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CONDITION THAT ALL IRREDUCIBLE REPRESENTATIONS OF A COMPACT LIE GROUP, IF RESTRICTED TO A SUBGROUP, CONTAIN NO REPRESENTATION MORE THAN ONCE

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1. One of the results of the theory of the irreducible representations of the unitary group in n dimensions U_n is that these representations, if restricted to the subgroup U_{n-1} leaving a vector (let us say the unit vector e_1 along the first coordinate axis) invariant, do not contain any irreducible representation of this U_{n-1} more than once (see [**1**, Chapter X and Equation (10.21)]; the irreducible representations of the unitary group were first determined by I. Schur in his doctoral dissertation (Berlin, 1901)). Some time ago, a criterion for this situation was derived for finite groups [**3**] and the purpose of the present article is to prove the aforementioned result for compact Lie groups, and to apply it to the theory of the representations of U_n.

The physicist's interest in the question whether a group and a corresponding subgroup satisfy the criterion of the title of this article is motivated by the phenomenon of "symmetry breaking". Let us consider a quantum mechanical operator—usually the Hamiltonian—which is invariant under a group of operators. When the operators of the group are applied to the characteristic vectors of a characteristic value of the quantum mechanical operator, these vectors transform, except for the rare case of "accidental degeneracy", according to an irreducible representation of the invariance group. The physicist says that the characteristic vectors of each characteristic value "belong" to the different rows of the irreducible representation in question. If the original quantum mechanical operator is now modified by a small perturbation which is not invariant under the whole group but only under a subgroup, the original characteristic value splits up in as many characteristic values as are irreducible representations of the subgroup contained in the irreducible representation of the whole group to which this characteristic value belongs. If each irreducible representation of the subgroup is contained only once in the irreducible representation of the full group, only one new characteristic value belonging to any irreducible representation of the subgroup will arise from any characteristic value of the original quantum mechanical operator. The corresponding characteristic vectors belong to the irreducible representations of the subgroup and are completely determined by this criterion. This is not the case if several characteristic values of the new problem arise from a characteristic value of

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the original problem, i.e., if the irreducible representation of the original, full group contains any irreducible representation of the subgroup more than once. The knowledge of the characteristic vectors of the new problem, containing the small perturbation, reduces the calculation of the characteristic values of the new problem to the calculation of a matrix element or scalar product. The calculation of the splitting of the atomic energy levels by a magnetic field (Zeeman effect) is an example herefor; the original group is R_3 (or U_2) in this case, the subgroup is R_2 (or U_1). There are, however, also other questions which can be answered with relative ease if the characteristic vectors of the new problem are known.

In order to formulate the criterion for the situation given in the title of this article, one introduces the concept of subclasses. These consist of the elements of the group (in our case U_n) which can be transformed into each other by an element of the subgroup (in our case U_{n-1}). In the case of finite groups, it was shown that the criterion for no irreducible representation of the whole group to contain any representation of the subgroup more than once is that the subclasses commute. In the case of continuous groups, the subclasses are, as a rule, manifolds of lower dimensionality than the whole group. As a result, the commuting nature of the subclasses is less easily defined than in the case of finite groups. In the case of the latter, the statement that two subclasses C_1 and C_2 commute means that every element R of the group appears as often in the complex C_1C_2 as in the complex C_2C_1. Both these complexes are, naturally, finite if the whole group is finite. If the group is continuous, the sets C_1C_2 and C_2C_1 are, as a rule, continuous and infinite manifolds and the frequency of the occurrence of an element R in them is not defined. Hence, the criterion for finite groups must be reformulated and this is the principal subject of the present article, leading to Equation (11). The f in this equation is an arbirtary numerical function on the group. Naturally, this equation is also an expression for the identity of all complexes C_1C_2 and C_2C_1 in the case of finite groups, C_1 and C_2 being subclasses, if the invariant integrations over the subgroup—indicated by the $\int du$ and $\int ds$ symbols—are replaced by summations over the elements of the subgroup.

In the third part of the article, (11) will be shown to hold for all pairs of elements R and T of U_n if the integrations over u and s are the invariant integrations over the subgroup U_{n-1} which leaves a vector e_1 invariant. Thus, a proof for the result of the theory of the representations of U_n, mentioned in the first paragraph, will be produced.

Lastly, a sufficient but not necessary property of subgroups is given which assures that the irreducible representations of the whole group, if restricted to the subgroup in question, do not contain any representation of the latter more than once.

