BULL. AUSTRAL. MATH. SOC. 16A49, 16A35
VOL. 39 (1989) [339–342]

MORITA DUALITY AND ARTINIAN LEFT DUO RINGS

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We characterise artinian left duo rings via Morita duality, and show that there is a large class of such rings that possess a duality.

Azumaya [2] and Morita [7] proved that there is a duality between the finitely generated left modules over a ring R and the finitely generated right modules over some ring S if and only if R is left artinian and the indecomposable injective left R-modules are all finitely generated. Azumaya [3] proved that his exact rings have duality and conjectured that they have self-duality.

A ring is called *left duo (right duo)* if each left (right) ideal is a two-sided ideal, and it is *duo* if it is both left and right duo. Habeb [6] observed that artinian duo rings are exact. In [10], we verified Azumaya's conjecture for a large class of artinian duo rings. (for example, those with Jacobson radical square zero.) Rosenberg and Zelinsky [8] gave an example of a local left artinian left duo ring without duality. In this example the ring is not right artinian. Motivated by this example, we study (two-sided) artinian left duo rings and Morita duality. A presentation of Morita duality can be found in Anderson and Fuller [1, Section 23, Section 24].

As Habeb [6] noted, artinian left duo rings are finite direct sums of local artinian left duo rings, therefore we need consider only local artinian left duo rings. Throughout this paper, R is a local artinian left duo ring with Jacobson radical $J \neq 0$. According to [8, Theorem 1 and Remark 1], we can assume that $J^2 = 0$. Let $D = R/J$ be the division ring. Then ${}_D J_D$ is a bi-vector space with finite dimensional on each side, since R is artinian. By [8, Theorem 1], the ring R has a duality if and only if ${}_D\operatorname{Hom}_D(J, D)$ is finitely generated.

Since R is left duo, $Da \supseteq aD$ for each $0 \neq a \in J$, so there is a ring monomorphism

$$\sigma_a : D \longrightarrow D$$

given by $\sigma_a(d)a = ad$ for all $d \in D$. And $\sigma_a(D)$ is a division subring of D.

Received 11 July 1988

LEMMA 1. *For each $0 \neq a \in J$, $D_{\sigma_a(D)}$ is finite dimensional.*

PROOF: Since R is artinian left duo, $(Da)_D$ is a finitely generated right D-module. Let

$$Da = \bigoplus_{j=1}^{m} (d_j a)D = \bigoplus_{j=1}^{m} d_j \sigma_a(D) a$$

for some $d_1, \ldots, d_m \in D$, and then $D = \sum_{j=1}^{m} d_j \sigma_a(D)$. ∎

Our main result gives a complete characterisation of artinian left duo rings via Morita duality.

THEOREM 2. *For the local artinian left duo ring R we have the following:*

(1) *if $\dim({}_D J) > 1$, then R has a duality; in this case D is a field;*

(2) *if $\dim({}_D J) = 1$; let $J = Da$ and $\sigma = \sigma_a$; then*

(i) if ${}_{\sigma(D)}D$ is finite dimensional, R has a duality; in particular, if D is a field then R has a duality;

(ii) if ${}_{\sigma(D)}D$ is not finite dimensional, R does not have a duality.

PROOF: (1) First we use an idea in the proof of [**6**, Theorem 3.11] to show that the monomorphism σ_a does not depend on a.

Let a and $b \in J$ be linearly independent over D. Then for each $d \in D$, we have

$$\sigma_a(d)a = ad,\ \sigma_b(d)b = bd, \text{ and } \sigma_{a+b}(d)(a+b) = (a+b)d.$$

So $\sigma_a(d) = \sigma_{a+b}(d) = \sigma_b(d)$, and hence $\sigma_a = \sigma_b$.

Now let $0 \neq a$ and $0 \neq b \in J$ be linearly dependent over D. Since $\dim({}_D J) > 1$, we can find $c \in J$ such that c is linearly independent of a whence linearly independent of b. From the discussion above, we have $\sigma_a = \sigma_c = \sigma_b$.

This proves that all the non-zero elements determine the same monomorphism, say, $\sigma: D \longrightarrow D$. Let $0 \neq d_1$, $d_2 \in D$ and $0 \neq a \in J$, then by the uniqueness of σ we have

$$d_1 \sigma(d_2) a = d_1 a d_2 = \sigma(d_2) d_1 a.$$

So $d_1 \sigma(d_2) = \sigma(d_2) d_1$, which says that $\sigma(D)$ is contained in the centre of D. In particular, $\sigma(D)$ is a subfield of D. But $D \cong \sigma(D)$, so D is a field. It follows from Lemma 1 that ${}_{\sigma(D)}D$ is finitely generated.

