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THE ETA INVARIANT AND NON-SINGULAR BILINEAR PRODUCTS ON R^n

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ABSTRACT. Milnor showed that non-singular bilinear products on R^n exist only if $n = 1, 2, 4, 8$ using topological methods. In this note, we give a proof of this result by purely analytical methods.

0. **Introduction**. Let $RP^{n-1} = S^{n-1}/Z_2$ denote real projective space of dimension $n - 1$ and let P be the tangential operator of the Pin^c complex over RP^{n-1} for $n - 1$ even; we refer to section 2 for details. In this note we will compute the eta invariant of P and thereby give a proof of Milnor's result concerning the non-existence of non-singular bilinear products on R^n for $n \neq 1, 2, 4, 8$ by purely analytical methods. The combinatorial expressions which arise in this calculation are very suggestive and the explicit calculation of the eta invariant for this example is of importance in its own right and has applications to equivariant bordism as noted in [2]. Atiyah et al. [1] used the eta invariant primarily in an odd dimensional context in computing the boundary correction term to the index theorem; this is an example of an even dimensional use.

The paper is divided into three sections. In section 1, we establish the notation and reduce the proof of Milnor's result to an equivalent statement regarding Z_2 equivariant maps from S^{n-1} to the general linear group $GL(j, C)$. We review the analytic facts regarding the eta invariant we shall need to settle this question. In section 2, we construct a suitable operator over real projective space. In section 3, we use complex variables to evaluate the eta invariant.

1. **Non-singular bilinear products on R^n**. Let $m: R^n \times R^n \to R^n$ be a bilinear product on R^n. Let $f(x)y = m(x, y)$ give the product. $f(x)$ is a linear map on R^n for fixed x and the pairing is non-singular if $f(x)$ is invertible for $x \neq 0$. If R^n is a real division algebra, then the algebra structure gives rise to such a non-singular bilinear pairing. Milnor's theorem [1] is:

THEOREM 1.1. *If there exists a non-singular bilinear product on R^n, then $n = 1, 2, 4, 8$.*

In particular, real division algebras can exist only in dimensions 1, 2, 4, 8. Since the

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real numbers, complex numbers, quaternions, and Cayley numbers provide suitable examples in these dimensions, Milnor's theorem is sharp.

We complexify to regard $f: S^{n-1} \to GL(n, C)$. From the bilinearity in the first factor, we conclude that f has the equivariance property $f(-x) = -f(x)$. In this paper, we will give a purely analytic proof of the well known result:

THEOREM 1.2. *Assume there exists a continuous map $f: S^{n-1} \to GL(j, C)$ such that $f(-x) = -f(x)$. Then:*

(a) Let $n - 1 = 2k$ be even. Then 2^k divides j.
(b) Let $n - 1 = 2k + 1$ be odd. Then 2^k divides j.

We use this result to prove theorem 1.1. Let m be such a non-singular bilinear product and let $f: S^{n-1} \to GL(n, C)$ be the associated map. Since $f(-x) = -f(x)$, we may apply theorem 1.2 to conclude that 2^k divides n:

n	$n-1$	k	conclusion
3	2	1	2 divides 3 (impossible − no such product exists)
4	3	1	2 divides 4 (such a product exists − take the quaternions)
5	4	2	4 divides 5 (impossible − no such product exists)
6	5	2	4 divides 6 (impossible − no such product exists)
7	6	3	8 divides 7 (impossible − no such product exists)
8	7	3	8 divides 8 (such a product exists − take the Cayley numbers)

It is immediate that if $n > 8$ then $2^{k(n)} > n$ so there exists no such products in this case. This completes the proof of theorem 1.1.

We use Clifford matrices to show theorem 1.2 is sharp:

LEMMA 1.3. *Let $n = 2k + 1$ and let $v = 2^k$. There exist skew-adjoint $v \times v$ complex matrices $\{e_1, \ldots, e_n\}$ such that $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{i,j}$*

PROOF. These matrices arise from the spin representations. If $n = 3$, define:

