THE ISOMETRIES OF H^p

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1. Let σ be the Lebesgue measure on the unit circle $|z| = 1$ with

$$\int d\sigma = 1$$

and let $\mathbf{L}^p$ be the space of complex-valued σ-measurable functions f such that

$$\int |f|^p \, d\sigma$$

is finite. $\mathbf{H}^p$ is the closure in $\mathbf{L}^p$ of the algebra of analytic polynomials

$$\sum_{n \geqslant 0} c_n z^n \qquad (|z| = 1).$$

When $1 \leqslant p$, $\mathbf{H}^p$ has a neat alternative description as the subspace in $\mathbf{L}^p$ of functions whose Fourier coefficients vanish for negative indices. We shall always mean by p a finite positive number.

A linear isometry of $\mathbf{H}^p$ is a linear transformation T of $\mathbf{H}^p$ into $\mathbf{H}^p$ such that

$$\int |Tf|^p \, d\sigma = \int |f|^p \, d\sigma.$$

The purpose of this paper is to describe the linear isometries of $\mathbf{H}^p$ when $p \neq 2$. As $\mathbf{H}^2$ is a Hilbert space, it has many isometries, and no more can be said about them than can be said about the isometries of any Hilbert space. DeLeeuw, Rudin, and Wermer in **(3)** have described all linear isometries of $\mathbf{H}^1$ *onto* $\mathbf{H}^1$. They discovered that such isometries come from conformal maps of the unit disk onto itself. We have found that the same is true of isometries of $\mathbf{H}^p$ onto $\mathbf{H}^p$, and that isometries of $\mathbf{H}^p$ into $\mathbf{H}^p$ come from inner functions.

The argument in **(3)** depends in an essential way on a result obtained by deLeeuw and Rudin in **(2)** about the extreme points of the unit ball of $\mathbf{H}^1$, and on the observation that an isometry of $\mathbf{H}^1$ onto $\mathbf{H}^1$ is a permutation of these extreme points. When isometries of $\mathbf{H}^1$ *into* $\mathbf{H}^1$ are considered, the argument in **(3)** is no longer applicable. When $1 < p$, every function on the boundary of the unit ball of $\mathbf{H}^p$ is an extreme point of this unit ball, and therefore the observation that these extreme points are permuted by an onto isometry will not give any information about the isometry. When $0 < p < 1$, we do not know anything about the geometry of the unit ball of $\mathbf{H}^p$ that will give any information about an isometry of $\mathbf{H}^p$. Banach **(1**, chap. 11**)** and Lamperti **(5)** have studied the isometries of $\mathbf{L}^p$. Their methods are not applicable to our problem.

It happens that a pair of elementary propositions can be used to get at the

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$\mathbf{H}^p$ isometries. These propositions are in the next section. A description of the linear isometries of $\mathbf{H}^p$ is in the third section.

2. In the following propositions σ_1 and σ_2 are positive measures defined on any two measurable spaces with

$$\int d\sigma_1 = \int d\sigma_2 = 1.$$

PROPOSITION 1. *Suppose that f_k is in $\mathbf{L}^p(\sigma_k)$ $(k = 1, 2)$ and that for all complex numbers z*

$$\int |1 + zf_1|^p \, d\sigma_1 = \int |1 + zf_2|^p \, d\sigma_2. \tag{1}$$

Then

$$\int |f_1|^2 \, d\sigma_1 = \int |f_2|^2 \, d\sigma_2.$$

Proof. If both

$$\int |f_1|^2 \, d\sigma_1, \qquad \int |f_2|^2 \, d\sigma_2$$

are infinite, there is nothing to show. Suppose then that

$$\int |f_1|^2 \, d\sigma_1$$

is finite. Consider

$$\frac{1}{2\pi} \int_0^{2\pi} |1 + ze^{ix}|^p \, dx - 1. \tag{2}$$

When $|z| < 1$,

$$(1 + ze^{ix})^{p/2} = \sum_{j \geqslant 0} \binom{p/2}{j} z^j e^{ijx}$$

and (2) is given by

$$(p^2/4)|z|^2 + \sum_{j \geqslant 2} \binom{p/2}{j}^2 |z|^{2j},$$

and therefore

$$r^{-2}\left(\frac{1}{2\pi} \int_0^{2\pi} |1 + rf_k e^{ix}|^p \, dx - 1\right) \rightarrow (p^2/4)|f_k|^2 \tag{3}$$

