EQUIVALENT FORMULATIONS OF THE BORSUK-ULAM THEOREM

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1. Introduction. Let R^0 denote a space consisting of just one point and for each positive integer n let R^n denote euclidean n-space. For each non-negative integer n let S^n denote the n-sphere

$$\left\{(x_1, \ldots, x_{n+1}) \in R^{n+1} : \sum_{i=1}^{n+1} x_i^{\,2} = 1\right\}.$$

In 1933 K. Borsuk published proofs of the following two theorems (**2**, p. 178).

THEOREM (Borsuk–Ulam). *If n is a non-negative integer and f is a continuous function from S^n into R^n, there is a point p in S^n such that $fp = f(-p)$.*

THEOREM (Lusternik–Schnirelmann). *If n is a non-negative integer, S^n cannot be covered by $n + 1$ closed sets, none containing a pair $(p, -p)$ of diametrically opposite points.*

Since both theorems are true, they are, of course, logically equivalent. But if their hypotheses are suitably weakened, the resulting statements can be shown to be equivalent in a more interesting sense.

Definition. A *T-space* is a topological space X such that T is a continuous involution on X, i.e., T is a homeomorphism from X onto X such that $TTp = p$ for each point p in X. If x is a point of the T-space, X, (x, Tx) is called an *antipodal pair.*

Example. If n is a non-negative integer and $T : S^n \to S^n$ is the map such that $Tp = -p$, $p \in S^n$, then S^n is a T-space.

For each non-negative integer n and each T-space X we let each of $B_n(X)$ and $L_n(X)$ denote a sentence:

$B_n(X)$. *If f is a continuous function from X into R^n, there is a point p in X such that $fp = fTp$.*

$L_n(X)$. *If each of $C_1, \ldots, C_{n+1}$ is a closed subset of X and contains no antipodal pair, then*

$$X \neq \bigcup_{i=1}^{n+1} C_i.$$

It is known that, if X is a normal T-space and n is a non-negative integer,

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then $B_n(X)$ if and only if $L_n(X)$. A proof of this and similar theorems is contained herein. A number of previously published theorems concerning spheres are reformulated as properties that are equivalent to each other in normal T-spaces. A more exact description of the relation of this paper to previous work is contained in §3 below.

Definitions. Suppose X is a T-space. If $A \subseteq X$ and $A = TA$, A is called an *antipodal subset* of X. A subset S of X is said to be *severed* if there is a subset H of S, closed in S, such that $S = H \cup TH$ and $H \cap TH = \emptyset$. A collection of severed sets will be said to be *severed.*

Remarks. A severed set is antipodal.

If X is a T-space and A is an antipodal subset of X, the restriction of T to A makes the subspace A a T-space.

For reference purposes we list two theorems on normal spaces.

(1.1) *If H and K are disjoint closed subsets of a normal space X, there is a continuous function from X into the number interval* $[0, 1]$ *that assumes the value* 0 *on each point of H and the value* 1 *on each point of K* (**1**, p 74).

(1.2) *If $\{D_1, \ldots, D_r\}$ is a finite open cover of a normal space X, there is a collection $\{C_1, \ldots, C_r\}$ of closed sets such that $C_i \subseteq D_i$, $i \in \{1, \ldots, r\}$, and*

$$\bigcup_{i=1}^{r} C_i = X \qquad (\mathbf{1}, \text{p. } 73).$$

2. Principal definitions and theorems. For each non-negative integer n and each T-space X we define sentences $C_n(X)$ and $O_n(X)$.

$C_n(X)$. *X cannot be covered by fewer than $n + 1$ closed severed sets.*

$O_n(X)$. *X cannot be covered by fewer than $n + 1$ open severed sets.*

(2.1) *$C_n(X)$ implies $O_n(X)$, if X is a normal T-space and n is a non-negative integer.*

Proof. Suppose $r < n + 1$ and there is an open severed cover $\{D_1, \ldots, D_r\}$ of X. (1.2) implies the existence of a closed severed cover $\{C_1, \ldots, C_r\}$ of X, which contradicts $C_n(X)$.

For each T-space X we define a sentence $Y_n(X)$ recursively.

$Y_0(X)$. $X \neq \emptyset$.