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.1}$$

and verify these satisfy the given commutation relations. We now use induction. Let $n > 3$ and choose $\{f_1, \ldots, f_{n-2}\}$ which are $2^{k-1} \times 2^{k-1}$ skew-adjoint matrices satisfying the given commutation relations. Let $g_1 = f_1 \otimes I_2, \ldots, g_{n-3} = f_{n-3} \otimes I_2$, $g_{n-2} = i \cdot f_{n-2} \otimes e_1$, $g_{n-1} = i \cdot f_{n-2} \otimes e_2$, and $g_n = i \cdot f_{n-2} \otimes e_3$. It is immediate that the g_ν satisfy the conditions of lemma 1.3 which completes the proof. To show theorem 1.2 is sharp, it suffices to consider $n = 2k + 2$. Let $j = 2^k \cdot u$ and let $f(x_1, \ldots, x_{2k+2}) = x_1 e_1 \otimes I_u + \cdots + x_{2k+1} e_{2k+1} \otimes I_u + x_{2k+2} I_v \otimes I_u$. Consequently $f(x) \cdot f(x)^* = |x|^2 \cdot I_j$ so $f: S^{n-1} \to GL(j, C)$ and $f(-x) = -f(x)$.

If $f: S^{n-1} \to GL(j, C)$ exists with $f(-x) = -f(x)$, then restriction defines a similar map on S^{n-2}. It therefore suffices to prove theorem 1.2 for the case $n - 1 = 2k$. Let L denote the complexification of the classifying bundle over RP^{n-1}; $L = S^{n-1} \times$

$C/(z, w) = (-z, -w)$. Let $j \cdot L$ denote the direct sum of j copies of L and let $j \cdot 1$ denote the direct sum of j copies of the trivial bundle. A trivialization for $j \cdot L$ is a function $f: S^{n-1} \to GL(j, C)$ so $f(-x) = -f(x)$. This shows:

LEMMA 1.4. *Let $n - 1 = 2k$. Then there exists $f: S^{n-1} \to GL(j, C)$ with $f(-x) = -f(x)$ if and only if there is a bundle isomorphism between $j \cdot L$ and $j \cdot 1$.*

This is, of course, the usual starting point for the standard topological proof of theorem 1.1 using K-theory. Lemmas 1.3 and 1.4 show $L - 1$ is a torsion element of order at most 2^k in $\tilde{K}(RP^{2k})$. Theorem 1.2 shows the order of $L - 1$ is exactly 2^k. Since $\tilde{K}(RP^{2k})$ is generated by $L - 1$, these results are equivalent to a large slice of the K-theoretic calculation $\tilde{K}(RP^{2k}) = Z_{2^k}$.

In this paper, we will use the theory of differential equations to replace K-theory. Let M be a compact smooth manifold without boundary of even dimension $m = 2k$. We refer to Gilkey [4] for details regarding the analytic facts cited here; they are consequences of the calculus of pseudo-differential operators depending upon a complex parameter developed by Seeley [5]. Let $P: C^\infty(V) \to P^\infty(V)$ be a first order self-adjoint elliptic differential operator over M, and let $\{\phi_\nu, \lambda_\nu\}_{\nu=1}^\infty$ be a complete spectral resolution of P. The ϕ_ν are smooth functions forming a complete orthonormal basis of $L^2(V)$ with $P\phi_\nu = \lambda_\nu \phi_\nu$ for $\lambda_\nu \in R$. Order the sequence so $|\lambda_1| \leq |\lambda_2| \leq \cdots$ then there exists $C > 0$ so that $|\lambda_\nu| \geq C \cdot \nu^{1/m}$, so the eigenvalues grow fairly rapidly in absolute value. Define

$$\eta(s, P) = \{\Sigma_\nu \operatorname{sign}(\lambda_\nu) |\lambda_\nu|^{-s} + \dim \operatorname{Ker}(P)\}/2. \tag{1.2}$$

as a measure of the spectral asymmetry of P. Because m is *even* and P is an *odd* order operator, certain local invariants vanish which do not vanish in general.

LEMMA 1.5. *Let M be a compact even dimensional manifold without boundary and let $P: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic first order differential operator. Let $\eta(s, P)$ be as defined by equation (1.2). $\eta(s, P)$ is holomorphic for* $\operatorname{Re}(s) > m$ *and admits a meromorphic extension to C with isolated simple poles on the real axis; the origin is a regular value. Define $\eta(P) = \eta(0, P) \in R/Z$.* If P_t *is a smooth 1-parameter family of such operators, then $\eta(P_t)$ is independent of t.*

REMARK. The eta invariant has integer jumps as eigenvalues cross the origin; when reduced mod Z it becomes independent of the parameter t. For *odd* dimensional manifolds, the analogous invariant plays a crucial role in the signature theorem for manifolds with boundary of Atiyah et al. [1].

