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ON WEIGHTED POLYNOMIAL APPROXIMATION WITH GAPS

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Abstract. Let α be a nonnegative continuous function on $\mathbb{R}$. In this paper, the author obtains a necessary and sufficient condition for polynomials with gaps to be dense in C_α, where C_α is the weighted Banach space of complex continuous functions f on $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity.

§1. Introduction

Let $\alpha(t)$ be a nonnegative continuous function on $\mathbb{R}$, which is, henceforth, called a weight, defined on $\mathbb{R}$. We usually suppose that

$$\lim_{|t|\to\infty}\frac{\alpha(t)}{\log|t|}=\infty. \tag{1}$$

Given a weight $\alpha(t)$, we consider the weighted Banach space C_α consisting of complex continuous functions $f(t)$ on $\mathbb{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity, and define

$$\|f\|_\alpha=\sup\{|f(t)e^{-\alpha(t)}| : t\in\mathbb{R}\}$$

for $f\in C_\alpha$. The classical Bernstein problem on weighted polynomial approximation is as follows: determine whether or not the polynomials are dense in the space C_α in the norm $\|\ \ \|_\alpha$; see [2], [3]. In this direction we mention one result, for example, the sufficiency of the above problem was obtained by S. Izumi and T. Kawata in 1937 in [7]. Later on, several authors obtained this result in different forms (for example, T. Hall ([6]), de Branges ([5]) and A. Borichev ([2], [3])).

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THEOREM A. ([2], [3], [5], [6], [7]) *Suppose that* $\alpha(t)$ *is an even function satisfying* (1) *and* $\alpha(e^t)$ *is a convex function on* $\mathbb{R}$. *Then a necessary and sufficient condition for polynomials to be dense in the space* C_α *is*

$$\int_{-\infty}^{+\infty} \frac{\alpha(t)}{1+t^2}\, dt = \infty. \tag{2}$$

The Müntz Theorem ([4]) naturally leads us to consider the density of polynomials with gaps in the space C_α. Denote by $M(\Lambda)$ the space of polynomials with gaps which are finite linear combinations of the system $\{t^\lambda : \lambda \in \Lambda\}$, where $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ is a sequence of strictly increasing positive integers. The condition (1) guarantees that $M(\Lambda)$ is a subspace of C_α; we then ask whether $M(\Lambda)$ is dense in C_α in the norm $\| \ \|_\alpha$ - this is the so-called weighted polynomial approximation with gaps, which is similar to the classical Bernstein problem on weighted polynomial approximation. Motivated by Bernstein's problem and Malliavin's Method ([8]), we find a necessary and sufficient condition for $M(\Lambda)$ to be dense in C_α. The main result is as follows.

THEOREM. *Suppose that* $\alpha(t)$ *is an even function satisfying* (1) *and* $\alpha(e^t)$ *is convex function on* $\mathbb{R}$. *Let* $\Lambda = \{\lambda_n : n \in \mathbb{R}\}$ *is a sequence of strictly increasing positive integers and let*

$$\Lambda(r) = \begin{cases} 2\sum_{\lambda_n \le r} \frac{1}{\lambda_n}\,, & \text{if } r \ge \lambda_1 \\ 0\,, & \text{otherwise}, \end{cases} \tag{3}$$

$k(r) = \Lambda(r) - \log^+ r$, $\log^+ r = \max\{\log r, 0\}$, $\widetilde{k}(r) = \inf\{k(r') : r' \ge r\}$.
If

$$\int_{0}^{+\infty} \frac{\alpha(\exp\{\widetilde{k}(t) - a\})}{1+t^2}\, dt = \infty \tag{4}$$

for each $a \in \mathbb{R}$, *then* $M(\Lambda)$ *is dense in* C_α.

Conversely, if the sequence Λ *contains all of the positive odd integers, then for* $M(\Lambda)$ *to be dense in* C_α, *it is necessary that* (4) *holds for each* $a \in \mathbb{R}$.

Remark. Since $\sum_{n \le r} \frac{1}{n} - \log r$ converges to Euler's constant γ, as $r \to \infty$, the condition (4) is equivalent to the condition (2) in the case that $\Lambda = \mathbb{N} = \{1, 2, \dots\}$. Therefore our theorem is a generalization of Theorem A. If

Λ contains all of the positive odd integers $2\mathbb{N}-1=\{2k-1 : k=1,2,\dots\}$, then $\widetilde{k}(r)=\widetilde{\Lambda}(r)+O(1)$ $(r\to\infty)$, where $\widetilde{\Lambda}(r)$ is defined by (3) with Λ replaced by $\widetilde{\Lambda}=\{\lambda+1 : \lambda\in\Lambda,\ \lambda \text{ even}\}$. In this case, $\widetilde{k}(r)$ in the integral of (4) can be replaced by $\widetilde{\Lambda}(r)$. Moreover, we conjecture that the condition (4) is also necessary for polynomials with gaps to be dense in the space C_α, if we remove the the restriction that Λ contains all of the positive odd integers.