$Y_n(X)$, $n > 0$. *If F is a closed subset of X such that $F \cup TF = X$, then* $Y_{n-1}(F \cap TF)$.

$Y_n(X)$, $n > 0$, can be reformulated thus:

$Y_n(X)$, $n > 0$. *If D is an open severed subset of X, then* $Y_{n-1}(X - D)$.

Since $\emptyset$ is an open severed set, we have

(2.2) *$Y_n(X)$ implies $Y_{n-1}(X)$ if X is a T-space and n is a positive integer.*

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(2.3) *$O_n(X)$ implies $Y_n(X)$ if X is a normal T-space and n is a non-negative integer.*

Proof. Induction on n. The case $n = 0$ is trivial. Suppose (a) $n > 0$, (b) X is a normal T-space, (c) $O_n(x)$, (d) $O_{n-1}(Z)$ implies $Y_{n-1}(Z)$ for all normal T-spaces Z, (e) D_n is an open severed subset of X, (f) each of $E_1, \ldots, E_{n-1}$ is a closed severed set and

$$(g) \qquad X - D_n = \bigcup_{i=1}^{n-1} E_i.$$

Since (b), there is a severed set D_i, open in X, such that $E_i \subseteq D_i$,

$$i \in \{1, \ldots, n-1\}.$$

$\{D_1, \ldots, D_n\}$ is a covering of X by fewer than $n + 1$ open severed sets, which contradicts (c). Hence (f) is false, i.e., $C_{n-1}(X - D_n)$. By (2.1) $O_{n-1}(X - D_n)$ and by (d) $Y_{n-1}(X - D_n)$, which proves $Y_n(X)$.

If n is a non-negative integer and X is a T-space, we define

$A_n(X)$. *If each of $C_1, \ldots, C_{n+2}$ is a closed subset of X,*

$$\bigcup_{i=1}^{n+2} C_i = X, \qquad \bigcup_{i=1}^{n+2} (C_i \cap TC_i) = \emptyset,$$

and $j \in \{1, \ldots, n+1\}$, then there is a point p in X such that

$$p \in \bigcap_{i=1}^{j} C_i \quad and \quad Tp \in \bigcap_{i=j+1}^{n+2} C_i.$$

(2.4) *$Y_n(X)$ implies $A_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. Induction on n. The case $n = 0$ is trivial. Suppose that $n > 0$, X is a T-space with a closed cover $\{C_1, \ldots, C_{n+2}\}$, no C_i contains an antipodal pair, $Y_n(X)$, $j \in \{1, \ldots, n\}$, and $Y_{n-1}(Z)$ implies $A_{n-1}(Z)$ for any T-space Z. Let

$$F = \left[\left(\bigcup_{i=1}^{n} C_i\right) \cap \left(\bigcup_{i=1}^{n+1} TC_i\right)\right] \cup C_{n+2}.$$

Then

$$F \cup TF = X \quad \text{and} \quad F \cap TF = \bigcup_{i=1}^{n+1} D_i,$$

where

$$D_i = C_i \cap F \cap TF, \qquad i \in \{1, \ldots, n\},$$

and

$$D_{n+1} = C_{n+1} \cap C_{n+2} \cap F \cap TF.$$

Since $D_i \subseteq C_i$, D_i contains no antipodal pair, $i \in \{1, \ldots, n+1\}$. Since $Y_n(X)$, $Y_{n-1}(F \cap TF)$. Accordingly $A_{n-1}(F \cap TF)$. There is a point p such

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that

$$p \in \bigcap_{i=1}^{j} D_i \subseteq \bigcap_{i=1}^{j} C_i, Tp \in \bigcap_{i=j+1}^{n+1} D_i \subseteq \bigcap_{i=j+1}^{n+2} C_i.$$

The case $j = n + 1$ is a permutation of the case $j = 1$.

(2.5) *$A_n(X)$ implies $L_n(X)$ if X is a T-space and n is a non-negative integer.* This statement is obvious.