Let W be a vector bundle over M and let P_W denote the operator P with coefficients in W. Although P is not uniquely defined, it is well defined up to homotopy so lemma 1.5 implies $\eta(P_W) \in R/Z$ is an invariant of W. If there is a bundle isomorphism between two bundles W_1 and W_2, then $\eta(P_{W_1}) = \eta(P_{W_2})$. In sections 2 and 3, we will prove:

LEMMA 1.6. *Let $M = RP^{2k}$ and let $v = 2^k$. There exists $P: C^\infty(1^v) \to C^\infty(1^v)$ over M which is a self-adjoint elliptic first order differential operator so that $\eta(P) - \eta(P_L) = 2^{-k}$.*

Thus if there exists a bundle isomorphism between $j \cdot L$ and $j \cdot 1$, then $2^{-k} \cdot j = 0$ in R/Z so that 2^k divides j. This completes the proof of theorem 1.2.

2. **The tangential operator of the Pin^c complex**. We now come to the heart of the matter. We will construct an operator P over RP^{2k} such that if $m = 2k$,

$$(2.1)\quad \eta(s, P) - \eta(s, P_L) = 2^{k-1} \sum_{j=0}^{\infty} (-1)^j \binom{m+j+2}{m-2} ((2j + m - 1)/2)^{-s}$$

We will evaluate equation (2.1) at $s = 0$ in section 3 to complete the proof of lemma 1.6. In fact P is the tangential operator of the Pin^c complex over RP^{2k}; as we shall not need this fact, we omit a proof and refer to Gilkey [4] for details. Let $v = 2^k$, let $m = 2k$, and let $\{e_i\}$ for $1 \le i \le 2k + 1$ be $v \times v$ complex matrices as given in lemma 1.3. Define:

$$(2.2)\qquad D = \Sigma_i e_i \partial/\partial x_i \quad \text{on} \quad C^\infty(R^{m+1} \times C^v)$$
$$\Delta_e = -\Sigma_i \partial^2/\partial x_i^2 \quad \text{on} \quad C^\infty(R^{m+1})$$

Both Δ_e and D are self adjoint and $D^2 = \Delta_e \cdot I_v$. Let r denote the Euclidean length and let $\theta \in S^m$. Introduce spherical coordinates (r, θ) on $R^{m+1} - 0$. Let $\{ds_e^2, \text{dvol}_e, \Delta_e\}$ and $\{ds_\theta^2, \text{dvol}_\theta, \Delta_\theta\}$ denote the metric, volume, and Laplacian on R^{m+1} and S^m respectively. Then

$$ds_e^2 = dr^2 + r^2\, ds_\theta^2$$
$$(2.3)\qquad \text{dvol}_e = r^m\, dr \cdot \text{dvol}_\theta$$
$$\Delta_e = -\partial^2/\partial r^2 - m \cdot r^{-1} \cdot \partial/\partial r + r^{-2} \cdot \Delta_\theta$$

We use these identities to decompose D in spherical coordinates:

LEMMA 2.1.

(a) $D = e(\theta) \cdot \partial/\partial r + r^{-1} \cdot A$ where A is an invariantly defined first order tangential operator on $C^\infty(S^m \times C^v)$ without constant term.
(b) Let A^* denote the adjoint over S^m, then $A^* = A - m \cdot e(\theta)$.
(c) $A \cdot e(\theta) + e(\theta) \cdot A = -m \otimes I_v$ and $\Delta_\theta \cdot I_v = A^2 - e(\theta)A$.

PROOF. Introduce local coordinates $(\theta_1, \ldots, \theta_m)$ on S^m and decompose $D = f(r, \theta)\partial/\partial r + \Sigma_j f_j(r, \theta)\partial/\partial\theta_j$. $D(1) = 0$ so there is no constant term.

$$(2.4)\qquad D(r^2) = 2 \cdot \Sigma_i x_i e_i = 2 \cdot e(r, \theta) = 2r \cdot e(\theta) = 2r \cdot f(r, \theta)$$

so the coefficient of $\partial/\partial r$ is $e(\theta)$. Let c be a positive scalar. Change coordinates $x \to cx$ and $r \to cr$; θ is unchanged and $D \to c^{-1}D$. Thus the $f_j(r, \theta)$ are homogeneous of order -1 in r. Define $A = \Sigma_j f_j(1, \theta)\partial/\partial\theta_j$, then A is a tangential differential operator

