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ON THE LEVITZKI RADICAL

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1. **Introduction.** The Levitzki radical, which is fundamental in the study of algebras satisfying a polynomial identity, has been shown to exist in the varieties of alternative and Jordan algebras (see Zhevlakov [8], Zwier [9], and Tsai [7]—for an important application of this radical to alternative algebras satisfying a polynomial identity, see Slater [6]). In fact, Hartley [4] even investigated local nilpotence for Lie algebras, though this property can not be radical in the sense of Kurosh-Amitsur [3] for these algebras.

Naturally, the existence or non-existence of a satisfactory theory of local nilpotence in a variety of algebras will depend on the particular identities defining the variety. Zwier [9], by looking at the general form of each of a set I of identities, was able to give a sufficiency condition for the existence of the Levitzki radical in the variety $\mathscr{V}(I)$ of algebras defined by the set I. Recently the author [2] gave conditions on the existence of this radical by studying for each $A \in \mathscr{V}(I)$ its universal enveloping algebra $\mathfrak{U}(A)$.

In this paper we sharpen the main result of [2] to derive a very simple sufficiency condition for the existence of the Levitzki radical in what Zwier [9] calls 2-varieties. For such a variety of algebras, over an operator ring Φ, our condition is just that the associative algebra $\mathfrak{U}(A)$ be nilpotent for $A=\Phi x$, $x^2=0$. This result allows us to describe explicitly, in terms of defining identities, which 2-varieties of power-associative algebras carry a satisfactory theory of the Levitzki radical.

It is pointed out that a 2-variety is similar to what Albert [1] previously called a variety of almost alternative algebras, and these include Lie, associative, and alternative algebras. We determine the defining identities of the largest 2-variety of power-associative algebras in which local nilpotence is not a Kurosh-Amitsur radical. This variety is in a sense a generalization of Lie algebras. Following: A structure theory for the finite dimensional algebras of this class will be given elsewhere.

With this paper and the previous one [2], we hope to have shown that universal enveloping algebras are useful for reasons other than the historical one of studying irreducible representations of algebras. In the present context we feel a general theory of algebras satisfying a polynomial identity could be developed for a variety whose universal enveloping algebras satisfy a condition such as the one above.

2. **Main results.** Throughout this paper Φ is a fixed commutative associative ring with unity and $\mathscr{V}(I)$ is a variety of non-associative Φ-algebras that satisfy a set I of identities (which are homogeneous in the sense of [2]).

If $A \in \mathscr{V}(I)$ and M is a Φ-space which admits bilinear compositions ma and $am(\in M)$ for $a \in A$, $m \in M$ then $A \oplus M$ can be made into an algebra by defining $(a+m)(b+n)=ab+mb+an$ for $a, b \in A$, $m, n \in M$. If $A \oplus M \in \mathscr{V}(I)$ then M is called a bimodule for A. Such a bimodule defines a pair of linear maps $S, T: A \to \mathrm{Hom}(M, M)$, where $S_a: m \to ma$ and $T_a: m \to am$ for $a \in A$ and $m \in M$. The pair $\rho=(S, T)$ is called the representation defined by the bimodule M. As in the case of associative algebras, the notions of representation and bimodule, are equivalent (see [5]).

For $A \in \mathscr{V}(I)$, $\mathfrak{U}(A)$ will denote the universal enveloping algebra for the representations of A. The associative algebra $\mathfrak{U}(A)$ is characterised by the fact that there is a (canonical) representation $\rho^*=(S^*, T^*)$ of A in $\mathfrak{U}(A)$ such that for any representation $\rho=(S, T)$ of A in an associative algebra X there is a unique homomorphism $\psi: \mathfrak{U}(A) \to X$ for which $\rho=\rho^*\psi$, in the sense that $S=S^*\psi$ and $T=T^*\psi$.

It is important to realize that $\mathfrak{U}(A)$ is an extrinsic object—it depends not only on the algebra A but also on the variety $\mathscr{V}(I)$ which contains A. For example, if $A=\Phi x$, $x^2=0$, then A belongs to the variety of associative algebras, as well as the variety of Lie algebras. Thinking of A as an associative algebra, $\mathfrak{U}(A)^3=0$. However, if A is regarded as a member of the variety of Lie algebras, then $\mathfrak{U}(A)=\Phi[t]$, the ring of polynomials in t with zero constant term.