(2.6) *$L_n(X)$ implies $C_n(X)$ if X is a normal T-space and n is a non-negative integer.*

Proof. The case $n = 0$ is trivial. Suppose (a) $n > 0$, (b) X is a normal T-space such that $L_n(X)$, (c) each of $A_1, \ldots, A_n$ is a closed subset of X containing no antipodal pair, and (d) contrary to the desired conclusion,

$$\bigcup_{i=1}^{n} (A_i \cup TA_i) = X.$$

Since X is normal, for each i in $\{1, \ldots, n\}$, there is an open set D_i containing A_i such that $\text{Cl}\, D_i \cap \text{Cl}\, TD_i = \emptyset$. Let

$$Q = \left(\bigcup_{i=1}^{n} TA_i\right) - \left(\bigcup_{i=1}^{n} D_i\right).$$

Q is closed. Suppose p is a point of X not in any of $\text{Cl}\, D_1, \ldots, \text{Cl}\, D_n$. p is not in any of the A_i's. By (d),

$$p \in \bigcup_{i=1}^{n} TA_i, \qquad p \in Q.$$

Thus

$$X = Q \cup \bigcup_{i=1}^{n} \text{Cl}\, D_i.$$

Suppose $x \in Q$. Then $Tx \in A_j \subseteq D_j$ for some j. Tx is not in Q. None of $\text{Cl}\, D_1, \ldots, \text{Cl}\, D_n$, Q contains an antipodal pair. (b) is contradicted and (d) is false.

If n is a non-negative integer and X is a T-space, we define

$F_n(X)$. *Suppose m is a positive integer, $\{C_1, \ldots, C_m\}$ is a closed cover of X, and*

$$\bigcup_{i=1}^{m} (C_i \cap TC_i) = \emptyset.$$

Then $m \geqslant n + 2$ and there are $n + 2$ integers, $k_0, \ldots, k_{n+1}$ such that

$$1 \leqslant k_0 < \ldots < k_{n+1} \leqslant m$$

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and

$$\bigcap_{i=0}^{n+1} T^i C_{k_i} \neq \emptyset.$$

(2.7) *$Y_n(X)$ implies $F_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. Induction on n. The case $n = 0$ is trivial. Suppose $n > 0$, $Y_n(X)$, and the hypothesis of $F_n(X)$. Since $Y_0(X)$ (cf. (2.2)), we have $F_0(X)$ and $m \geqslant 2$. Let

$$F = \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} (C_i \cap TC_j).$$

Since $F \cup TF = X$, the inductive hypothesis gives $F_{n-1}(F \cap TF)$. Let

$$K_i = \left(C_i \cap \bigcup_{j=i+1}^{m} TC_j\right) \cap TF, \qquad i \in \{1, \ldots, m-1\}.$$

Then

$$\bigcup_{i=1}^{m-1} (K_i \cap TK_i) = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{m-1} K_i = F \cap TF.$$

Since $F_{n-1}(F \cap TF)$, $m - 1 \geqslant (n-1) + 2$ (whence $m \geqslant n + 2$) and there are $n + 1$ integers $k_0, \ldots, k_n$ such that $1 \leqslant k_0 < \ldots < k_n \leqslant m - 1$ and

$$\emptyset \neq \bigcap_{i=0}^{n} T^i K_{k_i} \subseteq \left(\bigcap_{i=0}^{n-1} T^i C_{k_i}\right) \cap \left(T^n C_{k_n} \cap T^n\left(\bigcup_{j=k_n+1}^{m} TC_j\right)\right),$$

which implies the desired conclusion.

(2.8) *$F_n(X)$ implies $L_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

If n is a non-negative integer and X is a T-space, we define

$P_n(X)$. *Suppose m is a positive integer, each of $A_1, \ldots, A_m$ is a closed subset of X,*

$$\bigcup_{i=1}^{m} (A_i \cap TA_i) = \emptyset \quad \textit{and} \quad \bigcup_{i=1}^{m} (A_i \cup TA_i) = X.$$

Then $m \geqslant n + 1$ and there is an integer sequence $k_0, \ldots, k_n$ such that

$$1 \leqslant k_0 < \ldots < k_n \leqslant m$$

and

$$\bigcap_{i=0}^{n} T^i A_{k_i} \neq \emptyset.$$

(2.9) *$Y_n(X)$ implies $P_n(X)$ if X is a T-space and n is a non-negative integer.*

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Proof. The case $n = 0$ is trivial. Suppose $n > 0$, $Y_n(X)$, and the hypothesis of $P_n(X)$. Let

$$Z = \left(\bigcup_{i=1}^{m} A_i\right) \cap T\left(\bigcup_{i=1}^{m} A_i\right).$$