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on S^m and $D = e(\theta)\partial/\partial r + r^{-1}\cdot A$. This proves (a). D is self-adjoint and $\text{dvol}_e = r^m\cdot dr\cdot \text{dvol}_\theta$. Therefore:

$$e(\theta)\cdot\partial/\partial r + r^{-1}\cdot A = D = D^* = r^{-m}e(\theta)\cdot\partial/\partial r\cdot r^m + r^{-1}\cdot A^* \tag{2.5}$$

We solve for A^* to prove (b). Finally

$$\begin{aligned} D^2 &= -\partial^2/\partial r^2 + e(\theta)\cdot\partial/\partial r\cdot r^{-1}\cdot A + r^{-1}\cdot A\cdot e(\theta)\cdot\partial/\partial r + r^{-2}\cdot A^2 \\ &= -\partial^2/\partial r^2 + r^{-1}\{e(\theta)\cdot A + A\cdot e(\theta)\}\partial/\partial r - r^{-2}e(\theta)\cdot A + r^{-2}\cdot A^2 \\ &= \Delta_e\otimes I_v = \{-\partial^2/\partial r^2 - mr^{-1}\cdot\partial/\partial r + r^{-2}\cdot\Delta_\theta\}\otimes I_v \end{aligned} \tag{2.6}$$

We equate tangential and radial parts in this equation to complete the proof.

We can now construct the desired self-adjoint differential operator on S^m

LEMMA 2.2. *Let A be as in Lemma 2.1. Let $B = e(\theta)\cdot A$ and $P = B + (m-1)/2$ on $C^\infty(S^m\times C^v)$. Then P and B are self-adjoint first order elliptic differential operators which are invariant under the antipodal action and descend to operators on $C^\infty(RP^m\times C^V)$. $P^2 = \{\Delta_\theta + ((m-1)/2)^2\}\otimes I_v$.*

PROOF. By lemma 2.1,

$$\begin{aligned} B^* &= A^*\cdot e(\theta)^* = \{A - m\cdot e(\theta)\}\cdot\{-e(\theta)\} \\ &= -A\cdot e(\theta) + m = e(\theta)\cdot A = B \end{aligned} \tag{2.7}$$

so B is self adjoint. Furthermore

$$\begin{aligned} B^2 &= e(\theta)\cdot A\cdot e(\theta)\cdot A = e(\theta)\{-e(\theta)\cdot A - m\}\cdot A \\ &= A^2 - e(\theta)\cdot A - (m-1)B = \Delta_\theta\otimes I_v - (m-1) \end{aligned} \tag{2.8}$$

so that

$$\begin{aligned} (B + (m-1)/2)^2 &= B^2 + (m-1)B + ((m-1)/2)^2 \\ &= \{\Delta_\theta + ((m-1)/2)^2\}\otimes I_v \end{aligned} \tag{2.9}$$

which completes the proof of lemma 2.2.

The eigenvalues of Δ_θ can be computed using spherical harmonics. Let $S(m, j)$ denote the vector space of homogeneous polynomials of degree in the variables $\{x_1, \ldots, x_{m+1}\}$ and let $H(m, j) = \{f\in S(m, j) : \Delta_e(f) = 0\}$ be the subspace of harmonic polynomials. We restrict these elements to the sphere to obtain a complete orthogonal decomposition $L^2(S^m) = \oplus_j H(m, j)$. It is immediate from equation 2.3 that if $f\in H(m, j)$, then $\Delta_\theta(f) = j(j + m - 1)f$ so this gives the complete spectral resolution of Δ_θ on S^m. The multiplicities are given by

$$\begin{aligned} \dim H(m, j) &= \dim S(m, j) - \dim S(m, j-2) \\ &= \binom{m+j}{m} - \binom{m+j-2}{m}. \end{aligned} \tag{2.10}$$

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Let $f \in H(m, j, v) = H(m, v) \otimes C^v$. Lemma 2.2 implies

$$P^2(f) = \{j(j + m - 1) + (m - 1)^2/4\}f = \{(2j + m - 1)/2\}^2 f. \tag{2.11}$$

These eigenvalues are distinct values of j so $H(m, j, v)$ must be invariant under P and the eigenvalues of P on this space are $\pm(2j + m - 1)/2$. Decompose

$$H(m, j, v) = H^+(m, j, v) \oplus H^-(m, j, v) \tag{2.12}$$

into the positive and negative eigenvalues of P.