2. Let G be a compact Lie group, and let H be a subgroup of G. Consider now those functions $a(R)$ defined and continuous on the group space, for which

$\int a(R)D(R)dR$ commutes with $D(s)$ for all s in H, and for all irreducible representations D of G. The dR denotes the invariant integration over the group. Now suppose that the matrix $C^{(0)}$ commutes with all $D^{(0)}(s)$ for a single $D^{(0)}$, all s in H. Then the matrix $C^{(0)}$ can be written as $\int f(R)D^{(0)}(R)dR$, where

$$f(R) = \frac{l}{v}\sum_{i,k} C_{ik}^{(0)}D_{ki}^{(0)}(R^{-1}), \tag{1}$$

where l is the dimension of $D^{(0)}$ and $v = \int dR$ is the group volume, since

$$\begin{aligned} &\frac{l}{v}\int\sum_{i,k} C_{ik}^{(0)}D_{ki}^{(0)}(R^{-1})D_{jl}^{(0)}(R)dR \\ &= \frac{l}{v}\sum_{i,k} C_{ik}^{(0)}\frac{v}{l}\delta_{kl}\delta_{ji} \\ &= C_{jl}^{(0)}. \end{aligned} \tag{2}$$

It is clear that $f(R)$ is continuous on the group, since it is a linear functional of the matrix elements of a group representation.

Consider now the matrix $C^{(m)}$, defined by

$$\begin{aligned} C_{ij}^{(m)} &= \int f(R)D_{ij}^{(m)}(R)dR \\ &= \int \frac{l}{v}\sum_{n,k} C_{nk}^{(0)}D_{kn}^{(0)}(R^{-1})D_{ij}^{(m)}(R)dR \\ &= \begin{cases} 0, & m \neq 0 \\ C_{ij}^{(0)}, & m = 0. \end{cases} \end{aligned} \tag{3}$$

The matrix $C^{(m)}$ is equal to either $C^{(0)}$ or zero, so $\int f(R)D_{ij}^{(m)}(R)dR$ certainly commutes with $D^{(m)}(s)$, and so $f(R)$ is contained in the set of $a(R)$ defined above. Hence, any matrix which commutes with the $D^{(j)}(s)$, for every s but any single j, is contained in the set $\int a(R)D(R)dR$.

The requirement that $\int a(R)D(R)dR$ commute with $D(s)$ for s in the subgroup H imposes certain conditions on the functions $a(R)$. In particular,

$$\int a(R)D(R)D(s)dR = \int a(R)D(s)D(R)dR; \tag{4}$$

multiplication of (4) by $D(s^{-1})$ on the right yields

$$\int a(R)D(R)dR = \int a(R)\, D(sRs^{-1})dR \tag{4a}$$

and replacing the variable R on the right by $s^{-1}Rs$ then gives

$$\int a(R)D(R)dR = \int a(s^{-1}Rs)D(R)dR. \tag{5}$$

Since this holds for all irreducible representations D, the completeness of the matrix elements of the irreducible representations requires that

$$a(R) = a(s^{-1}Rs). \tag{5a}$$

We will call the set of elements of the group which can be written in the form $s^{-1}Rs$ the *subclass* containing the element R. Evidently, the product of two subclasses contains full subclasses. A subclass either is contained in the subgroup, in which case it consists of exactly one class of the subgroup, or contains no elements therein.

According to (5a), then, the functions in our set a are simply the continuous, subclass-valued functions on G. Any such function can be written in the form

$$a(R) = \int \tilde{a}(s^{-1}Rs)ds, \tag{6}$$

where $\tilde{a}$ is any continuous function on the group.

Consider now the $\int a(R)D(R)dR$. These matrices commute with all the $D(s)$. If the representation $D(R)$ is considered as a representation of the subgroup H, then all the matrices which commute with the $D(s)$ will commute with each other, if and only if $D(s)$ contains no irreducible representation of H more than once. Thus, the condition that $D(R)$ restricted to H contain no irreducible representation of H more than once is equivalent to the condition that the $\int a(R)D(R)dR$ commute with each other.

