

NON-EXPANSIVE DERIVED HORSESHOES

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Abstract

For a two parameter family of C^3 diffeomorphisms having a homoclinic orbit of tangency derived from a horseshoe, the relationship between the measure of the parameter values at which the diffeomorphism (restricted to a certain compact invariant set containing the horseshoe) is not expansive and the Hausdorff dimension of the horseshoe associated to the homoclinic orbit of tangency is investigated. This is a simple application of the Newhouse-Palis-Takens-Yoccoz theory.

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0. Introduction

We say that a homeomorphism f of a compact metric space (X, d) is *expansive* if there is a constant $\epsilon > 0$ such that for every $\mathbf{x}, \mathbf{y} \in X$, $d(f^n(\mathbf{x}), f^n(\mathbf{y})) \leq \epsilon$ ($\forall n \in \mathbf{Z}$) implies $\mathbf{x} = \mathbf{y}$. This notion is the most fundamental property in the stability theory of dynamical systems and the property has been investigated by several researchers. In light of the bifurcation theory, the non-expansivity of the Hénon family $H_{a,b}(x, y) = (y, y^2 - bx - a)$, $(x, y) \in \mathbf{R}^2$ was investigated by Milnor [1] and it is proved that for every determinant b of $H_{a,b}$, there exists a countable set $\Sigma_b \subset \mathbf{R}$ such that $H_{a,b}$ can be expansive (on a certain bounded invariant set containing the non-wandering set) only if $h(H_{a,b}) \in \Sigma_b$. Here $h(H_{a,b})$ denotes the topological entropy of $H_{a,b}$.

In this paper, we construct a C^3 two parameter family $\{f_{s,t}\}_{(s,t) \in (-\eta, \eta)^2}$ exhibiting a homoclinic orbit of tangency derived from a Smale's horseshoe on a sphere S^2 in

Section 1 (see also [4, pp. 93–99]), and investigate the non-expansivity of the family restricted to a certain compact invariant subset containing the horseshoe. We call the above family the *two parameter horseshoe family* for convenience. The two parameter horseshoe family which is well known as a prototype of the theory developed in [2–5] satisfies the following properties ([4, Chapter 5]):

- for every $s \in (-\eta, \eta)$, there is a (t -independent) hyperbolic basic set (horseshoe) $\Lambda(s)$ whose Hausdorff dimension $HD(\Lambda(s))$ varies depending only on parameter s ;
- for every $s \in (-\eta, \eta)$, there is a single orbit $\mathcal{O}(s)$ of homoclinic tangency (associated with $\Lambda(s)$) along $t = 0$, and this tangency has a quadratic order of contact and unfolds generically for $t > 0$ into two transversal intersections;
- for every $(s, t) \in (-\eta, \eta)^2$, there is an $f_{s,t}$ -invariant compact subset $X_{s,t}$ of the non-wandering set of $f_{s,t}$ such that for every $s \in (-\eta, \eta)$, $X_{s,0} = \Lambda(s) \cup \mathcal{O}(s)$ and $X_{s,t} = \Lambda(s)$ if $t \in (-\eta, 0)$.

Thus for every $s \in (-\eta, \eta)$, $f_{s,t} : X_{s,t} \rightarrow X_{s,t}$ is expansive when $t \in (-\eta, 0]$. Notice that by choosing suitable expanding and contracting rates of a horseshoe, for any $\varepsilon \in (0, 2)$, we can construct a two parameter horseshoe family $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$ satisfying $HD(\Lambda(0)) = \varepsilon$.

Our aim is to study the relationship between the measure of the parameter values $t \in (0, \eta)$ such that $f_{s,t} : X_{s,t} \rightarrow X_{s,t}$ is not expansive and the Hausdorff dimension of $\Lambda(s)$. At first, by using Newhouse’s results stated in [2] and [6] we show that $f_{s,t} : X_{s,t} \rightarrow X_{s,t}$ is not expansive for lots of $t > 0$.

THEOREM A. *For any two parameter horseshoe family $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$ and for any $s \in (-\eta, \eta)$, there is a sequence of intervals $\{I_n\}_{n=0}^\infty \subset (0, \eta)$ in t -parameter space such that $I_n \rightarrow 0$ as $n \rightarrow \infty$ and $f_{s,t} : X_{s,t} \rightarrow X_{s,t}$ is not expansive for $t \in I_n$.*

For any family $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$ and for any $s \in (-\eta, \eta)$, denote by $NE(s)$ the set of all parameter values $t > 0$ such that $f_{s,t} : X_{s,t} \rightarrow X_{s,t}$ is not expansive. Then, by applying the results proved by Palis and Takens [3] and Palis and Yoccoz [5] the following is obtained.