§2. Proof of Theorem

In order to prove our theorem, we need some technical lemmas (Hereafter we denote a positive constant by A, not necessarily the same at each occurrence).

Lemma 1. ([8]) *Let $\beta(t)$ be a nonnegative convex function on $\mathbb{R}$ such that $\beta(\log|t|)$ satisfies* (1)*, and assume that*

$$\beta^*(t)=\sup\{xt-\beta(x) : x\in\mathbb{R}\},\quad t\in\mathbb{R} \tag{5}$$

is the Young transform of the function $\beta(x)$; see [9]*. Let $\widetilde{k}(r)$ be a increasing function on $[0,\infty)$ and there exist a positive constant A such that*

$$\widetilde{k}(R)-\widetilde{k}(r)\le A(\log R-\log r+1) \tag{6}$$

for $R>r>1$. Let $f(z)$ be an analytic function in $\mathbb{C}_+$ and there exist a positive constant A such that

$$|f(z)|\le A\exp\{Ax+\beta(x)-x\widetilde{k}(|z|)\},\quad z=x+iy\in\mathbb{C}_+. \tag{7}$$

If

$$\int_1^{+\infty}\frac{\beta^*(\widetilde{k}(t)-a)}{1+t^2}\,dt=\infty \tag{8}$$

for each real number a, then $f(z)\equiv 0$.

Lemma 2. ([1]) *If Λ is a sequence of increasing positive integrals, then the function*

$$G_\Lambda(z)=\prod_{n=1}^{\infty}\left(\frac{\lambda_n-z}{\lambda_n+z}\right)\exp\left(\frac{2z}{\lambda_n}\right) \tag{9}$$

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is analytic in the closed right half plane $\overline{\mathbb{C}}_+ = \{z = x + iy : x \geq 0\}$, *and there exists a positive constant* A *such that*

$$\big|\log|G_\Lambda(z)| - x\Lambda(|z|)\big| \leq Ax, \quad z = x + iy \in D, \tag{10}$$

where $\Lambda(r)$ *is defined by* (3) *and* $D = \{z \in \overline{\mathbb{C}}_+ : |z - \lambda_n| \geq \frac{1}{8},\, n \in \mathbb{R}\}$.

Proof of Theorem. By the Hahn-Banach theorem, we need to prove that if T is a bounded linear functional on C_α and $T(t^\lambda) = 0$ for $\lambda \in \Lambda$, then $T = 0$. So let T be a bounded linear functional on C_α and $T(t^\lambda) = 0$ for $\lambda \in \Lambda$; then by the Riesz representation theorem, there exists a complex measure μ such that

$$\int_{-\infty}^{+\infty} e^{\alpha(t)}\, d|\mu|(t) = \|T\|,$$

and

$$T(h) = \int_{-\infty}^{+\infty} h(t)\, d\mu(t)$$

for $h \in C_\alpha$. Therefore the function

$$f_0(z) = e^{\frac{\pi}{2}iz} \int_0^{+\infty} t^z\, d\mu(t) + e^{-\frac{\pi}{2}iz} \int_{-\infty}^0 |t|^z\, d\mu(t)$$

is analytic in the open right half-plane $\mathbb{C}_+$, continuous in the closed right half-plane $\overline{\mathbb{C}}_+ = \{z = x + iy : x \geq 0\}$, $f_0(\lambda) = 0$, $\lambda \in \Lambda$ and

$$|f_0(z)| \leq \|T\| \exp\big\{\beta(x) + \tfrac{\pi}{2}|y|\big\}$$

for $z = x + iy \in \mathbb{C}_+$, where

$$\beta(x) = \sup\{x \log t - \alpha(t) : t > 0\}$$

is the Young transform of the convex function $\alpha(e^s)$. Let $G_\Lambda(z)$ be defined by (9) and $\Gamma(z)$ be The Gamma function. By (10) and the Stirling asymptotic formula, we see that there exists a positive constant A such that the function