A 2-variety is defined to have the property that whenever $A \in \mathscr{V}(I)$ and J is an ideal of A, so is J^2 an ideal of A. By using the homogeneity of I it was shown in [2, p. 30] that a 2-variety $\mathscr{V}(I)$ satisfies two identities of degree three, namely

$$(1) \qquad (x_1x_2)x_3 = \alpha_1(x_3x_1)x_2+\alpha_2(x_1x_3)x_2+\alpha_3x_2(x_3x_1)+\alpha_4x_2(x_1x_3) \\ +\alpha_5(x_3x_2)x_1+\alpha_6(x_2x_3)x_1+\alpha_7x_1(x_3x_2)+\alpha_8x_1(x_2x_3)$$

and

$$(2) \qquad x_3(x_1x_2) = \beta_1(x_3x_1)x_2+\beta_2(x_1x_3)x_2+\beta_3x_2(x_3x_1)+\beta_4x_2(x_1x_3) \\ +\beta_5(x_3x_2)x_1+\beta_6(x_2x_3)x_1+\beta_7x_1(x_3x_2)+\beta_8x_1(x_2x_3),$$

where $\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8$ are scalars from Φ and independent of x_1, x_2, and x_3.

It follows from these relations that a 2-variety is a generalization of almost alternative algebras in the sense of Albert [1].

For representations we have the following

Lemma. *Let $\mathscr{V}(I)$ be a 2-variety and $A \in \mathscr{V}(I)$. Then for a representation $\rho=(S, T)$ of A and $a, b \in A$, the product Y_aY_b, where Y is ambiguously S or T, can be written as a linear combination of terms of the form Y_bY_a and Y_u, where $u=ab$ or ba and $Y=S$ or T.*

Proof. Suppose M is the Φ-space providing the representation ρ. Then $A \oplus M \in \mathscr{V}(I)$ and the identities above hold for $A \oplus M$. If we put $x_2=a$, $x_3=b$ and choose $x_1 \in M$ then $(x_1x_2)x_3=(x_1)S_aS_b$, and by (1), S_aS_b is a linear combination of the terms $Y_b Y_a$ and Y_u. Equally, by choosing $x_2 \in M$, $x_1=a$ and $x_3=b$ we find from (1) that $T_a S_b$ can be put in the form required. From similar substitutions in (2) we find in all four cases $Y_a Y_b$ to be a linear combination of $Y_b Y_a$'s and Y_u's.

An algebra A of a 2-variety is called prime if whenever J and K are ideals of A such that $JK=0$, either $J=0$ or $K=0$.

Now we may state the main result of this paper.

THEOREM. *Let $\mathscr{V}(I)$ be a 2-variety with the property that $\mathfrak{U}(A)$ is nilpotent for each* 1*-dimensional trivial algebra $A \in \mathscr{V}(I)$, that is, for $A=\Phi x$, $x^2=0$. Then the property of being locally nilpotent is a Kurosh-Amitsur radical property in $\mathscr{V}(I)$ and each semi-simple algebra in $\mathscr{V}(I)$ is a subdirect sum of prime semi-simple algebras.*

Proof. We shall show by induction on n that if $B \in V(I)$ is generated by $x_1, \ldots, x_n$ and $B^2=0$ then $\mathfrak{U}(B)$ is nilpotent; the theorem will then follow because of [2, Theorem 2.7]. For $n=1$ there is nothing to prove. Assuming $n>1$, let $J=\Phi x_1+\cdots+\Phi x_{n-1}$. Evidently J is an ideal of B and $\mathfrak{U}(B/J)$ is nilpotent because of our assumption. Moreover, if $\rho^*=(S^*, T^*)$ is the canonical representation of B in $\mathfrak{U}(B)$ and J^* is the ideal of $\mathfrak{U}(B)$ generated by the elements S_x^*, T_x^*, $x \in J$, then $\mathfrak{U}(B/J) \simeq \mathfrak{U}(B)/J^*$ because of [5, page 88]. Hence $\mathfrak{U}(B)^m \subseteq J^*$ for some integer m.

Let J^{**} be the subalgebra of $\mathfrak{U}(B)$ generated by the elements S_x^*, T_x^*, $x \in J$. Certainly the maps $x \to S_x^*$, $x \to T_x^*$ define a representation of J in J^{**}; hence if $\rho'=(S', T')$ is the canonical representation of J in $\mathfrak{U}(J)$ then $\rho^*=\rho'\psi$ for a homomorphism $\psi:\mathfrak{U}(J) \to J^{**}$. However, $\mathfrak{U}(J)$ is nilpotent by induction and as ψ is clearly onto, J^{**} is nilpotent.

