ON THE EQUATION $ax - xb = c$

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Beniamino Segre, in his memorial lecture of 1958 [5], [6], inaugurated the study of non-linear geometry in three dimensions over a division ring. In his treatment of sections of quadrics by planes, he is naturally led to consider conics and the problem of tangency. Now in the commutative case the locus of intersection of a quadric and a plane containing a generator is the line-pair consisting of this generator and one from the other family. Such a plane is then the tangent plane of the point of intersection of the two generators. Segre extends this notion to the non-commutative case, where the locus of intersection is not always a line-pair. He joins up the remaining points of intersection in pairs, and calls the points where the lines so formed cut the base generator, the 'points of contact' of the plane (π) and the quadric (Q). A line in π is called a 'tangent' if it passes through a point of contact, but does not contain any of the points of intersection of Q and π.

The question now arises whether there are lines in π which do *not* pass through a point of contact, and yet meet Q but once, in the base generator. Such lines, as Segre shows, indeed exist provided that in the division ring under consideration we can find insoluble equations of the form

(1) $$ax - xb = c, \text{ with } a \text{ not conjugate to } b.$$

This note gives an example of such an equation.

The equation, with its non-conjugacy condition, has been mentioned independently by D. W. Barnes in the consideration of double-six configurations over a division ring, which is clearly a related problem. It might be noted here that equation (1) is soluble if either a or b is algebraic over the centre of the division ring [4], [1].

Example of an insoluble equation of the form $ax - xb = c$. Let F be the field of rational functions in one indeterminate, λ, over say, the complex numbers. We then consider the set of extended formal power series $\{\sum_{i \geqq n} a_i t^i : a_i \in F, \text{ and } t \text{ an indeterminate}\}$. It can be shown [2], [3] that this set forms a division ring under the usual definitions of equality and addition, and with multiplication defined by: $ta = a^\sigma t$, where σ is any automorphism of F. We will take σ to be the non-trivial automorphism defined by: $(\varphi(\lambda))^\sigma = \varphi(\lambda+1)$ for $\varphi(\lambda) \in F$. We will denote the division ring so formed by $F(t, \sigma)$.

Let $a = \lambda t$,
$b = 1t$, *and*
$c = 1t$, *be elements of* $F(t, \sigma)$. *Then a is not conjugate to b, and there does not exist an element x in* $F(t, \sigma)$ *such that* $ax - xb = c$.

PROOF. If a is conjugate to b, $ay - yb = 0$ for some non-zero $y \in F(t, \sigma)$. Let $y = \sum_{i \geq k} \theta_i(\lambda) t^i$. Then $\sum_{i \geqq k} (\lambda\theta_i(\lambda+1) - \theta_i(\lambda)) t^{i+1} = 0$, so that $\lambda\theta_i(\lambda+1) - \theta_i(\lambda) = 0$. Inspection of the degrees of the polynomials constituting the numerators and denominators of this equation shows that the only solution is $\theta_i(\lambda) = 0$ for all i. Hence $y = 0$, and a and b are not conjugate.

Now suppose that $\sum_{i \geqq k} \varphi_i(\lambda) t^i$ satisfies $ax - xb = c$. Then, as before, we obtain $\varphi_i(\lambda) = 0$ for $i \neq 0$, but $\lambda\varphi_0(\lambda+1) - \varphi_0(\lambda) = 1$. This equation has no solution in F. For suppose $\varphi_0(\lambda) = p(\lambda)/q(\lambda)$, where p and q are polynomials whose highest common factor has degree zero, is a solution. Then $\lambda p(\lambda+1) q(\lambda) - p(\lambda) q(\lambda+1) = q(\lambda) q(\lambda+1)$. If now λ_0 is a root of $q(\lambda)$ it must also be a root of $q(\lambda+1)$, as p and q are coprime. So $\lambda_0 + 1$ is a root of $q(\lambda)$. It follows by induction that if $q(\lambda)$ has a zero at λ_0 it will have zeros at $\lambda_0 + n$ for all positive integers n. This is impossible, so $q(\lambda)$ is a constant, say 1. We now have $\lambda p(\lambda+1) = p(\lambda) + 1$, which an inspection of degrees shows to be impossible.

References

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