THEOREM B. *For any two parameter horseshoe family $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$,*

- (1) *if $HD(\Lambda(0)) < 1$, then there exists a constant $\eta_0 > 0$ such that for all $s \in (-\eta_0, \eta_0)$,*

$$\lim_{t \rightarrow 0} \frac{m(NE(s) \cap [0, t])}{t} = 0,$$

- (2) *if $HD(\Lambda(0)) > 1$, then there are constants $\eta_0 > 0$ and $c > 0$ such that for*

almost all $s \in (-\eta_0, \eta_0)$,

$$\limsup_{t \rightarrow 0} \frac{m(NE(s) \cap [0, t])}{t} > c.$$

Here m is the Lebesgue measure.

Theorem A is also true for the case when $\{f_{s,t}\}_{(s,t) \in (-\eta, \eta)^2}$ is C^2 . To see this, we need to repeat the argument given in [6, Sections 7 and 8] for the C^2 family near $\mathcal{O}(s)$. In this paper, to avoid such a technical routine, we shall assume a C^3 differentiability. On the other hand, for a proof of Theorem B we do not need a C^3 differentiability. The conclusions are proved in Section 3 under the C^2 assumption.

1. Two parameter family derived from a horseshoe

In this section, we construct a two parameter horseshoe family on S^2 stated in the introduction by following [8, Section II]. Take a small $\alpha_0 > 0$ and put $I = [0, 1 + \alpha_0] \subset \mathbf{R}$. Fix $0 < \nu < \lambda^{-1} < (2|I|)^{-1}$. For β_0 sufficiently small so that $0 < 2\beta_0 \ll 1 - 2\lambda^{-1}$, let

$$H_0 = I \times [0, \lambda^{-1} + \beta_0], \quad H_1 = I \times [1 - \lambda^{-1} - \beta_0, 1]$$

be two horizontal strips in \mathbf{R}^2 and $y_0 = 1 + \beta_0\lambda/2$. Put a horizontal strip $H_2 = [0, 1] \times [y_0 - \gamma_0, y_0 + \gamma_0]$ and $x_0 = 1 + \alpha_0/2$, where $0 < 2\gamma_0 \ll \min\{\beta_0\lambda, \alpha_0\}$. Then $[x_0 - \gamma_0, x_0 + \gamma_0] \times [0, 1] \subset (I \times [0, 1]) \setminus [0, 1]^2$. Finally, set a vertical line $\ell = \{x_0\} \times [0, 1]$ and denote $(x_0, 0)$ by \mathbf{x}_0 .

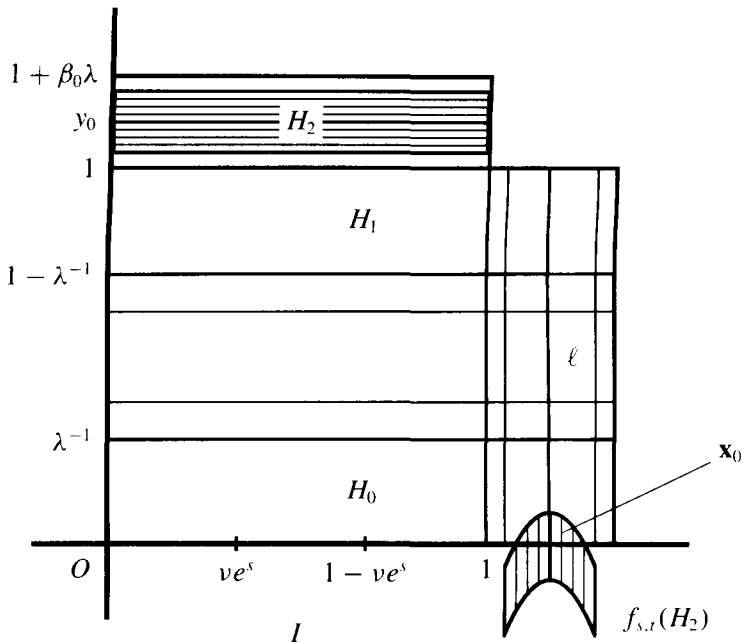
Fix $0 < 1 - 2\lambda^{-1} < c_0 < 1$. Since $S^2 \approx \mathbf{R}^2 \cup \{\infty\}$, for a sufficiently small $\eta > 0$, we can construct a two parameter family of C^3 diffeomorphisms $f_{s,t} : S^2 \rightarrow S^2$ ($(s, t) \in (-\eta, \eta)^2$) (that is, $F(x, s, t) : S^2 \times (-\eta, \eta)^2 \rightarrow S^2$ is a C^3 map such that $F(\cdot, s, t) = f_{s,t}(\cdot)$ is a diffeomorphism) satisfying the following five assertions (for example [4, pp. 93–99]).

- $f_{s,t}(x, y) = \begin{cases} (ve^x, \lambda y) & \text{if } (x, y) \in H_0, \\ (1 - ve^x, \lambda(1 - y)) & \text{if } (x, y) \in H_1, \\ (x_0 - y_0 + y, t - c_0x - (y - y_0)^2/2) & \text{if } (x, y) \in H_2. \end{cases}$

- There are a sink $\mathbf{p}_1 = f_{s,t}(\mathbf{p}_1)$ and a source $\mathbf{p}_2 = f_{s,t}(\mathbf{p}_2) = \{\infty\}$ for all $(s, t) \in (-\eta, \eta)^2$.