Writing $a(R)$ as in (6), we have, instead of

$$\left(\int a(R)D(R)dR\right)\left(\int a'(T)D(T)dT\right) = \left(\int a'(T)D(T)dT\right)\left(\int a(R)D(R)dR\right) \tag{7}$$

the equation

$$\begin{aligned} &\iiiint \tilde{a}(s^{-1}Rs)\tilde{a}'(u^{-1}Tu)D(RT)dsdudRdT \\ &= \iiiint \tilde{a}(s^{-1}Rs)\tilde{a}'(u^{-1}Tu)D(TR)dsdudRdT, \end{aligned} \tag{8}$$

or

$$\begin{aligned} &\iiiint \tilde{a}(R)\tilde{a}'(T)D(sRs^{-1}uTu^{-1})dsdudRdT \\ &= \iiiint \tilde{a}(R)\tilde{a}'(T)D(uTu^{-1}sRs^{-1})dsdudRdT. \end{aligned} \tag{9}$$

Since $\tilde{a}$ and $\tilde{a}'$ are arbitrary functions, this requires that for any D

$$\iint D(sRs^{-1}uTu^{-1})dsdu = \iint D(uTu^{-1}sRs^{-1})dsdu. \tag{10}$$

But since the matrix elements of the irreducible representations form a complete set of functions on group space, (10) says that any function integrated over the set $sRs^{-1}\ uTu^{-1}$ is equal to the same function integrated over the set $uTu^{-1}\ sRs^{-1}$:

THEOREM 1. *The necessary and sufficient condition for no irreducible representations of a compact Lie group to contain any representation of a subgroup H more than once is the validity of*

$$\iint f(sRs^{-1}uTu^{-1})dsdu = \iint f(uTu^{-1}sRs^{-1})dsdu \tag{11}$$

for all elements R, T of the full group and any continuous function f on this group.

The ds and du in (11) denote the invariant integration over the subgroup H.

Theorem 1 corresponds to the criterion for finite groups that the subclasses commute.

3. We want to show next that Equation (11) holds when the group G is U_n, and the subgroup H is U_{n-1}.

In order to do this, it will be demonstrated, first, that if a subclass contains a transformation R, it also contains its transpose R^T. In order to demonstrate this, we transform R to the principal axes

$$R = V^{\dagger} \omega V, \tag{12}$$

where ω is a diagonal matrix with diagonal elements of modulus 1, V is unitary. The cross denotes hermitean adjoint. The transpose of (12) reads $R^T = V^T \omega \bar{V}$, but we may write as well (the bar denotes complex conjugation)

$$R^T = V^T \eta \omega \bar{\eta} \bar{V}, \tag{13}$$

where η is an arbitrary diagonal matrix the diagonal elements of which are also of modulus 1 so that $\eta\bar{\eta} = 1$ and also $\eta\omega\bar{\eta} = \omega$. One can express ω by means of both (12) and (13); equating the two expressions gives

$$VRV^{\dagger} = \bar{\eta}\bar{V}R^T V^T \eta$$

so that

$$V^T \eta V R (V^T \eta V)^{-1} = R^T; \tag{14}$$

i.e., the unitary matrix $V^T \eta V$ transforms R into its transpose R^T and this is true for any η as defined above. It may be observed, parenthetically, that these matrices are symmetrical. This observation represents a special case of a more general theorem due to O. Taussky and H. Zassenhaus [**2**]. Equation (14) now leads to Lemma 1.

LEMMA 1. *Any subclass of U_n with respect to U_{n-1} which contains R also contains R^T.*

In order to prove this, it is necessary only to prove that η in (14) can be so chosen that $V^T \eta V$ leaves e_1 invariant. This will be the case if

$$(V^T \eta V)_{i1} = \delta_{i1} \qquad (i = 1, 2, \ldots, n), \tag{15}$$

or

$$\sum_l V_{li} \eta_l V_{l1} = \delta_{i1}, \tag{15a}$$

and this will be satisfied if one chooses

$$\eta_l = \bar{V}_{l1} / V_{l1} \tag{16}$$

if $V_{l1} \neq 0$ and arbitrarily, though of modulus 1, if $V_{l1} = 0$. With these η_l one has

$$(V^T \eta V)_{i1} = \sum_l V_{li} \bar{V}_{l1} = \delta_{i1}. \tag{17}$$

This then proves Lemma 1.

A further lemma concerns the group integral.