when $r \rightarrow 0$. Now (2) is non-negative for all z. Thus, the left side of (3) is non-negative and we may apply the Fatou-Lebesgue theorem to (3). Then

$$(p^2/4) \int |f_2|^2 \, d\sigma_2$$

is not greater than the lower limit when $r \rightarrow 0$ of

$$r^{-2}\left(\frac{1}{2\pi} \int \int_0^{2\pi} |1 + rf_k e^{ix}|^p \, dx \, d\sigma_k - 1\right) \tag{4}$$

with $k = 2$. Now (2) is bounded by $A|z|^2$ when $|z| < 1$, and by $A|z|^p$ when $|z| \geqslant 1$, where A depends only on p. Therefore, (2) is bounded by $A|z|^2 + A|z|^p$, and by $A|z|^2$ if $0 < p \leqslant 2$. Thus, the left side of (3) is bounded by

$$A|f_k|^2 + A|r|^{p-2}|f_k|^p$$

if $2 \leqslant p$, and by

$$A|f_k|^2$$

if $0 < p \leqslant 2$, and we may apply the dominated convergence theorem to (3) with $k = 1$. Then

$$(p^2/4)\int |f_1|^2\, d\sigma_1$$

is the limit when $r \to 0$ of (4) with $k = 1$. But (4) does not depend on k because of our assumption (1), and we find that

$$\int |f_2|^2\, d\sigma_2 \leqslant \int |f_1|^2\, d\sigma_1.$$

But now

$$\int |f_2|^2\, d\sigma_2$$

is finite, and so the reverse inequality is also true.

PROPOSITION 2. *Let* $\mathbf{A}$ *be a subalgebra of* $\mathbf{L}^\infty(\sigma_1)$ *that contains constants, and let* T *be a linear transformation of* $\mathbf{A}$ *into* $\mathbf{L}^\infty(\sigma_2)$ *with* $T\,1 = 1$. *Suppose that* $p \neq 2$ *and*

$$\int |Tf|^p\, d\sigma_2 = \int |f|^p\, d\sigma_1$$

for all f *in* $\mathbf{A}$. *Then* T *is multiplicative:*

$$T(fg) = Tf\; Tg. \tag{5}$$

Proof. Let f, g belong to $\mathbf{A}$. Since $T(1 + zf) = 1 + zTf$, we have

$$\int |1 + zTf|^p\, d\sigma_2 = \int |1 + zf|^p\, d\sigma_1 \tag{6}$$

for all complex numbers z. When $|z|$ is small, $|1 + zf|^p$ is given by

$$\sum_{j,k \geqslant 0} \binom{p/2}{j}\binom{p/2}{k} z^j \bar{z}^k f^j \bar{f}^k$$

and the right side of (6) is given by

$$\sum_{j,k \geqslant 0} \binom{p/2}{j}\binom{p/2}{k} z^j \bar{z}^k \int f^j \bar{f}^k\, d\sigma_1. \tag{7}$$

Similarly, when $|z|$ is small, the left side of (6) is given by

$$\sum_{j,k \geqslant 0} \binom{p/2}{j}\binom{p/2}{k} z^j \bar{z}^k \int (Tf)^j\, (\overline{Tf})^k\, d\sigma_2. \tag{8}$$

As (7) and (8) have the same values when $|z|$ is small, they must have the same coefficients, and we find that

$$\binom{p/2}{j}\binom{p/2}{k} \int (Tf)^j\, (\overline{Tf})^k\, d\sigma_2 = \binom{p/2}{j}\binom{p/2}{k} \int f^j \bar{f}^k\, d\sigma_1. \tag{9}$$