Let $D = \sum_{j=1}^{m} \sigma(D) d_j$ and $J = \bigoplus_{i=1}^{n} Da_i$ for some $d_1, \ldots, d_m \in D$ and $a_1, \ldots, a_n \in J$. By the uniqueness of σ, we have

$$\sigma(d)a_i = a_i d$$

for all $d \in D$ and all i, $1 \leqslant i \leqslant n$. Since $Da_i \supseteq a_i D$, we have

$$_D\operatorname{Hom}_D(J, D) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_D(Da_i, D)$$

as left D-modules. So we only need to show that each $_D\operatorname{Hom}_D(Da_i, D)$ is finitely generated. To do so, let $a = a_i$ be fixed.

We define $f_j \in \operatorname{Hom}_D(Da, D)$ by $f_j(a) = d_j$, $1 \leqslant j \leqslant m$, and claim that $_D\operatorname{Hom}_D(Da, D) = \sum_{j=1}^{m} Df_j$. In fact, for each $f \in \operatorname{Hom}_D(Da, D)$, we have $f(a) = \sum_{j=1}^{m} \sigma(t_j)d_j$ for some $t_j \in D$. Then

$$\begin{aligned}\left(\sum_{j=1}^{m} t_j f_j\right)(a) &= \sum_{j=1}^{m} f_j(at_j) = \sum_{j=1}^{m} f_j(\sigma(t_j)a)\\ &= \sum_{j=1}^{m} \sigma(t_j)f_j(a) = \sum_{j=1}^{m} \sigma(t_j)d_j = f(a).\end{aligned}$$

Therefore $f = \sum_{j=1}^{m} t_j f_j$ which proves our claim. It follows that $_D\operatorname{Hom}_D(J, D)$ is finitely generated.

(2) (i) Since $_{\sigma(D)}D$ is finitely generated, we let $D = \sum_{j=1}^{m} \sigma(D)d_j$ for some $d_1, \ldots, d_m \in D$. Defining the same f_j's as that given in the proof of (1), one shows that

$$_D\operatorname{Hom}_D(J, D) = {}_D\operatorname{Hom}_D(Da, D) = \sum_{j=1}^{m} Df_j,$$

which is finitely generated. If D is a field, then by Lemma 1 $_{\sigma(D)}D$ is finite dimensional.

(2) (ii) For $f_1, \ldots, f_n \in \operatorname{Hom}_D(Da, D)$, let $f_i(a) = d_i$. Since $_{\sigma(D)}D$ is not finitely generated, there is a $d \in D \setminus \left(\sum_{i=1}^{n} \sigma(D)d_i\right)$. Define $f \in \operatorname{Hom}_D(Da, D)$ by $f(a) = d$. Then $f \notin \sum_{i=1}^{n} Df_i$ and so $_D\operatorname{Hom}_D(Da, D)$ is not finitely generated. It follows that R does not have a duality. ∎

It should be noted that artinian left duo rings arise much more frequently than artinian duo rings. A typical example is Dlab and Ringel's "exceptional (1, 2) ring" [5, Proposition II.3.3]. These rings belong to the class of artinian left duo rings described in our Theorem 2(2)(i).

Cohn [4] (respectively, Schofield [9]) has shown that there is a division ring D with a division subring C such that, $\dim(D_C)$ is finite but $\dim({}_C D) = \infty$ (respectively, both $\dim(D_C)$ and $\dim({}_C D)$ are finite but different). From this one constructs an artinian ring $\begin{bmatrix} D & D \\ 0 & C \end{bmatrix}$ without a duality (respectively, which has a duality but without self-duality). The author is unable to answer the following questions.

Question 1 (respectively, 2): Do there exist division rings D and C desciibed above with $D \cong C$ as rings?

If Question 1 (respectively, 2) is true, using the same method as in [8, p.375] we can construct a local artinian left duo ring without a duality (respectively, with a duality but without self-duality). Simply let ${}_D J = {}_D D$ as left D-module, and as a right D-module, define $jd = jg(d)$ in D, where $g : D \cong C$. And set $R = D \times J$ with multiplication

$$(d_1, j_1)(d_2, j_2) = (d_1 d_2, d_1 j_2 + j_1 d_2).$$

Then R is the ring we need. If Question 1 is false, the set of the rings in Theorem 2(2) (ii) is empty. In this event we conclude that every artinian left duo ring has a Morita duality.

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