Since $Y_n(X)$, $Y_{n-1}(Z)$. By (2.7), $F_{n-1}(Z)$. The sets $Z \cap A_i$, $i \in \{1, \ldots, m\}$, satisfy the hypothesis of $F_{n-1}(Z)$. Hence $m \geqslant (n-1) + 2$ and there is an integer sequence $k_0, \ldots, k_n$ such that $1 \leqslant k_0 < \ldots < k_n \leqslant m$ and

$$\emptyset \neq \bigcap_{i=0}^{n} T^i(A_{k_i} \cap Z) \subseteq \bigcap_{i=0}^{n} T^i A_{k_i}.$$

$P_n(X)$ implies a statement, $R_n(X)$, which in appearance is slightly stronger than $P_n(X)$. If n is a non-negative integer and X is a T-space, we define

$R_n(X)$. *Suppose m is a positive integer, $F_1, F_{-1}, \ldots, F_m, F_{-m}$ are $2m$ closed subsets of X,*

$$\bigcup_{i=1}^{m} (F_i \cap F_{-i}) = \emptyset$$

and, for any point p in X, there is an i in $\{\pm 1, \ldots, \pm m\}$ such that $p \in F_i$ and $Tp \in F_{-i}$. Then $m \geqslant n + 1$ and there is an integer sequence $k_0, \ldots, k_n$ such that $1 \leqslant k_0 < \ldots < k_n \leqslant m$ and

$$\bigcap_{i=0}^{n} F_{(-1)^i k_i} \neq \emptyset.$$

(2.10) *$P_n(X)$ implies $R_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. Suppose $P_n(X)$ and the hypothesis of $R_n(X)$. Define A_i to be $F_i \cap TF_{-i}$. $A_1, \ldots, A_m$ satisfy the hypothesis of $P_n(X)$.

(2.11) *$R_n(X)$ implies $P_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

If n is a non-negative integer and X is a T-space, we define

$V_n(X)$. *If each of $B_1, \ldots, B_{n+1}$ is a closed subset of X containing no antipodal pair and*

$$\bigcup_{i=1}^{n+1} (B_i \cup TB_i) = X,$$

then

$$\bigcap_{i=1}^{n+1} B_i \neq \emptyset.$$

(2.12) *$P_n(X)$ implies $V_n(X)$ if n is a non-negative integer and X is a T-space.*

Proof. Set $m = n + 1$ and let $A_i = T^{i+1}B_i$, $i \in \{1, \ldots, m\}$.

If n is a non-negative integer and X is a T-space, we define

$T_n(X)$. *Suppose that each of* $A_1, A_{-1}, \ldots, A_{n+1}, A_{-n-1}$ *is a closed subset of* X *containing no antipodal pair,* $A_i \cap A_{-1} = \emptyset$, $i \in \{1, \ldots, n+1\}$, *and*

$$\bigcup_{i=1}^{n+1} (A_i \cup A_{-i}) = X.$$

Then

$$\bigcap_{i=1}^{n+1} A_i \neq \emptyset.$$

(2.13) $V_n(X)$ *implies* $T_n(X)$ *if* X *is a* T-*space and* n *is a non-negative integer.*

Proof. Suppose (a) each of $A_1, A_{-1}, \ldots, A_{n+1}, A_{-n-1}$ is a closed subset of the T-space X,

(b) $$\bigcup_{i=1}^{n+1} (A_i \cap TA_i) = \bigcup_{i=1}^{n+1} (A_{-i} \cap TA_{-i}) = \bigcup_{i=1}^{n+1} (A_i \cap A_{-i}) = \emptyset,$$

(c) $$X = \bigcup_{i=1}^{n+1} (A_i \cup A_{-i})$$

and (d) $V_n(X)$.