- The non-wandering set $\Omega(f_{s,t}) = \Lambda(s) \cup \{\mathbf{p}_1, \mathbf{p}_2\}$ of $f_{s,t}$ is hyperbolic for all $t \in (-\eta, 0)$ and $s \in (-\eta, \eta)$, where $\Lambda(s) \subset H_0 \cup H_1$ is a horseshoe of $f_{s,t}$. Note that $\Lambda(s)$ is independent of the parameter t .

- $\Omega(f_{s,0}) = \Lambda(s) \cup \{\mathbf{p}_1, \mathbf{p}_2\} \cup \mathcal{O}(s)$ for all $s \in (-\eta, \eta)$, where $\mathcal{O}(s) = \bigcup_{n=-\infty}^{\infty} f_{s,0}^n(\mathbf{x}_0)$.



- $\Lambda(0) \cup \mathcal{O}(0)$ is the maximal $f_{0,0}$ -invariant set in some neighborhood of it.

The horseshoe $\Lambda(s)$ of $f_{s,t}$ is a direct product of a horizontal Cantor set $\Lambda_1(s) \subset [0, 1] \times \{0\}$ and a vertical Cantor set $\Lambda_2 \subset \{0\} \times [0, 1]$ that is, $\Lambda(s) = \Lambda_1(s) \times \Lambda_2$. Denote by $\tau(C)$ a thickness of a Cantor set $C \subset \mathbf{R}$ introduced in [2], and by $HD(C)$ the Hausdorff dimension of C . Then, these quantities are also independent of t . Actually,

$$\tau(\Lambda_1(s)) = \frac{ve^s}{1 - 2ve^s}, \quad \tau(\Lambda_2) = \frac{1}{\lambda - 2}$$

and

$$HD(\Lambda_1(s)) = -\frac{\log 2}{s + \log v}, \quad HD(\Lambda_2) = \frac{\log 2}{\log \lambda}$$

(see [2, pp. 106–107]). Recall that $HD(\Lambda(s)) = HD(\Lambda_1(s)) + HD(\Lambda_2)$. Let

$$\Gamma_1(s) = \{(x, y_0) : (x, 0) \in \Lambda_1(s)\} \subset [0, 1] \times \{y_0\} \quad (s \in (-\eta, \eta))$$

and

$$\Gamma_2 = \{(x_0, y) : (0, y) \in \Lambda_2\} \subset \ell.$$

Then, these Cantor sets satisfy $\tau(\Gamma_1(s)) = \tau(\Lambda_1(s))$ and $\tau(\Gamma_2) = \tau(\Lambda_2)$. Furthermore, since $f_{s,t}$ is an affine map on $[0, 1] \times \{y_0\}$, we see that $\tau(f_{s,t}(\Gamma_1(s))) = \tau(\Gamma_1(s))$ for all $(s, t) \in (-\eta, \eta)^2$.

Let d be a usual metric on S^2 , and denote the stable manifold and the unstable manifold of $\mathbf{x} \in \Lambda(s)$ by

$$W^s(\mathbf{x}; s, t) = \{ \mathbf{y} \in S^2 : d(f_{s,t}^n(\mathbf{x}), f_{s,t}^n(\mathbf{y})) \rightarrow 0 \text{ as } n \rightarrow \infty \},$$

$$W^u(\mathbf{x}; s, t) = \{ \mathbf{y} \in S^2 : d(f_{s,t}^{-n}(\mathbf{x}), f_{s,t}^{-n}(\mathbf{y})) \rightarrow 0 \text{ as } n \rightarrow \infty \}$$

for $(s, t) \in (-\eta, \eta)^2$. Then $W^s(O; s, t)$ and $W^u(O; s, t)$ have a quadratic homoclinic tangency at $\mathbf{x}_0 \in \ell$ along $t = 0$ for all $s \in (-\eta, \eta)$. Here $O = (0, 0)$.

For our family, since S^2 is compact, there are compact sets (a filtration) $\emptyset \neq M_1 \subset M_2 \subset S^2$ for $f_{0,0}$ (see [3, p. 365]) such that

- (1) $\mathbf{p}_1 \in M_1, \Lambda(0) \cup \mathcal{O}(0) \subset M_2 \setminus \text{int}M_1, \mathbf{p}_2 \in S^2 \setminus \text{int}M_2,$
- (2) $f_{0,0}(M_i) \subset \text{int}M_i$ for $i = 1, 2,$
- (3) $\bigcap_{n=-\infty}^{\infty} f_{0,0}^n(M_2 \setminus \text{int}M_1) = \Lambda(0) \cup \mathcal{O}(0).$

Since η is small, we may assume that $f_{s,t}(M_1) \subset \text{int}M_1, f_{s,t}^{-1}((\text{int}M_2)^c) \subset M_2^c, \Omega(f_{s,t}) \cap M_1 = \{\mathbf{p}_1\}$ and $\Omega(f_{s,t}) \cap M_2^c = \{\mathbf{p}_2\}$ for all $(s, t) \in (-\eta, \eta)^2$. Finally, for all $(s, t) \in (-\eta, \eta)^2$, let $X_{s,t}$ be the set of all non-wandering points of $f_{s,t}$ contained in $M_2 \setminus \text{int}M_1$.