$$f(z) = \frac{f_0(z)}{G(z)\Gamma(1+z)}$$

satisfies

$$|f(z)| \leq A \exp\{\beta(x) - x\widetilde{k}(|z|) + Ax\},$$

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where $\widetilde{k}(r) = \inf\{\Lambda(r') - \log^+ r' : r' \geq r\}$ satisfies (6) with $A = 1$. We may assume, without loss of generality, that $\alpha(1) = 0$. As is known, $\beta(x)$ is a convex nonnegative function which also satisfies $\beta(0) = 0$ and

$$\sup\{xs - \beta(x) : x \geq 0\} = \alpha(e^s). \tag{11}$$

We see from Lemma 1 and (4) that $f(z) \equiv 0$, so $f_0(z) \equiv 0$. In particular $f_0(n) = 0$, $n = 0, 1, 2, \ldots$. Therefore $T(t^n) = 0$, $n = 0, 1, 2, \ldots$. Since the condition (3) implies the condition (2), $T = 0$ by Theorem A. This completes the proof of the necessity of the theorem.

Conversely, assume that the sequence Λ contains all of the odd positive integers $2\mathbb{N} - 1$, then $\widetilde{k}(r) = \widetilde{\Lambda}(r) + O(1)$ $(r \to \infty)$, where $\widetilde{\Lambda}(r)$ is defined by (3) with Λ replaced by $\widetilde{\Lambda} = \{\lambda + 1 : \lambda \in \Lambda,\ \lambda \text{ even}\}$. In this case, $\widetilde{k}(r)$ in the integral of (4) can be replaced by $\widetilde{\Lambda}(r)$. $\widetilde{k}(r) = k(r) + O(1)$ $(r \to \infty)$. Assume that there exists a real number a such that the integral

$$\int_0^\infty \frac{\alpha(\exp\{\widetilde{\Lambda}(t) - a\})}{1 + t^2}\, dt < \infty.$$

Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\exp\{\widetilde{\Lambda}(t) - a\})$ for $t \geq 0$ and let $u(z)$ be the the Poisson integral of $\varphi(t)$, i.e.,

$$u(x + iy) = \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t)}{x^2 + (y - t)^2}\, dt.$$

Then $u(x + iy)$ is harmonic in the half-plane $\mathbb{C}_+$ and there exists an analytic function $g_1(z)$ on $\mathbb{C}_+$ satisfying

$$\begin{aligned} \operatorname{Re} g_1(z) = u(z) &\geq \frac{4x}{\pi} \int_{|t| \geq |z|} \frac{\varphi(t)}{x^2 + (y - t)^2}\, dt \\ &\geq \varphi(|z|) = \alpha(\exp\{\widetilde{\Lambda}(|z|) - a\}) \\ &\geq (x - 1)(\widetilde{\Lambda}(|z|) - a) - \beta(x - 1), \end{aligned}$$

where $z = x + iy$, $r = |z|$, $x > 1$. Let

$$g_0(z) = \frac{G_{\widetilde{\Lambda}}(z)}{(1 + z)^N} \exp\{-g_1(z) - Nz - N\},$$

where N is a large positive integer and $G_{\widetilde{\Lambda}}(z)$ is defined by (9). By (9) and (10), we have $g_0(\lambda + 1) = 0$ for $\lambda \in \Lambda$, λ even and

$$|g_0(z)| \leq \frac{1}{1 + |z|^2} \exp\{\beta(x - 1) - x\}, \quad z \in \mathbb{C}_+. \tag{12}$$

Let

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0\Big(\frac{1}{2} + iy\Big) t^{-(\frac{1}{2}+iy)}\, dy.$$

Then $h_0(t)$ is continuous on $[0, +\infty)$. By the Cauchy formula,

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(x + iy) t^{-(x+iy)}\, dy \tag{13}$$

for $x > 0$. We obtain from (11), (12) and (13) that

$$|h_0(t)| \le \exp(-\alpha(t) - |\log t|)$$

and

$$g_0(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} h_0(t) t^{z-1}\, dt$$

for $x > 0$. We extend the function $h_0(t)$ to an even function by letting $h_0(t) = h_0(-t)$ for $t < 0$. Therefore the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)h(t)\, dt \quad (h \in C_\alpha)$$

satisfies $T(t^\lambda) = 0$ for $\lambda \in \Lambda$, and

$$\|T\| = \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} |h_0(t)| e^{\alpha(t)}\, dt > 0.$$

By the Riesz representation theorem, the space $M(\Lambda)$ is not dense in C_α. This completes the proof of the theorem. $\square$

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