Let $B_1 = A_1$ and, if $2 \leqslant i \leqslant n+1$, let

$$B_i = A_i \cup \left(TA_{-i} \cap \bigcup_{j=1}^{i-1} A_{-j} \right).$$

Then

$$\bigcup_{j=1}^{n+1} (B_j \cup TB_j) = X.$$

Since A_i does not intersect either TA_i or A_{-i} (see (b)), B_i does not intersect TB_i, $i \in \{1, \ldots, n+1\}$. By (d)

$$\bigcap_{i=1}^{n+1} B_i \neq \emptyset.$$

To complete the proof it will suffice to prove that

$$\bigcap_{i=1}^{n+1} B_i = \bigcap_{i=1}^{n+1} Ai.$$

By an induction on i it will be shown that

$$\bigcap_{j=1}^{i} B_j = \bigcap_{j=1}^{i} A_j, \qquad i \in \{1, \ldots, n+1\}.$$

If $i = 1$, the assertion holds by the definition of B_1. Suppose $2 \leqslant i \leqslant n+1$ and it is known that

$$\bigcap_{j=1}^{i-1} B_j = \bigcap_{j=1}^{i-1} A_j.$$

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Then

$$\bigcap_{j=1}^{i} B_j = \left(\bigcap_{j=1}^{i-1} A_j\right) \cap \left[A_i \cup \left(TA_{-i} \cap \bigcup_{j=1}^{i-1} A_{-j}\right)\right]$$

$$= \left(\bigcap_{j=1}^{i} A_j\right) \cup \left[\left(\bigcap_{j=1}^{i-1} A_j\right) \cap TA_{-i} \cap \left(\bigcup_{j=1}^{i-1} A_{-j}\right)\right].$$

By (b),

$$\left(\bigcap_{j=1}^{i-1} A_j\right) \cap \left(\bigcup_{j=1}^{i-1} A_{-j}\right) = \emptyset.$$

Hence

$$\bigcap_{j=1}^{i} B_j = \bigcap_{j=1}^{i} A_j.$$

(2.14) *$T_n(X)$ implies $V_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

(2.15) *$V_n(X)$ implies $C_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

If n is a non-negative integer and X is a T-space, we define

$H_n(X)$. If X is covered by a finite collection of closed severed sets, some $n + 1$ members of the collection have a point in common.

(2.16) *$P_n(X)$ implies $H_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

If n is a non-negative integer and X is a T-space, we define

$J_n(X)$. If X is covered by a finite collection of open severed sets, some $n + 1$ of them have a point in common.

(2.17) *$H_n(X)$ implies $J_n(X)$ if X is a normal T-space and n is a non-negative integer.*

Proof. Suppose (a) X is a normal T-space, (b) m is a positive integer, (c) each of $D_1, \ldots, D_m$ is an open severed subset of X,

(d)
$$\bigcup_{i=1}^{m} D_i = X$$

and (e) $H_n(X)$.

(1.2) implies there are closed severed sets $C_1, \ldots, C_m$ such that $C_i \subseteq D_i$, $i \in \{1, \ldots, m\}$, and

$$\bigcup_{i=1}^{m} C_i = X.$$

By (e) there is a subset A of $\{1, \ldots, m\}$ of cardinality $n + 1$ such that

$$\emptyset \neq \bigcap_{i \in A} C_i \subseteq \bigcap_{i \in A} D_i.$$

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(2.18) *$J_n(X)$ implies $O_n(X)$ if X is a T-space and n is a non-negative integer.*

This is obvious.

If n is a non-negative integer and X is a T-space, we define the sentence $Z_n(X)$.

$Z_0(X)$. $X \neq \emptyset$.

$Z_n(X)$, $n > 0$. *There is no continuous map f from X into S^{n-1} such that $Tfp = fTp$ for each p in X.*

(2.19) *$O_n(X)$ implies $Z_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. This is obvious if $n = 0$. Suppose $n > 0$, X is a T-space such that $O_n(X)$, and f is a continuous function from X into S^{n-1} such that $Tfp = fTp$ for each p in X. S^{n-1} is covered by the n severed open sets $D_1, \ldots, D_n$, where

$$D_i = \{(x_1, \ldots, x_n) \in S^{n-1} : x_i \neq 0\}, \qquad i \in \{1, \ldots, n\}.$$

$\{f^{-1}D_1, \ldots, f^{-1}D_n\}$ is a covering of X by open severed sets, contrary to $O_n(X)$.