LEMMA 2.3.

$$\text{Dim}\, H^+(m, j, v) - H^-(m, j, v) = 2^k \cdot \binom{m + j - 2}{m - 2}.$$

PROOF. Decompose $B = \Sigma_{i<j} B_{ij} e_i e_j + B_0 \cdot I$ into scalar operators times matrices. The leading symbol of B anti-commutes with $e(\theta)$ and is trace free. As $e_i e_j$ is tracefree, B_0 is a 0^{th} order operator. B vanishes on constants so $B_0 = 0$. From the commutation relations $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{i,j}$, we see

$$B_{ij} = (B + e_i B e_i + e_j B e_j + e_i e_j B e_j e_i)/4. \tag{2.13}$$

Thus $H(m, j, v)$ is $B_{i,j}$ invariant. As this is a scalar operator, $H(m, j)$ is B_{ij} invariant. As $e_i \cdot e_j$ is trace free for $i < j$, $B_{i,j} \cdot e_i \cdot e_j$ is trace free and B itself is trace free acting on $H(m, j, v)$. Therefore

$$\begin{aligned} \text{Tr}(P \text{ on } H(m, j, v)) &= \text{Tr}(B + (m - 1)/2) \\ &= (m - 1)/2 \cdot \dim H(m, j, v) \\ &= (m - 1)/2 \cdot v \cdot \left\{\binom{m + j}{j} - \binom{m + j - 2}{j - 2}\right\} \\ &= (2j + m - 1)/2 \cdot \{\dim H^+(m, j, v) - \dim H^-(m, j, v)\} \end{aligned} \tag{2.14}$$

We solve equation (2.14) for $\dim H^+ - \dim H^-$ to complete the proof of Lemma 3.3.

Let j be even and let $f \in H^{\pm}(m, j, v)$. Since $f(x) = -f(x)$, f descends to define a section to $C^\infty(RP^m \times C^v)$ over RP^m and is an eigenfunction of P over RP^m. If j is odd, then $f(-x) = -f(x)$, so f descends to define a section to $C^\infty((RP^m \times C^v) \otimes L)$ and is an eigenfunction of P_L. Since these functions give a complete orthogonal basis for L^2, this gives the complete spectral resolution for P and P_L over RP^m. This shows:

LEMMA 2.4. *Let P be the operator defined in lemma 2.2. P induces an operator we continue to denote by P over RP^m and*

$$\begin{aligned} \eta(s, P) - \eta(s, P_L) = 2^{k-1} \cdot \sum_{j=0}^{\infty} (-1)^j \cdot \binom{m + j - 2}{m - 2} \\ \cdot \{(2j + m - 1)/2\}^{-s}. \end{aligned} \tag{2.15}$$

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3. **The calculation of $\eta(P) - \eta(P_L)$.** In this section, we will use the technique of Abel summation to evaluate the series given in equation (2.15). Let $m = 2k$ and let $z \in C$. Define

$$\eta_m(z, s) = 2^{k-1} \sum_{j=0}^{\infty} z^j \cdot \binom{m+j-2}{m-2} \cdot ((2j + m - 1)/2)^{-s} \tag{3.1}$$

We will show $\eta_m(-1, 0) = 2^{-k}$ to complete the proof of theorem 1.6.

Let $a > 0$ and let p be a poynomial of degree u. Define

$$\zeta_{a,p}(z, s) = \sum_{j=0}^{\infty} z^j \cdot p(j) \cdot (j + a)^{-s}. \tag{3.2}$$

We let ζ_a correspond to the polynomial $p(z) = z$. The sum in equation (3.2) converges absolutely for $|z| \leq 1$ and for $\mathrm{Re}(s) > u + 1$ to a holomorphic function of (z, s). Let $\Omega = (C - [1, \infty)) \times C$ then:

LEMMA 3.1.

(a) $\zeta_{a,p}(z, s)$ *has a holomorphic extension to* Ω.
(b) $\eta_m(z, s)$ *has a holomorphic extension to* Ω *with* $\eta_m(z, 0) = 2^{k-1}(1 - z)^{1-m}$.