If p is an even integer, the binomial coefficient $\binom{p/2}{j}$ vanishes when $j > p/2$, and (9) does not say anything when $j > p/2$ or $k > p/2$. Anyway $\binom{p/2}{j} \neq 0$

for $j = 0, 1, 2$ since $p \neq 2$, and we have

$$\int (Tf)^2\, \overline{Tf}\, d\sigma_2 = \int f^2\, \bar{f}\, d\sigma_1 \tag{10}$$

and

$$\int Tf\, \overline{Tf}\, d\sigma_2 = \int f\, \bar{f}\, d\sigma_1. \tag{11}$$

Now (11) gives

$$\int Tg\, \overline{Tf}\, d\sigma_2 = \int g\bar{f}\, d\sigma_1$$

and therefore, as **A** is an algebra,

$$\int T(f^2)\overline{Tf}\, d\sigma_2 = \int f^2\, \bar{f}\, d\sigma_1. \tag{12}$$

From (10) and (12) we get

$$\int (Tf)^2\, \overline{Tf}\, d\sigma_2 = \int T(f^2)\overline{Tf}\, d\sigma_2.$$

Replace f by $zf + g$ in this identity. Then both sides are polynomials in z and $\bar{z}$, and identifying coefficients we find that

$$\int (Tf)^2\, \overline{Tg}\, d\sigma_2 = \int T(f^2)\overline{Tg}\, d\sigma_2.$$

With this we find that

$$\int |(Tf)^2 - T(f^2)|^2\, d\sigma_2 = 0$$

and so

$$T(f^2) = (Tf)^2. \tag{13}$$

(5) can be obtained from (13) by replacing f by $f + g$.

3. $\mathbf{H}^\infty$ is the subspace in $\mathbf{L}^\infty$ of functions whose Fourier coefficients vanish for negative indices. A function ϕ in $\mathbf{H}^\infty$ is said to be an inner function if $|\phi| = 1$. An inner function plays two roles: it is a vector in $\mathbf{H}^\infty$ and it is an analytic transformation of the unit circle into itself. The important role in this drama is as an analytic transformation. Suppose ϕ is a non-constant inner function and let μ be the Borel measure on the unit circle induced by ϕ:

$$\mu X = \sigma(\phi^{-1}X)$$

for Borel subsets X of the unit circle. Then

$$\int g\, d\mu = \int g(\phi)\, d\sigma$$

when g is a Borel-measurable function, and the Fourier-Stieltjes coefficient of μ at n is

$$\int \bar{\phi}^n\, d\sigma.$$

As averaging with σ is a multiplicative linear functional on $\mathbf{H}^\infty$,

$$\int \phi^n\, d\sigma = \left(\int \phi\, d\sigma\right)^n$$

when $n > 0$, and thus the Fourier-Stieltjes coefficient of μ at $n > 0$ is

$$\left(\int \bar{\phi}\, d\sigma\right)^n.$$

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Therefore

(14) $$\mu = P\sigma,$$

where P is the Poisson kernel:

$$P(z) = (1 - |a|^2)/|1 - az|^2 \qquad (a = \int \bar{\phi}\, d\sigma).$$

We shall call P the Poisson kernel induced by ϕ. (14) will be needed later. Because μ is absolutely continuous with respect to σ, the definition of μ shows that ϕ^{-1} takes subsets of the unit circle with σ measure zero into sets of the same kind. Thus, $f(\phi)$ is defined when f is in $\mathbf{L}^p$. We shall denote by $\mathbf{\Sigma}$ the collection of σ-measurable subsets of the unit circle, and by $\mathbf{\Sigma}(\phi)$ the collection of sets $X\Delta Z$, where $X = \phi^{-1}Y$ with Y in $\mathbf{\Sigma}$ and Z is in $\mathbf{\Sigma}$ with $\sigma Z = 0$. Then $\mathbf{\Sigma}(\phi)$ is contained in $\mathbf{\Sigma}$.