2. Non-expansive sequence and proof of Theorem A

Let f be a homeomorphism of a compact metric space (X, d) . We say that a sequence of points $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^{\infty} \subset X \times X \setminus \Delta$ is a *non-expansive sequence of f* if for every $\epsilon > 0$ there exists $i = i(\epsilon) > 0$ such that $d(f^n(\mathbf{x}_i), f^n(\mathbf{y}_i)) < \epsilon$ for all $n \in \mathbf{Z}$. Here Δ is the diagonal set in $X \times X$. Theorem A follows from the next proposition.

PROPOSITION. *For any two parameter horseshoe family $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$ and for any $s \in (-\eta, \eta)$, there is a sequence of intervals $\{I_n\}_{n=0}^{\infty} \subset (0, \eta)$ in t -parameter space ($I_n \rightarrow 0$ as $n \rightarrow \infty$) such that for every $t \in I_n$, there is a non-expansive sequence $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^{\infty} \subset X_{s,t} \times X_{s,t}$ of $f_{s,t}$.*

Let $\{f_{s,t}\}_{(s,t) \in (-\eta,\eta)^2}$ be a two parameter horseshoe family. To prove the proposition, it is enough to analyse the diffeomorphisms $f_{s,t} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ($(s, t) \in (-\eta, \eta)^2$). For simplicity, we give a proof for the case when $s = 0$ (the other case follows in a similar way). Denote $f_{0,t}$ by $f_t, \Lambda_1(0)$ by $\Lambda_1, W^s(\mathbf{x}; 0, t)$ by $W^s(\mathbf{x}; t)$ and so on. To prove the proposition, we show that there is a sequence of intervals $\{I_n\}_{n=0}^{\infty}$ in t -parameter space ($I_n \rightarrow 0$ as $n \rightarrow \infty$) with the property that for every $t \in I_n$, there are two curves $\gamma^s = \gamma^s(t), \gamma^u = \gamma^u(t)$ in the stable manifold $W^s(\Lambda; t)$ and unstable manifold

$W^u(\Lambda; t)$ of $\Lambda = \Lambda_1 \times \Lambda_2$ having a quadratic tangency at a point $\mathbf{z} \in \gamma^u \cap \gamma^s$ and there is a sequence of curves $\gamma_i^u = \gamma_i^u(t)$ such that

- (a) γ_i^u is in an unstable manifold $W^u(\Lambda; t)$, and γ_i^u converges to γ^u as $i \rightarrow \infty$,
- (b) γ_i^u meets γ^s transversely, and $\gamma_i^u \cap \gamma^s$ contains two distinct points $\mathbf{x}_i, \mathbf{y}_i$ which approach \mathbf{z} as $i \rightarrow \infty$.

Then, from (b) we can see that $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^\infty$ is a non-expansive sequence of f_t . To do this, we prove a lemma which is a slight modification of [2, Lemma 4]. For basic definitions on a Cantor set and its thickness see [2, pp. 106–108] or [4, pp. 61–64].

Let $C, C' \subset \mathbf{R}$ be Cantor sets and let $\{U_j\}_{j=-2}^\infty, \{U'_j\}_{j=-2}^\infty$ be the gaps of C, C' . If $\tau(C)\tau(C') > 1$, then there are defining sequences $\{C_i\}_{i=0}^\infty, \{C'_i\}_{i=0}^\infty$ of C, C' such that $\tau(\{C_i\}_{i=0}^\infty)\tau(\{C'_i\}_{i=0}^\infty) > 1$. We say that $x \in C$ is a *right-limit-point* (respectively *left-limit-point*) of C if there is a sequence $\{x_n\} \subset C$ such that $x_n > x$ (respectively $x_n < x$) and $x_n \rightarrow x$ as $n \rightarrow \infty$. Clearly, every $x \in C$ is either a right-limit-point or a left-limit-point of C .

LEMMA 2.1 (see [2, Lemma 4]). *Let C and C' be Cantor sets in \mathbf{R} with C in no C' -gap closure and C' in no C -gap closure, and let $\tau(C)\tau(C') > 1$. Then there exists $x \in C \cap C'$ which is either a right-limit-point of both C and C' or a left-limit-point of both C and C' .*