PROOF. Let

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt. \tag{3.3}$$

$\Gamma(s)$ is holomorphic and non-zero for $\mathrm{Re}(s) > 0$. The functional equation $s\Gamma(s) = \Gamma(s + 1)$ gives a meromorphic extension to C with isolated simple poles at the non-positive integers. We use the Mellin transform

$$\Gamma(s)^{-1} \cdot \int_0^{\infty} t^{s-1} e^{-\lambda t}\, dt = \Gamma(s)^{-1} \cdot \lambda^{-s} \int_0^{\infty} (\lambda t)^{s-1} e^{-\lambda t}\, d(\lambda t) = \lambda^{-s}. \tag{3.4}$$

to express

$$\begin{aligned} \zeta_a(z, s) = \sum_{j=0}^{\infty} z^j (j + a)^{-s} &= \Gamma(s)^{-1} \sum_{j=0}^{\infty} \int_0^{\infty} z^j \cdot e^{-jt} \cdot e^{-at} \cdot t^{s-1}\, dt \\ &= \Gamma(s)^{-1} \int_0^{\infty} \{1/(1 - ze^{-t})\} \cdot e^{-at} \cdot t^{s-1}\, dt \end{aligned} \tag{3.5}$$

after interchanging the order of integration and summation. The sums and integrals converge absolutely for $|z| < 1$ and $\mathrm{Re}(s) > 1$. We restrict to $z \in C - [1, \infty)$ so the integrand is well defined. We decompose the integral into pieces to obtain a holomorphic extension to Ω. The integral from 1 to ∞ converges absolutely for all $(z, s) \in \Omega$ to define a holomorphic function. Let $f(t) = e^{-at}/(1 - ze^{-t})$ and expand f in a Taylor series about $t = 0$:

$$f(t) = \sum_{j=0}^{N} c_j(a, z) t^j + r_N(t, a, z) \tag{3.6}$$

where $|r_N(t, a, z)| \leq c(N, a, z)t^N$. Since r_N decays in t, the integral from 0 to 1 of $r_N t^{s-1}\, dt$ defines a function which is holomorphic for $\operatorname{Re}(s) > -N$. Therefore:

$$(3.7) \qquad \zeta_a(z, s) = \Gamma(s)^{-1}\left\{\sum_{j=0}^{N} c_j(a, z)(j + s)^{-1} + R_N(a, z, s)\right\}$$

where c_j and r_N are holomorphic for $\operatorname{Re}(s) > -N$. Γ^{-1} is holomorphic with simple zeros at the non-positive integers. These simple zeros cancel the simple poles of $(j + s)^{-1}$ which gives the desired holomorphic extension if $p(z) = z$. More generally, decompose $p(j) = \Sigma_\nu c_\nu (j + a)^\nu$ so

$$(3.8) \qquad \zeta_{a,p}(z, s) = \sum_{\nu=0}^{j} c_\nu \zeta_z(z, s - \nu)$$

This proves 3.1(a). Let $a = (m - 1)/2$ and let $p(j) = \binom{m + j - 2}{m - 2}$. Then $\zeta_{a,p}(z, s)$ $= \eta_m(z, s)$ has a holomorphic extension to Ω. If we can establish that $\zeta_m(z, 0) =$ $(1 - z)^{1-m}$ for $|z| < 1$, then this identity holds for all $z \in C - [1, \infty)$ by analytic continuation. Differentiate the series $1/(1 - z) = \Sigma_j z^j$ $m - 2$ times to see

$$(3.10) \qquad (m - 2)!(1 - z)^{1-m} = \sum_{j=0}^{\infty} j(j - 1)\cdots(j - m + 3)z^{j-(m-2)}$$
$$= \sum_{\nu=0}^{\infty} (\nu + (m - 2))\cdots(\nu + 1)z^\nu.$$
$$(1 - z)^{1-m} = \sum_{\nu=0}^{\infty} \binom{\nu + m - 2}{m - 2} z^\nu$$

which completes the proof of lemma 3.1. Let $z = -1$, then

$$(3.11) \quad \eta(P) - \eta(P_L) = 2^{k-1}\sum_{k=0}^{\infty} z^k \binom{m + j - 2}{m - 2}((2j + m - 1)/2^{-s}|_{s=0,\, z=-1}$$

which complete the proof of theorem 1.6.

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