THEOREM 1. *Suppose that $p \neq 2$ and T is a linear isometry of $\mathbf{H}^p$ into $\mathbf{H}^p$. Then there is a non-constant inner function ϕ and a function F in $\mathbf{H}^p$ such that*

(15) $$Tf = Ff(\phi).$$

ϕ and F are related by:

(16) $$\int_X |F|^p\, d\sigma = \int_X 1/P(\phi)\, d\sigma$$

when X is in $\mathbf{\Sigma}(\phi)$, where P is the Poisson kernel induced by ϕ. Conversely, when a non-constant inner function ϕ and a function F in $\mathbf{H}^p$ are related by (16), (15) *defines a linear isometry of $\mathbf{H}^p$ into $\mathbf{H}^p$.*

Proof. We shall require certain properties of analytic functions. A convenient reference is Hoffman's book **(4)**, which also contains the deLeeuw-Rudin-Wermer solution of the $\mathbf{H}^1$ isometry problem.

Suppose T is a linear isometry of $\mathbf{H}^p$. To use our propositions, we need an isometry that takes 1 to 1. Let $F = T1$, and let ν be the measure $d\nu = |F|^p\, d\sigma$. As F is in $\mathbf{H}^p$ and $F \neq 0$, F cannot vanish on any set of positive σ-measure, and thus ν and σ are mutually absolutely continuous. Define a linear transformation S from $\mathbf{H}^p$ to $\mathbf{L}^p(\nu)$ by $Sf = Tf/F$. Then S is an isometry of $\mathbf{H}^p$ into $\mathbf{L}^p(\nu)$ with $S1 = 1$. Let χ be the famous inner function $\chi(z) = z$. Then

$$\int |S(\chi^n)|^p\, d\nu = 1,$$

and (Proposition 1)

$$\int |S(\chi^n)|^2\, d\nu = 1,$$

and thus $|S(\chi^n)| = 1$ as $p \neq 2$. This tells us that S takes the algebra generated by χ into $\mathbf{L}^\infty(\nu)$, and now we know that S is multiplicative on this algebra (Proposition 2). Then, when f is a polynomial, $S(f(\chi)) = f(S\chi)$ and

(17) $$T(f(\chi)) = Ff(\phi)$$

where $\phi = S\chi$.

Because F is in $\mathbf{H}^p$, $F = MG$, where M is an inner function and G is an outer

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function in $\mathbf{H}^p$. The property of outer functions we require is that if f is in $\mathbf{L}^\infty$ and fG is in $\mathbf{H}^p$, then f is in $\mathbf{H}^\infty$. Thus, because of (17), $M\phi^n$ is an inner function when $n \geqslant 0$.

ϕ is an inner function. There are several ways to see this. One method is to pass to the interior of the unit disk and look at the Blaschke products and singular parts of M and $M\phi$. We prefer a less direct method, based on invariant subspaces, that keeps things on the unit circle. Let $\mathbf{B}$ be the closed subspace of $\mathbf{L}^2$ spanned by $\chi^j \phi^k$ $(j, k \geqslant 0)$. Then $\mathbf{B}$ is invariant under multiplication by χ, but not by $\bar{\chi}$ as $M\mathbf{B}$ is contained in $\mathbf{H}^2$. Therefore, $\mathbf{B} = \psi\mathbf{H}^2$ where $|\psi| = 1$. Now $\chi^j \phi^k \psi$ is in $\mathbf{B}$ for $j, k \geqslant 0$, and, too, ψ can be approximated by polynomials in χ and ϕ. Thus ψ^2 is in $\mathbf{B}$, and therefore $\psi^2 = \psi g$ where g is in $\mathbf{H}^2$. Then ψ is in $\mathbf{H}^2$ and $\mathbf{B} = \mathbf{H}^2$. Thus ϕ is inner as ϕ is in $\mathbf{B}$.

We have a non-constant inner function ϕ and a function F in $\mathbf{H}^p$ such that Tf is given by (15) when f is in the algebra generated by χ. As this algebra is dense in $\mathbf{H}^p$ and T is bounded, Tf is given by (15) for all f in $\mathbf{H}^p$.