PROOF. Let $\{U_j\}_{j=-2}^\infty, \{U'_j\}_{j=-2}^\infty$ be the gaps of C, C' , and let $\{C_i\}_{i=0}^\infty, \{C'_i\}_{i=0}^\infty$ be defining sequences of C, C' such that $\tau(\{C_i\}_{i=0}^\infty)\tau(\{C'_i\}_{i=0}^\infty) > 1$. Put $c_0 = C_0$. Then, by [2, Gap Lemma] there is a component c_1 of $c_0 \cap C_1$ such that c_1 is not contained in \bar{U}_j for all $j \geq -2$. Thus $c_1 \cap C'_1 \neq \emptyset$. By induction, for every $i > 0$ we can find a component $c_i \subset c_{i-1} \cap C_i$ such that c_i is not contained in \bar{U}'_j for all $j \geq -2$ and $c_i \cap C'_i \neq \emptyset$. Thus there is $x \in \bigcap_{i=0}^\infty (c_i \cap C'_i) \subset \bigcap_{j=0}^\infty (C_j \cap C'_j) = C \cap C'$ since $C'_0 \supset C'_1 \supset C'_2 \supset \dots$. Suppose that x is a right-limit-point of C but not a left-limit-point of C . Then there is a gap U_{j_0} such that $\bar{U}_{j_0} = [y, x]$ for some y . If x is not a right-limit-point of C' , then there is a gap U'_{j_1} such that $\bar{U}'_{j_1} = [x, z]$ for some z . Since $\text{diam}(c_i) \rightarrow 0$ as $i \rightarrow \infty$, $U_{j_0} \cap c_i = \emptyset$ if $i > 0$ is large enough. Thus, we can find $i > 0$ such that $c_i \subset \bar{U}'_{j_1}$ since $x \in \bigcap_{i=0}^\infty c_i$. This is a contradiction. Therefore x is also a right-limit-point of C' . The other case follows in a similar way. \square

LEMMA 2.2. *If $\tau(\Lambda_1)\tau(\Lambda_2) > 1$, then there is a non-expansive sequence $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^\infty \subset X_t \times X_t$ of f_t for all $t \in (0, \eta)$.*

PROOF. Suppose $\tau(\Lambda_1)\tau(\Lambda_2) > 1$. Let $\Gamma_1 \subset [0, 1] \times \{y_0\}$ and $\Gamma_2 \subset \ell$ be Cantor sets as stated in Section 1. Then, for all $t \in (0, \eta)$, there exists $\mathbf{z}_t \in f_t(\Gamma_1) \cap \Gamma_2 \neq \emptyset$ by Lemma 2.1 since $\tau(\ell \cap f_t(\Gamma_1))\tau(\ell \cap \Gamma_2) > 1$. Fix $t > 0$ and take $\mathbf{z}_1 = (z_1, 0) \in \Lambda_1$,

$\mathbf{z}_2 = (0, z_2) \in \Lambda_2$ such that

$$\mathbf{z}_t \in f_t(\{z_1\} \times [0, 1 + \beta_0\lambda]) \cap (I \times \{z_2\}).$$

Note that $f_t(\{z_1\} \times [0, 1 + \beta_0\lambda])$ is a curve (say γ^u) in an unstable manifold of $W^u(\Lambda, t)$ and $I \times \{z_2\}$ is a curve (say γ^s) in a stable manifold of $W^s(\Lambda, t)$. Then, by Lemma 2.1, we may assume that \mathbf{z}_1 (respectively \mathbf{z}_2) is a limit point of $([0, z_1] \times \{0\}) \cap \Lambda_1$ (respectively $(\{0\} \times [z_2, 1]) \cap \Lambda_2$) (the other case follows in a similar way). Thus, for every $i > 0$, there are $\mathbf{z}_i^1 = (z_i^1, 0) \in \Lambda_1$ near \mathbf{z}_1 and non-wandering (distinct) points $\{\mathbf{x}_i, \mathbf{y}_i\} \in f_t(\{z_1^i\} \times [0, 1 + \beta_0\lambda]) \cap (I \times \{z_2\})$ near \mathbf{z}_t such that $|f_t^n(\mathbf{x}_i) - f_t^n(\mathbf{y}_i)|_{\mathbb{R}^2} \leq 1/i$ for all $n \in \mathbb{Z}$ since

$$\begin{aligned} |f_t^n(\mathbf{x}) - f_t^n(\mathbf{y})|_{\mathbb{R}^2} &\leq \nu^n |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^2} \quad \text{if } \mathbf{x}, \mathbf{y} \in I \times \{z_2\}, \\ |f_t^{-n}(\mathbf{x}) - f_t^{-n}(\mathbf{y})|_{\mathbb{R}^2} &\leq \lambda^{-n} |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^2} \quad \text{if } \mathbf{x}, \mathbf{y} \in \{z_1^i\} \times [0, 1 + \beta_0\lambda] \end{aligned}$$

for all $n > 0$. Here $|\cdot|_{\mathbb{R}^2}$ is a usual metric on \mathbb{R}^2 . Note that $f_t(\{z_1^i\} \times [0, 1 + \beta_0\lambda])$ is a curve (say γ_i^u) in an unstable manifold of $W^u(\Lambda, t)$. The proof is completed. \square

PROOF OF PROPOSITION. Let $f_t : X_t \rightarrow X_t$ ($t \in (-\eta, \eta)$) be as before. By the proof of [6, Section 8, Proposition 8.1] (see also [6, Theorem D]), for every $n > 0$, there exist $0 < t_n < 1/n$ and a hyperbolic basic set $\Lambda_{t_n}^{**} \subset X_{t_n}$ of f_{t_n} such that

