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DIVERGENCE OF FOURIER SERIES

BY
DANIEL M. OBERLIN

ABSTRACT. This note contains a strengthened version of the following well-known theorem: there exists a continuous function whose Fourier series diverges at a point.

The following theorem was first proved by P. du Bois Reymond.

THEOREM 1. *There exists a continuous function whose Fourier series diverges at a point.*

Since then several other proofs of Theorem 1 have been given. A particularly simple one, which has been reproduced in several popular texts, is due to Lebesgue. The ingredients are the unboundedness of the Lebesgue constants

$$\int_0^{2\pi} \left| \sum_{j=-n}^{n} e^{ijt} \right| dt$$

and the Baire category theorem (in the guise of the uniform boundedness principle). The purpose of this paper is to use these same two ingredients to prove the following strengthened version of Theorem 1.

THEOREM 2. *Suppose $K \subseteq [0, 2\pi)$ is a closed set having positive Lebesgue measure. Almost every point x of K has the following property: there exists a continuous function f on K such that if $F \in L^\infty([0, 2\pi))$ coincides with f on K, then the partial sums of the Fourier series of F are unbounded at x.*

Theorem 1 is recovered by taking F continuous on $[0, 2\pi]$ (and such that $F(0) = F(2\pi)$). Theorem 2 is interesting also because of the contrast it presents with the following result.

THEOREM 3. (See [2].) *Suppose $K \subseteq [0, 2\pi)$ is a closed set having Lebesgue measure zero. If f is a continuous function on K, then there exists a continuous function F on $[0, 2\pi)$ which coincides with f on K and has a uniformly convergent (on $[0, 2\pi)$) Fourier series.*

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Theorem 2 is an immediate consequence of Lemmas 1 and 2 below. In their statements and proofs we adopt the notations and terminologies of [1]. In particular

$$T=[0,2\pi),\ \|f\|_1=\frac{1}{2\pi}\int_T |f|,\quad \text{and}\quad D_n(t)=\sum_{j=-n}^{n} e^{ijt}.$$

LEMMA 1. *Suppose that $K\subseteq T$ is a closed set of positive Lebesgue measure. Almost every $x\in K$ is such that the following holds: Fix a positive number M. There exists $f\in C(K)$ with $\|f\|_\infty\le 1$ such that if $F\in L^\infty(T)$ satisfies $\|F\|_\infty\le M$ and $F|_K=f$, then $\sup_n |S_n(F,x)|>M$.*

LEMMA 2. *Suppose $K\subseteq T$ is closed and fix $x\in T$. If every $f\in C(K)$ has an extension $F\in L^\infty(T)$ with $\sup_n|S_n(F,x)|<\infty$, then there is $M<\infty$ such that the following holds: if $f\in C(K)$ has $\|f\|_\infty\le 1$, there is an extension $F\in L^\infty(T)$ of f with $\|F\|_\infty\le M$ and $\sup_n |S_n(F,x)|\le M$.*

Proof of Lemma 1. Put $k_n(t)=|D_n(t)|/\|D_n\|_1$. Then

$$k_n\ge 0,\ \frac{1}{2\pi}\int_T k_n=1,\quad \text{and}\quad \lim_{n\to\infty}\int_\delta^{2\pi-\delta} k_n=0\quad \text{if}\quad 0<\delta<\pi$$

since

$$k_n(t)\le\frac{1}{\|D_n\|_1\sin(\frac{t}{2})}.$$

Thus the sequence $\{k_n\}$ forms a summability kernel. It follows that if χ_K denotes the characteristic function of K, then $\|k_n*\chi_K-\chi_K\|_1\to 0$. Since a sequence which converges in L^1 has a subsequence which converges almost everywhere, there is some subsequence $\{k_{n_j}\}$ such that $k_{n_j}*\chi_K(x)\to 1$ for almost all $x\in K$. That is, for almost all $x\in K$,