Now

$$\int |F|^p |f(\phi)|^p \, d\sigma = \int |f|^p \, d\sigma \tag{18}$$

when f is in $\mathbf{H}^p$. Let $X = \phi^{-1} Y$ where Y is in $\mathbf{\Sigma}$. We get, from (18),

$$\int_X |F|^p \, d\sigma = \int_Y d\sigma \tag{19}$$

because the characteristic function of Y can be approximated by the moduli of functions in $\mathbf{H}^p$, and we get from (14)

$$\int_Y d\sigma = \int_Y 1/P \, d\mu = \int_X 1/P(\phi) \, d\sigma. \tag{20}$$

We get (16) from (19) and (20).

Conversely, when a non-constant inner function ϕ and a function F in $\mathbf{H}^p$ are related by (16), we get (19) from (16) and (20), and (19) shows that T, defined by (15), is an isometry of $\mathbf{L}^p$ into $\mathbf{L}^p$ when T is restricted to simple functions. Then T is an isometry of $\mathbf{L}^p$. Now T takes the algebra generated by χ into $\mathbf{H}^p$, and therefore T takes $\mathbf{H}^p$ into $\mathbf{H}^p$.

We shall say that an inner function ϕ is a conformal map of the unit disk onto itself if ϕ is (a.e.) the restriction of such a function to the unit circle.

THEOREM 2. *Suppose that $p \neq 2$ and T is a linear isometry of $\mathbf{H}^p$ onto $\mathbf{H}^p$. Then*

$$Tf = b(d\phi/dz)^{1/p} f(\phi) \tag{21}$$

where ϕ is a conformal map of the unit disk onto itself and b is a unimodular complex number. Conversely, (21) *defines a linear isometry of $\mathbf{H}^p$ onto $\mathbf{H}^p$.*

When $p = 1$, Theorem 2 is the theorem of deLeeuw, Rudin, and Wermer **(3)**.

Proof. As T and T^{-1} are linear isometries of $\mathbf{H}^p$, $Tf = Ff(\phi)$ and $T^{-1}f = Gf(\psi)$. Then from $TT^{-1}f = T^{-1}Tf = f$ we find that

$$F\,G(\phi)\,f(\psi(\phi)) = G\,F(\psi)\,f(\phi(\psi)) = f.$$

Taking $f = 1$ shows that $F\,G(\phi) = G\,F(\psi) = 1$, and this in turn shows that $f(\psi(\phi)) = f(\phi(\psi)) = f$, and this shows that ϕ is a conformal map of the unit disk onto itself. Now, because ϕ is a conformal map of the unit disk onto itself, $\mathbf{\Sigma}(\phi) = \mathbf{\Sigma}$ and $|d\phi/dz| = 1/P(\phi)$ where P is the Poisson kernel induced by ϕ, and (16) becomes:

$$\int_X |F|^p\,d\sigma = \int_X |d\phi/dz|\,d\sigma$$

when X is in $\mathbf{\Sigma}$. Thus F and $(d\phi/dz)^{1/p}$ have the same modulus. F is an outer function, for otherwise the non-constant inner factor of F would divide every function in $\mathbf{H}^p$, and as $(d\phi/dz)^{1/p}$ is also an outer function and has the same modulus as F, $F = b(d\phi/dz)^{1/p}$, where b is a unimodular complex number.

4. Theorem 1 tells us that an isometry of $\mathbf{H}^p$ comes from an inner function, and that every inner function generates many isometries of $\mathbf{H}^p$. However, it does not tell us anything about the functions F in $\mathbf{H}^p$ that are related by (16) to an inner function ϕ. There is, of course, an exception: when ϕ is a conformal map of the unit disk onto itself, the modulus of F must be $1/P(\phi)$. But *when ϕ is not a conformal map of the unit disk onto itself, the modulus of F is not determined by ϕ.* Here are the details.