- $\Lambda \subset \Lambda_{t_n}^{**}$,
- $\tau^s(\Lambda_{t_n}^{**})\tau^u(\Lambda_{t_n}^{**}) > 1$,
- $\Lambda_{t_n}^{**}$ has a persistent quadratic tangency of $W^s(\Lambda_{t_n}^{**}; t_n)$ and $W^u(\Lambda_{t_n}^{**}; t_n)$ at $\mathbf{q}_n (\notin \Lambda_{t_n}^{**})$; that is, $\Lambda_{t_n}^{**}$ is a wild hyperbolic set (see [6, p. 438] for the definition).

Here

$$\tau^s(\Lambda_{t_n}^{**}) = \limsup_{\varepsilon \rightarrow 0} \{\tau(\gamma_\varepsilon \cap W^s(\Lambda_{t_n}^{**}; t_n))\}$$

is the stable thickness of $\Lambda_{t_n}^{**}$, γ is any C^1 arc transverse to $W^s(\Lambda_{t_n}^{**}; t_n)$ at \mathbf{q} and γ_ε is the arc of length ε in γ centered at \mathbf{q} . The unstable thickness, $\tau^u(\Lambda_{t_n}^{**})$, is defined in a similar manner (for example [2, 4 and 6]). Newhouse [2] shows that $\tau^s(\Lambda_{t_n}^{**})$ and $\tau^u(\Lambda_{t_n}^{**})$ are independent of \mathbf{q} and γ . We can find a C^1 curve ℓ_n around \mathbf{q}_n such that $W^s(\Lambda_{t_n}^{**}; t_n)$ and $W^u(\Lambda_{t_n}^{**}; t_n)$ have a quadratic tangencies along ℓ_n (see [6, p. 457]). Then for a small enough curve $\ell_{n\varepsilon}$ around \mathbf{q}_n , $\tau(\ell_{n\varepsilon} \cap W^s(\Lambda_{t_n}^{**}; t_n))\tau(\ell_{n\varepsilon} \cap W^u(\Lambda_{t_n}^{**}; t_n)) > 1$. Note that two Cantor sets $\ell_{n\varepsilon} \cap W^s(\Lambda_{t_n}^{**}; t_n)$ and $\ell_{n\varepsilon} \cap W^u(\Lambda_{t_n}^{**}; t_n)$ overlap around \mathbf{q}_n (see [6, p. 456, (8.2b)]). Since $\Lambda_{t_n}^{**}$ is a wild hyperbolic set, these properties are persistent with respect to C^2 topology. Hence, by the same technique used in the proof of Lemma 2.2 we can find an interval $I_n \subset (0, \eta)$ containing t_n such that the situations (a) and (b) hold for f_t and Λ_t^{**} for all $t \in I_n$. Here Λ_t^{**} is the continuation of $\Lambda_{t_n}^{**}$. Thus for every $t \in I_n$, there is a non-expansive sequence of f_t in X_t . If necessary, by arranging an index n we may suppose that $\{I_n\}_{n=0}^\infty$ is a sequence of intervals decreasing towards 0 as $n \rightarrow \infty$. \square

3. Proof of Theorem B

As stated above we prove the conclusions under the C^2 assumption. Let M be a closed surface and $\text{Diff}^2(M)$ be the set of all C^2 diffeomorphisms of M endowed with a C^2 topology. We say that $f \in \text{Diff}^2(M)$ is *persistently hyperbolic* if there is a neighborhood $\mathcal{U}(f) \subset \text{Diff}^2(M)$ such that for every $g \in \mathcal{U}(f)$, the non-wandering set of g is hyperbolic.

PROOF OF (1). Let $f_{s,t}$ be a two parameter horseshoe family and $\Lambda(s) = \Lambda_1(s) \times \Lambda_2$ be as before. Notice that $f_{s,t}$ is linear near O . Clearly, $f_{s,t}$ satisfies [3, Conditions (i), (ii) and (iii), pp. 339–340] for $s \in (-\eta, \eta)$. For all $s \in (-\eta, \eta)$, we denote by $B(s)$ the set of all parameter values $t > 0$ such that $f_{s,t}$ is not persistently hyperbolic. Since $HD(\Lambda_1(0)) + HD(\Lambda_2) < 1$ and $HD(\Lambda_1(s))$ depends continuously on s , by [3, Theorem, p. 340] there exists $\eta_0 > 0$ such that for all $s \in (-\eta_0, \eta_0)$,

$$\lim_{t \rightarrow 0} \frac{m(B(s) \cap [0, t])}{t} = 0.$$

Here m denotes the Lebesgue measure. Fix $s \in (-\eta_0, \eta_0)$. Then, for every $0 < t < \eta_0$ and $t' \in B(s)^c \cap [0, t]$, $f_{s,t'}$ is persistently hyperbolic and thus $f_{s,t'} : X_{s,t'} \rightarrow X_{s,t'}$ is expansive. Hence $NE(s) \subset B(s)$ and assertion (1) is proved. \square