We claim

$$J^{**}\mathfrak{U}(B) \subseteq \mathfrak{U}(B)J^{**}. \tag{3}$$

Indeed, if $a \in J$, $b \in B$ and $Y=S^*$ or T^* then $Y_a Y_b$ is a linear combination of $Y_b Y_a$'s because of the lemma and the fact that $B^2=0$. Since $\mathfrak{U}(B)$ is generated by the elements S_x^*, T_x^*, $x \in B$ (see [5, page 88]), we have now (3).

An immediate consequence of (3) is that $J^{**}+\mathfrak{U}(B)J^{**}$ is an ideal of $\mathfrak{U}(B)$. Clearly $J^*=J^{**}+\mathfrak{U}(B)J^{**}$, and $\mathfrak{U}(B)^m \subseteq J^{**}+\mathfrak{U}(B)J^{**}$. Therefore $\mathfrak{U}(B)^{m+1} \subseteq \mathfrak{U}(B)J^{**}$. From this relation and (3) it follows easily that $(\mathfrak{U}(B)^{m+1})^k \subseteq \mathfrak{U}(B)J^{**k}$. Since J^{**} is nilpotent, $\mathfrak{U}(B)$ is nilpotent.

With the above theorem we can determine explicitly, in terms of the general identities (1) and (2), which 2-varieties of power-associative algebras have a satisfactory theory of local nilpotence. Recall that in a power-associative algebra each element generates an associative subalgebra.

THEOREM. *Let $\mathscr{V}(I)$ be a variety of power-associative algebras over a field Φ satisfying the identities* (1) *and* (2). *Then local nilpotence is a Kurosh-Amitsur*

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radical property for $\mathscr{V}(I)$ *unless*

$$1+\alpha_1-\alpha_2-\alpha_3+\alpha_4 = -1+\alpha_5-\alpha_6-\alpha_7+\alpha_8 = -1+\beta_1-\beta_2-\beta_3+\beta_4$$
$$= 1+\beta_5-\beta_6-\beta_7+\beta_8 = 0,$$

Proof. We must show $\mathfrak{U}(A)$ nilpotent for $A=\Phi x$, $x^2=0$. Let $\rho=(S, T)$ be the canonical representation of A in $\mathfrak{U}(A)$ and M be the bimodule associated with the representation ρ. As in the proof of the Lemma, we apply the identities (1) and (2) to the algebra $A\oplus M$. For example, in (1) choose $x_1 \in M$ and $x_2=x_3=x$. Then

$$S_x^2 = \alpha_1 T_x S_x+\alpha_2 S_x^2+\alpha_3 T_x^2+\alpha_4 S_x T_x \tag{4}$$

Also in (1) choose $x_2 \in M$ and $x_1=x_3=x$, so that

$$T_x S_x = \alpha_5 T_x S_x+\alpha_6 S_x^2+\alpha_7 T_x^2+\alpha_8 S_x T_x. \tag{5}$$

In (2) choose $x_1 \in M$, and $x_2=x_3=x$. Then

$$S_x T_x = \beta_1 T_x S_x+\beta_2 S_x^2+\beta_3 T_x^2+\beta_4 S_x T_x. \tag{6}$$

Finally, letting $x_2 \in M$ and $x_1=x_3=x$ in (2), we have

$$T_x^2 = \beta_5 T_x S_x+\beta_6 S_x^2+\beta_7 T_x^2+\beta_8 S_x T_x. \tag{7}$$

The algebras in $\mathscr{V}(I)$ are power-associative, hence satisfy the identities $a^2a=aa^2$, $(a^2a)a=a^2a^2$, and $a(aa^2)=a^2a^2$. Linearizing the first of these identities yields $ya \cdot a+ay \cdot a+a^2y=a \cdot ay+a \cdot ya+ya^2$. If we choose $y \in M$ and $a=x$, then we obtain from this relation