THEOREM. *Let ψ and ϕ be inner functions. Then $\mathbf{\Sigma}(\psi)$ is contained in $\mathbf{\Sigma}(\phi)$ if and only if there is an inner function π with $\psi = \pi(\phi)$.*

Proof. Assume $\mathbf{\Sigma}(\psi)$ is contained in $\mathbf{\Sigma}(\phi)$.

Suppose

$$\int \phi\,d\sigma = 0.$$

Then $Tf = f(\phi)$ is an isometry of $\mathbf{L}^2$ into $\mathbf{L}^2$. The characteristic function of a set in $\mathbf{\Sigma}(\phi)$ is in $T\mathbf{L}^2$, and therefore ψ is in $T\mathbf{L}^2$ because ψ can be approximated by linear combinations of characteristic functions of sets in $\mathbf{\Sigma}(\psi)$. Now

$$\int \psi\bar{\phi}^n\,d\sigma = 0$$

when $n < 0$, and therefore ψ is in $T\mathbf{H}^2$. Let π be the function in $\mathbf{H}^2$ such that $T\pi = \psi$. Then $|\pi| = 1$ because $\psi = \pi(\phi)$.

Now let

$$c = \int \phi\,d\sigma.$$

We can assume that $|c| < 1$, for otherwise ϕ is a constant, and as $\mathbf{\Sigma}(\psi)$ is contained in $\mathbf{\Sigma}(\phi)$, ψ is too. Let λ be a conformal map of the unit disk onto itself that takes c to 0. Then

$$\int \lambda(\phi)\,d\sigma = 0$$

and $\mathbf{\Sigma}(\lambda(\psi))$ is contained in $\mathbf{\Sigma}(\lambda(\phi))$, and thus there is an inner function π with $\lambda(\psi) = \pi(\lambda(\phi))$.

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COROLLARY. *Let ψ and ϕ be inner functions. Then* $\mathbf{\Sigma}(\psi) = \mathbf{\Sigma}(\phi)$ *if and only if there is a conformal map π of the unit disk onto itself with $\psi = \pi(\phi)$.* $\mathbf{\Sigma}(\phi) = \mathbf{\Sigma}$ *if and only if ϕ is a conformal map of the unit disk onto itself.*

Proof. Assume that $\mathbf{\Sigma}(\psi) = \mathbf{\Sigma}(\phi)$. There are inner functions λ and π with $\phi = \lambda(\psi)$ and $\psi = \pi(\phi)$. Then $\phi = \lambda(\pi(\phi))$ and $\psi = \pi(\lambda(\psi))$, and these identities show that λ and π are conformal maps of the unit disk onto itself when ϕ and ψ are not constants.

Suppose now that ϕ is a non-constant inner function and that ϕ is not a conformal map of the unit disk onto itself. The condition given by (16) on F is that the conditional expectation of $|F|^p$ given $\mathbf{\Sigma}(\phi)$ is $1/P(\phi)$:

$$E(|F|^p) = 1/P(\phi).$$

E is a linear projection of $\mathbf{L}^1$ onto the subspace in $\mathbf{L}^1$ of $\mathbf{\Sigma}(\phi)$-measurable functions that takes positive functions to positive functions. As $\mathbf{\Sigma}(\phi) \neq \mathbf{\Sigma}$, there are bounded and real $\mathbf{\Sigma}$-measurable functions $v \neq 0$ such that

$$E(v) = 0.$$

An example is $v = w - E(w)$, where w is the characteristic function of a set in $\mathbf{\Sigma}$ that is not in $\mathbf{\Sigma}(\phi)$. Then

$$E((1/P(\phi)) + \epsilon v) = 1/P(\phi)$$

and, when ϵ is small, $((1/P(\phi)) + \epsilon v)^{1/p}$ is the modulus of a function in $\mathbf{H}^p$.

Does (16) imply that F is bounded? We do not know. When F is on the boundary of the unit ball of $\mathbf{H}^p$, is there always an inner function ϕ such that F and ϕ are related by (16)?

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