To prove (2), we need to introduce some notation. Let $\eta > 0$ and $f_{s,t} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ($(s, t) \in (-\eta, \eta)^2$) be as before. For every $\mathbf{z}_1 = (z_1, 0) \in \Lambda_1(0)$, there is a C^2 function $z_1(s) : (-\eta, \eta) \rightarrow \mathbf{R}$ such that $z_1 = z_1(0)$ and $(z_1(s), 0) \in \Lambda_1(s)$ for $s \in (-\eta, \eta)$. We denote $(z_1(s), 0)$ by $\mathbf{z}_1(s)$. For $\mathbf{z}_2 = (0, z_2) \in \Lambda_2$, let $\mathbf{z}_2(s) \equiv (0, z_2)$ ($s \in (-\eta, \eta)$) for convenience. Let U_1 (respectively U_2) be a neighborhood of $O \in \Lambda_1(0)$ (respectively Λ_2). Pick a sufficiently small $\varepsilon > 0$ and put

$$G_1(\mathbf{z}_1)(x, s, t) = (x + x_0, t - c_0 z_1(s) - x^2) \quad (x \in [-\varepsilon, \varepsilon])$$

and

$$G_2(\mathbf{z}_2)(x, s, t) = (x + x_0, z_2) \quad (x \in [-\varepsilon, \varepsilon])$$

for $\mathbf{z}_i \in U_i$ ($i = 1, 2$) and $(s, t) \in (-\eta, \eta)^2$. Then $G_i : U_i \rightarrow C^2([-\varepsilon, \varepsilon] \times (-\eta, \eta), [-\varepsilon, \varepsilon])$ is a continuous map for $i = 1, 2$ and

$$G_1(\mathbf{z}_1)([-\varepsilon, \varepsilon], s, t) \subset W^u(\mathbf{z}_1(s); s, t) \quad \text{and} \quad G_2(\mathbf{z}_2)([-\varepsilon, \varepsilon], s, t) \subset W^s(\mathbf{z}_2; s, t).$$

Set $T(\mathbf{z}_1, \mathbf{z}_2)(s) = c_0 z_1(s) + z_2$ for $(\mathbf{z}_1, \mathbf{z}_2) \in U_1 \times U_2$ and $s \in (-\eta, \eta)$. Then $T : U_1 \times U_2 \rightarrow C^2((-\eta, \eta), (-\eta, \eta))$ is a continuous map such that

- if $t < T(\mathbf{z}_1, \mathbf{z}_2)(s)$, then the function $x \mapsto G_1(\mathbf{z}_1)(x, s, t) - G_2(\mathbf{z}_2)(x, s, t)$ is strictly positive in $[-\varepsilon, \varepsilon]$.
- if $t = T(\mathbf{z}_1, \mathbf{z}_2)(s)$, then the same function is positive with a single zero in $[-\varepsilon, \varepsilon]$.

Hence $W^u(f_{s,t}(\mathbf{z}_1(s)); s, t)$ and $W^s(\mathbf{z}_2; s, t)$ have a quadratic tangency if $t = T(\mathbf{z}_1, \mathbf{z}_2)(s)$ for $s \in (-\eta, \eta)$. For the results for the Lebesgue measure of the set of all these parameter values t , see [7] and its references.

For a sufficiently small $r > 0$ and $s \in (-\eta, \eta)$, define $B_s^i(r) = \{\mathbf{z}_i \in U_i : |z_i(s)| \leq r\}$ for $i = 1, 2$. Here $\mathbf{z}_1 = (z_1, 0)$, $\mathbf{z}_2 = (0, z_2)$. Then $0 \leq T(\mathbf{z}_1, \mathbf{z}_2)(s) = c_0 z_1(s) + z_2 \leq (c_0 + 1)r$ for $(\mathbf{z}_1, \mathbf{z}_2) \in B_s^1(r) \times B_s^2(r)$ and $s \in (-\eta, \eta)$.

PROOF OF (2). From $HD(\Lambda_1(0)) + HD(\Lambda_2) > 1$, we prove that there are three constants $\eta_0 > 0$, $c' > 0$ and $r_1 > 0$ satisfying the following two properties (i) and (ii) for almost all $s \in (-\eta_0, \eta_0)$;

- (i) for every $0 < r < r_1$, there is a subset $T'_s(r) \subset [0, (c_0 + 1)r]$ in t -parameter space such that

$$\limsup_{r \rightarrow 0} \frac{m(T'_s(r))}{r} > c',$$

- (ii) for every $t \in T'_s(r)$, there is a non-expansive sequence $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^\infty$ of $f_{s,t}$ in $X_{s,t} \times X_{s,t}$.

Here m is the Lebesgue measure of the set. If the above properties are established, then by setting $t = (c_0 + 1)r$, assertion (2) is easily checked because $\sup T'_s(r) \leq (c_0 + 1)r$ and $T'_s(r) \subset NE(s)$.

