyticity of the kernel $K_{e^{-s\widetilde{L}}}$ see, for example, [**1, 2**]. However, analytic hypoellipticity cannot be used directly to investigate the convolution kernel of $K_{e^{-s\widetilde{L}}}$ when $\operatorname{Re} s = 0$.

REFERENCES

[1]   M. Christ, 'Analytic hypoellipticity, representations of nilpotent groups, and a nonlinear eigenvalue problem', *Duke Math. J.* **72** (1993), 595–639.

[2]   M. Christ, 'Nonexistence of invariant analytic hypoelliptic differential operators on nilpotent groups of step greater than two', in *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, Princeton Math. Series **42** (Princeton Univ. Press, Princeton, NJ, 1995), **pp.** 127–145.

[3]   G.B. Folland and E.M. Stein, 'Estimates for the $\overline{\partial}_b$ complex and analysis on theHeisenberg group', *Comm. Pure Appl. Math.* **27** (1974), 429–522.

[4]   A. Sikora, 'On the $L^2 \to L^\infty$ norms of spectral multipliers of "quasi-homogeneous" operators on homogeneous groups', *Trans. Amer. Math. Soc.* **351** (1999), 3743–3755.

[5]   R.S. Strichartz, 'L^p harmonic analysis and Radon transforms on the Heisenberg group', *J. Funct. Anal.* **96** (1991), 350–406.

[6]   M.E. Taylor, *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs **22** (American Mathematical Society, Providence, R.I., 1986).

Department of Mathematical Sciences   Instytut Matematyczny
New Mexico State University   Uniwersytet Wrocławski
PO Box 30001   50-384 Wrocław
Las Cruces NM 88003   pl. Grunwaldzki 2/4
Untied States of America   Poland
e-mail: asikora@nmsu.edu   e-mail: zenek@math.uni.wroc.pl