$$S_x^2+T_x S_x = T_x^2+S_x T_x. \tag{8}$$

Similarly, from a linearization of $(a^2a)a=a^2a^2$,

$$S_x^3+T_x S_x^2 = 0, \tag{9}$$

while from $a(aa^2)=a^2a^2$ we get

$$T_x^3+S_x T_x^2 = 0. \tag{10}$$

Since $\mathfrak{U}(A)$ is generated by S_x and T_x, to show it is nilpotent it is sufficient to show $\mathfrak{U}(A)=N$, where N is the Levitzki radical of $\mathfrak{U}(A)$. Now, from (9), $(S_x^2+T_xS_x)S_x=0$ while $(S_x^2+T_xS_x)T_x=(T_x^2+S_xT_x)T_x=T_x^3+S_xT_x^2=0$ because of (8) and (10). Therefore $\Phi(S_x^2+T_xS_x)$ is a right ideal of $\mathfrak{U}(A)$ whose square is zero. However, N contains all the nilpotent right ideals of $\mathfrak{U}(A)$, hence

$$S_x^2+T_x S_x \in N. \tag{11}$$

Now it follows that $N+\Phi(S_x+T_x)/N$ is a trivial right ideal of $\mathfrak{U}(A)/N$. Indeed, the product of the cosets $N+\Phi(S_x+T_x)$ and $N+S_x$ is $N+\Phi(S_x^2+T_xS_x)$, which$=0$

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because of (11), while

$$[N+\Phi(S_x+T_x)][N+T_x]=N+\Phi(S_xT_x+T_x^2)=N+\Phi(S_x^2+T_xS_x)=0$$

because of (8) and (11). Since $\mathfrak{U}(A)/N$ is Levitzki semi-simple and contains no non-zero nilpotent right ideals it follows that $S_x+T_x \in N$, or

$$T_x \equiv -S_x(\text{mod } N). \tag{12}$$

Then from (4)–(7) we have

$$\begin{aligned}(1+\alpha_1-\alpha_2-\alpha_3+\alpha_4)S_x^2 &\equiv (-1+\alpha_5-\alpha_6-\alpha_7+\alpha_8)S_x^2\\ &\equiv (-1+\beta_1-\beta_2-\beta_3+\beta_4)S_x^2\\ &\equiv (1+\beta_5-\beta_6-\beta_7+\beta_8)S_x^2 \equiv 0 \text{ (mod } N).\end{aligned}$$

If one of the above coefficients is non-zero, we have $S_x^2\equiv 0$ (mod N) and $S_xT_x\equiv T_x^2\equiv 0$ (mod N) as well because of (12). However, then $[\mathfrak{U}(A)/N]^2=0$. whence $N=\mathfrak{U}(A)$, as claimed. Q.E.D.

The result above is the best possible, as the following will show.

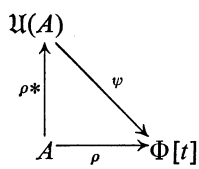
SCHOLIUM. *Let $\mathscr{V}(I)$ be the* 2-*variety defined by $a^2a=aa^2$, $(a^2a)a=a^2a^2$, $a(aa^2)=a^2a^2$, and the relations* (1) *and* (2). *Suppose further that*

$$\begin{aligned}1+\alpha_1-\alpha_2-\alpha_3+\alpha_4 &= -1+\alpha_5-\alpha_6-\alpha_7+\alpha_8 = -1+\beta_1-\beta_2-\beta_3+\beta_4\\ &= 1+\beta_5-\beta_6-\beta_7+\beta_8 = 0.\end{aligned}$$

Then local nilpotence is not a Kurosh-Amitsur radical property for $\mathscr{V}(I)$.

Proof. To prove the non-existence of the Levitzki radical in $\mathscr{V}(I)$, it is sufficient to show $\mathfrak{U}(A)$ is not nilpotent for $A=\Phi x$, $x^2=0$ because of [2, Theorem 2.7].

Let $\Phi[t]$ be the ring of polynomials in t with constant term zero. Define linear maps $S, T: A\to\Phi[t]$ by $S_{\alpha x}=\alpha t$ and $T_{\alpha x}=-\alpha t$ for $\alpha\in\Phi$. Then (8), (9), and (10) hold by inspection, while (4)–(7) are valid because of the assumed conditions on the α's and β's. Therefore $\rho=(S, T)$ is a representation of A in $\Phi[t]$ and there is a homomorphism $\psi:\mathfrak{U}(A)\to\Phi[t]$ for which the diagram



commutes, where ρ^* is the canonical representation of A in $\mathfrak{U}(A)$. Obviously ψ maps $\mathfrak{U}(A)$ onto $\Phi[t]$. Since, then, $\Phi[t]$ is not nilpotent, neither is $\mathfrak{U}(A)$ nilpotent.

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