Let us prove (i) and (ii). For every $r > 0$, $-\eta < s_0 < \eta$ and a subset L'_{s_0} of $B_{s_0}^1(r) \times B_{s_0}^2(r)$, put

$$T_s(r) = \bigcup_{(\mathbf{z}_1, \mathbf{z}_2) \in L'_{s_0}} T(\mathbf{z}_1, \mathbf{z}_2)(s) \quad (s \in (-\eta, \eta)).$$

Since $HD(\Lambda_1(0)) + HD(\Lambda_2) > 1$ and our family possesses three transversality conditions required in [5, pp. 93–95], by following the proof of the theorem stated in [5, Section 2], we have three constants $\eta_0 > 0$, $r_1 > 0$ and $c > 0$ with the property that for any $0 < r < r_1$ and $s_0 \in (-\eta_0, \eta_0)$, there exist subsets $L'_{s_0} \subset B_{s_0}^1(r) \times B_{s_0}^2(r)$ and $A^r_{s_0} \subset \{s \in (-\eta_0, \eta_0) : |s - s_0| < |\log r|^{-1}\}$ satisfying

$$m(A^r_{s_0}) \{ \inf_{s \in A^r_{s_0}} m(T_s(r)) \} > cr |\log r|^{-1}$$

(see [5, p. 95, Theorem]). The above assertion is established for a two parameter family of C^∞ diffeomorphisms possessing the three transversality conditions. In our

case, however, we can see that the assertion holds for $f_{s,t}$ because it is linear near O . Recall that $W^u(O; 0, t) \cap \Lambda_1(0)$ and $W^s(O; 0, t) \cap \Lambda_2$ are countable sets independent of t . Let $L^r_{s_0} = L^r_{s_0} \setminus \{W^u(O; 0, 0) \cap \Lambda_1(0)\} \times \{W^s(O; 0, 0) \cap \Lambda_2\}$ and put

$$T'_s(r) = \bigcup_{(z'_1, z'_2) \in L^r_{s_0}} T(z'_1, z'_2)(s) \subset T_s(r).$$

Then $m(T'_s(r)) = m(T_s(r))$ for $s \in (-\eta_0, \eta_0)$ since $T_s(r) \setminus T'_s(r)$ is a countable set. Thus we have $m(A^r_{s_0})\{\inf_{s \in A^r_{s_0}} m(T'_s(r))\} > cr|\log r|^{-1}$. Therefore, by the same method stated in [5, pp. 95–96], there is a constant $c' > 0$ such that

$$(*) \quad \limsup_{r \rightarrow 0} \frac{m(T'_s(r))}{r} > c'$$

for almost all $s \in (-\eta_0, \eta_0)$ and so (i) is proved.

To prove (ii), fix $s \in (-\eta_0, \eta_0)$ such that (*) is satisfied. We show that for every $t \in T'_s(r)$, there is a non-expansive sequence $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=0}^\infty$ of $f_{s,t}$ in $X_{s,t} \times X_{s,t}$. Let $\Gamma_1(s)$ and $\Gamma_2 (\subset \ell)$ be the Cantor sets defined in Section 1. Pick $t \in T'_s(r)$. Then there is

$$\mathbf{z}_t \in f_{s,t}(\Gamma_1(s)) \cap \Gamma_2 \setminus (W^s(O; s, t) \cap W^u(O; s, t)).$$

Take $\mathbf{z}_1 = (z_1, 0) \in \Lambda_1(s)$ and $\mathbf{z}_2 = (0, z_2) \in \Lambda_2$ such that $\mathbf{z}_t \in f_{s,t}(\{z_1\} \times [0, 1 + \beta_0\lambda]) \cap (I \times \{z_2\})$. Clearly, $\mathbf{z}_1 \notin W^u(O; s, t)$ or $\mathbf{z}_2 \notin W^s(O; s, t)$. It is easy to see that \mathbf{z}_1 is a limit point of $(\{0, z_1\} \times \{0\}) \cap \Lambda_1(s)$ for the case when $\mathbf{z}_1 \notin W^u(O; s, t)$. Thus, for every $i > 0$, there are $\mathbf{z}_1^i = (z_1^i, 0) \in \Lambda_1(s)$ near \mathbf{z}_1 and distinct points $\{\mathbf{x}_i, \mathbf{y}_i\} \in f_{s,t}(\{z_1^i\} \times [0, 1 + \beta_0\lambda]) \cap (I \times \{z_2\})$ near \mathbf{z}_t such that $|f_{s,t}''(\mathbf{x}_i) - f_{s,t}''(\mathbf{y}_i)|_{\mathbb{R}^2} \leq 1/i$ for all $n \in \mathbf{Z}$. For the case when $\mathbf{z}_2 \notin W^s(O; s, t)$, we can see that \mathbf{z}_2 is a limit point of $(\{0\} \times [0, z_2]) \cap \Lambda_2$, and so there is a non-expansive sequence. The proof of the theorem is completed. □

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