$$\frac{1}{2\pi}\int_K k_{n_j}(x-t)\,dt\to 1$$

and so

$$\frac{1}{2\pi}\int_{T\sim K} k_{n_j}(x-t)\,dt\to 0.$$

For such an x one can evidently pick $n(=n_j$ for some sufficiently large $j)$ such that

$$\|D_n\|_1\ge 3M,\ \frac{1}{2\pi}\int_{T\sim K} k_n(x-t)\,dt\le\frac{1}{3M},\quad \text{and}\quad \frac{1}{2\pi}\int_K k_n(x-t)\,dt>\tfrac{2}{3}.$$

Let $f\in C(K)$ be such that $\|f\|_\infty\le 1$ and

$$\frac{1}{2\pi\|D_n\|_1}\int_K f(t)D_n(x-t)\,dt>\tfrac{2}{3}.$$

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Suppose $F \in L^\infty(T)$ is such that $\|F\|_\infty \le M$ and $F|_K = f$. Then

$$\frac{S_n(F, x)}{\|D_n\|_1} = \frac{1}{2\pi \|D_n\|_1} \int_K f(t) D_n(x-t)\, dt + \frac{1}{2\pi \|D_n\|_1} \int_{T\sim K} F(t) D_n(x-t)\, dt.$$

The first term in the sum exceeds $\frac{2}{3}$ by the choice of f, while the second term has absolute value dominated by

$$\frac{\|F\|}{2\pi} \int_{T\sim K} k_n(x-t)\, dt.$$

By the choice of n and the fact that $\|F\|_\infty \le M$, this does not exceed $\frac{1}{3}$. Since $\|D_n\|_1 \ge 3M$, the inequality

$$S_n(F, x) > M$$

follows.

Proof of Lemma 2. For $N = 1, 2, \ldots$ let C_N be the set

$$\{f \in C(K)\text{: there exists } F \in L^\infty(T) \text{ with } F|_K = f; \|f\|_\infty, \sup_n |S_n(F, x)| \le N\}.$$

We begin by noting that each C_N is closed in $C(K)$: suppose $\{f_j\}$ is a sequence in C_N with $f_j \to f$ uniformly on K. For each j, let $F_j \in L^\infty(T)$ be such that

$$F_j|_K = f_j, \qquad \|F_j\|_\infty \le N, \quad \text{and} \quad \sup_n |S_n(F_j, x)| \le N.$$

Since the unit ball in $L^\infty(T)$ is a weak $-^*$ compact metrizable space, some subsequence $\{F_{j_l}\}$ of $\{F_j\}$ converges weak $-^*$ to some $F \in L^\infty(T)$ with $\|F\|_\infty \le N$ and

$$|S_n(F, x)| = \left| \frac{1}{2\pi} \int_T F(t) D_n(x-t)\, dt \right| \le N \quad \text{for each } n.$$

Also, if E is any measurable subset of K,

$$\int_E f = \lim_l \int_E f_{j_l} = \lim_l \int_E F_{j_l} = \int_E F.$$

This shows that $f = F$ a.e. on K.

Now the hypothesis of Lemma 2 is that $C(K)$ is the union of the closed sets C_N. Thus the Baire category theorem implies that there are $N, \delta > 0$, and $g \in C_N$ such that if $f \in C(K)$ has $\|f\|_\infty \le \delta$, then $f + g \in C_N$. It follows that to any such f there corresponds $F \in L^\infty(T)$ with

$$F|_K = f, \|F\|_\infty \le 2N, \quad \text{and} \quad \sup_n |S_n(F, x)| \le 2N.$$

This completes the proof of Lemma 2.

References

1. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley and Sons, New York, 1968.

2. D. Oberlin, *A Rudin-Carleson theorem for uniformly convergent Taylor series*, Michigan Math. J. **27** (1980), 309–313.

Florida State University
Tallahassee, Florida, 32306

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