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LEMMA 2. *If H is a compact Lie group, such that for all u in H there exists an element w of H such that $u^T = wuw^{-1}$, then, for any function f,*

$$\int f(u^T)du = \int f(u)du. \tag{18}$$

In order to prove Lemma 2, consider

$$\begin{aligned}\iint f(v^{-1}uv)dudv &= \int[\int f(v^{-1}uv)du]dv \\ &= \int[\int f(u)du]dv \\ &= V_H \int f(u)du\end{aligned} \tag{19}$$

where V_H is the group volume. But we have also

$$\iint f(v^{-1}uv)dudv = \int[f(v^{-1}uv)dv]dv. \tag{19a}$$

Choose now a w such that $u^T = wuw^{-1}$. Then

$$\begin{aligned}\iint f(v^{-1}uv)dudv &= \int[\int f(v^{-1}w^{-1}u^Twv)dv]du \\ &= \int[\int f(v^{-1}u^Tv)dv]du \\ &= \int[\int f(v^{-1}u^Tv)du]dv = V_H\int f(u^T)du,\end{aligned} \tag{20}$$

which proves the lemma.

We wish to show now that (11) holds for U_n with respect to the subgroup U_{n-1} which leaves a vector e_1 invariant. Replacing u by sq one obtains

$$\iint f(sRs^{-1}uTu^{-1})dsdu = \iint f(sRqTq^{-1}s^{-1})dsdq. \tag{21}$$

According to Lemma 1, every element of U_n and its transpose are in the same subclass. In particular, there must be an element z of U_{n-1} such that $RqTq^{-1} = z(RqTQ^{-1})^Tz^{-1}$. The z will be a function of q but not of s. Hence, s can be replaced by sz^{-1} so that

$$\begin{aligned}\iint f(sRs^{-1}uTu^{-1})dsdu &= \iint f(sz(RqTq^{-1})^Tz^{-1}s^{-1}dsdq \\ &= \iint f(s(RqTq^{-1})^Ts^{-1})dsdq \\ &= \iint f(s(q^T)^{-1}T^Tq^TR^Ts^{-1})dsdq.\end{aligned} \tag{22}$$

Because of Lemma 2, q^T can be replaced herein by q. There is, furthermore, some x in U_{n-1} such that $T^T = xTx^{-1}$ and some y such that $R^T = yRy^{-1}$. Hence (22) goes over into

$$\iint f(sRs^{-1}uTu^{-1})dsdu = \iint f(sq^{-1}xTx^{-1}qyRy^{-1}s^{-1})dsdq. \tag{23}$$

However, the integral over q remains invariant if one replaces q by xqs. After that, one can replace s by sy^{-1} and, finally, q by u^{-1}. Hence

$$\begin{aligned}\iint f(sRs^{-1}uTu^{-1})dsdu &= \iint f(q^{-1}TqsyRy^{-1}s^{-1})dsdq \\ &= \iint f(q^{-1}TqsRs^{-1})dsdq \\ &= \iint f(uTu^{-1}sRs^{-1})dsdu.\end{aligned} \tag{24}$$

This verifies (11) for our case leading to the result anticipated:

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THEOREM 2. *The irreducible representations of U_n do not contain any representation of U_{n-1} more than once.*

One easily verifies that *the same is true for the subgroup $U_1 \times SU_{n-1}$*. However, it is not true for SU_{n-1} alone, no matter whether the original group were U_n or SU_n. This is particularly obvious for $n = 2$: in this case SU_1 contains only the identity element, whereas SU_2 is not commutative and has many dimensional irreducible representations.

4. The preceding demonstration of Theorem 2 may appear complicated. However, the only essential point is that an anti-isomorphism $R \rightarrow R'$ (resulting in $R'T' = (TR)'$) can be generated by the elements s of the subgroup ($R' = sRs^{-1}$) on all elements R of the full group. In the case considered, the anti-isomorphism was $R' = R^T$. This observation renders it possible to generalize Theorem 2.

THEOREM 3. *If an anti-isomorphism of the full group $R \rightarrow R' = sRs^{-1}$ can be induced on all elements R of the full group by the elements s of the subgroup (s depending on R), then no irreducible representation of the full group contains any representation of the subgroup more than once.*

Theorem 3, in contrast to (11), gives a condition for its conclusion which is sufficient but not necessary.

REFERENCES

1. M. Hamermesh, *Group theory and its application to physical problems* (Addison Wesley, London, 1962).
2. O. Taussky and H. Zassenhaus, *On the similarity transformation between a matrix and its transpose*, Pacific J. Math. *9* (1959), 893–896.
3. E. P. Wigner, *Condition that the irreducible representations of a finite group, considered as representations of a subgroup, do not contain any representation more than once*, Spectroscopic and Group Theoretical Methods in Physics, F. Loebl, Editor (North Holland, Amsterdam, 1968).

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