For each non-negative integer n and T-space X we define

$E_n(X)$. *If f is a continuous function from X into R^n such that $fTp = -fp$ for each p in X, then, for some p in X, $fp = 0$.*

(2.20) *$Z_n(X)$ implies $E_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. Suppose n is a non-negative integer, X is a T-space such that $Z_n(X)$, and f is a continuous function from X into R^n such that $fTp = -fp \neq 0$ for each p in X. Let $g : (R^n - \{0\}) \to S^{n-1}$ be the function such that

$$g(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^2\right)^{-\frac{1}{2}} \cdot (x_1, \ldots, x_n).$$

Then $gf : X \to S^{n-1}$ is a continuous function such that $fTp = Tfp$, $p \in X$, which contradicts $Z_n(X)$.

(2.21) *$E_n(X)$ implies $B_n(X)$ if X is a T-space and n is a non-negative integer.*

Proof. This is obvious if $n = 0$. Suppose $n > 0$, X is a T-space such that $E_n(X)$, and $g : X \to R^n$ is continuous. Let $f : X \to R^n$ denote the function such that $fp = gp - gTp$, $p \in X$. For each p in X, $fp = -fTp$. Since $E_n(X)$, there is a p^* in X such that $fp^* = 0$. $gp^* = gTp^*$.

(2.22) *$B_n(X)$ implies $C_n(X)$ if X is a normal T-space and n is a non-negative integer.*

Proof. The case $n = 0$ is trivial. Suppose $n > 0$, X is a normal T-space, $B_n(X)$, each of $A_1, \ldots, A_n$ is a closed subset of X, and

$$\bigcup_{i=1}^{n} (A_i \cap TA_i) = \emptyset.$$

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By (1.1), for each i in $\{1, \ldots, n\}$, there is a continuous function f_i from X into $[0, 1]$ such that f_i assumes the value 0 at each point of A_i and the value 1 at each point of TA_i. $f = (f_1, \ldots, f_n)$ is a continuous function from X into euclidean n-space. Since $B_n(X)$, there is a point p in X such that $fp = fTp$. If, for some i, p were in A_i or TA_i, then $f_i p = f_i Tp$, which contradicts the construction of f_i. Hence p is not in

$$\bigcup_{i=1}^{n} (A_i \cup TA_i).$$

Most of the information contained in Theorems (2.1) to (2.22) is summarized by

(2.23) *If n is a non-negative integer and X is a normal T-space, then the following are equivalent:* $C_n(X)$, $O_n(X)$, $Y_n(X)$, $A_n(X)$, $L_n(X)$, $F_n(X)$, $P_n(X)$, $R_n(X)$, $V_n(X)$, $T_n(X)$, $H_n(X)$, $J_n(X)$, $Z_n(X)$, $E_n(X)$, $B_n(X)$.

3. Remarks. $L_n(S^n)$ was stated in 1930 (**8**, p. 26). $B_n(S^n)$ was stated in 1933 (**2**, Satz II, p. 178). A weakened form of $A_n(S^n)$ was stated in 1935 (**1**, Satz X, p. 487). $A_2(S^2)$ and $T_2(S^2)$ were stated and the higher-dimensional cases hinted at by Tucker in 1945 (**9**, pp. 295, 298–299). $R_n(S^n)$ and $F_n(S^n)$ were stated in 1952 (**4**, Theorem 1, p. 435; Theorem 2, p. 436). In 1960 Hadwiger stated $H_n(S^n)$ and, in a slightly different form, $J_n(S^n)$ (**6**, Satz 1, p. 52; Satz II, p. 53). $Y_n(X)$ was suggested by a theorem due to Yang (**10**, (4.1), p. 270). Yang proved the equivalence of various properties of bicompact Hausdorff T-spaces in which T has no fixed point (**10**, (4.4), (4.5), pp. 271–272). $Z_n(X)$ plays a fundamental role in (**11**). Conner and Floyd stated without proof the equivalence of some properties of normal fixed-point-free T-spaces (**3**, (3.4), p. 421). Most of these references have additional theorems on the n-sphere, which perhaps could have extended the list of properties considered here. A quick and elementary proof of $P_n(S^n)$ can be effected by first proving a combinatorial theorem due to Fan (**5**, Theorem 2, p. 370) and then passing to the continuous case in the usual manner (cf., e.g., Hadwiger's proof of $H_n(S^n)$; **6**, pp. 54–56). Homological properties of a T-space X sufficient to imply $B_n(X)$ have been given by Yang (**10**, (4.6), p. 272) and Jaworowski (**7**, Theorem 7, p. 252).

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