RESULTS ON COMMON FIXED POINTS ON COMPLETE METRIC SPACES

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The following theorem was proved in [**1**].

THEOREM 1. *Let S and T be continuous, commuting mappings of a complete, bounded metric space (X, d) into itself satisfying the inequality*

$$d(S^pT^{p'}x, S^qT^{q'}y) \leq c\,.\max\{d(S^rT^{r'}x, S^sT^{s'}y), d(S^rT^{r'}x, S^\rho T^{\rho'}x), d(S^sT^{s'}y, S^\sigma T^{\sigma'}y):$$
$$0 \leq r, \rho \leq p; 0 \leq r', \rho' \leq p'; 0 \leq s, \sigma \leq q; 0 \leq s', \sigma' \leq q'\}$$

for all x, y in X, where $0 \leq c < 1$ *and* $p, p', q, q' \geq 0$ *are fixed integers with* $p+p', q+q' \geq 1$. *Then S and T have a unique common fixed point z. Further, if p' or* $q'=0$, *then z is the unique fixed point of S and if p or* $q=0$, *then z is the unique fixed point of T.*

It was shown that the condition that S and T commute was necessary in this theorem. It is possible however that the condition that X be bounded is not necessary in this theorem. We now prove the following theorem which does not require S and T to commute or X to be bounded.

THEOREM 2. *Let S and T be continuous mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(S^px, T^qy) \leq c\,.\max\{d(S^rx, T^sy): 0 \leq r \leq p; 0 \leq s \leq q\} \tag{1}$$

for all x, y in X, where $0 \leq c < 1$ *and p, q are fixed positive integers. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and T.*

Proof. Let x be an arbitrary point in X and put

$$A = \max\{d(T^sx, T^qx): 0 \leq s \leq q\}.$$

Suppose that the sequence $\{S^nx : n = 1, 2, \ldots\}$ is unbounded. Then there exists an integer $n \geq p$ such that

$$d = d(S^nx, T^qx) \geq \max\{d(S^rx, T^qx): 0 \leq r \leq n\}$$

with

$$d > cA/(1-c).$$

Thus

$$d(S^rx, T^sx) \leq d(S^rx, T^qx) + d(T^qx, T^sx) \leq d + A$$

for $0 \leq r \leq n$ and $0 \leq s \leq q$. On using inequality (1), it now follows that

$$d = d(S^nx, T^qx) \leq c\,.\max\{d(S^rx, T^sx): n-p \leq r \leq n; 0 \leq s \leq q\} \leq c(d+A)$$

and so $d \leqslant cA/(1-c)$ giving a contradiction. This contradiction implies that the sequence $\{S^n x : n = 1, 2, \ldots\}$ must be bounded.

Similarly, we can prove that the sequence $\{T^n x : n = 1, 2, \ldots\}$ is bounded and so

$$M = \sup\{d(S^r x, T^s x) : r, s = 0, 1, 2, \ldots\}$$

is finite. Now for arbitrary $\varepsilon > 0$, choose a positive integer N such that $c^N M < \varepsilon$. It follows that for $m, n \geqslant N . \max\{p, q\}$

$$\begin{aligned} d(S^m x, T^n x) &\leqslant c . \max\{d(S^r x, T^s x) : m-p \leqslant r \leqslant m; n-q \leqslant s \leqslant n\} \\ &\leqslant c^2 . \max\{d(S^r x, T^s x) : m-2p \leqslant r \leqslant m; n-2q \leqslant s \leqslant n\} \\ &\leqslant c^N . \max\{d(S^r x, T^s x) : m-Np \leqslant r \leqslant m; n-Nq \leqslant s \leqslant n\} \\ &\leqslant c^N M < \varepsilon \end{aligned}$$

and so

$$d(S^m x, S^r x) \leqslant d(S^m x, T^n x) + d(T^n x, S^r x) < 2\varepsilon$$

for $m, n, r \geqslant N . \max\{p, q\}$. Thus $\{S^n x : n = 1, 2, \ldots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Further, since

$$d(S^n x, T^n x) < \varepsilon$$

for $n \geqslant N . \max\{p, q\}$. the sequence $\{T^n x : n = 1, 2, \ldots\}$ also converges to z. From the continuity of S and T it now follows immediately that z is a common fixed point of S and T.

Now suppose that w is a second fixed point of T. Then

$$\begin{aligned} d(z, w) &= d(S^p z, T^q w) \\ &\leqslant c . \max\{d(S^r z, T^s w) : 0 \leqslant r \leqslant p; 0 \leqslant s \leqslant q\} \\ &= cd(z, w) \end{aligned}$$

proving that $z = w$, since $c < 1$. Similarly we can prove that z is the unique fixed point of S. This completes the proof of the theorem.

Corollary 1. *Let S be a mapping and let T be a continuous mapping of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, T^q y) \leqslant c . \max\{d(S^r x, T^s y) : 0 \leqslant r \leqslant 1; 0 \leqslant s \leqslant q\}$$

for all, x, y in X, where $0 \leqslant c < 1$ and q is a fixed positive integer. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and T.

Proof. Let x be an arbitrary point in X. Then as in the proof of Theorem 2, the sequences $\{S^n x : n = 1, 2, \ldots\}$ and $\{T^n x : n = 1, 2, \ldots\}$ converge to a point z in X. Since T

is continuous, z is a fixed point of T. Further

$$\begin{aligned} d(Sz, z) &= d(Sz, T^q z) \\ &\leq c \,.\, \max\{d(S^r z, T^s z) : 0 \leq r \leq 1; 0 \leq s \leq q\} \\ &= cd(Sz, z) \end{aligned}$$

proving that $Sz = z$, since $c < 1$. Thus z is a common fixed point of S and T. The uniqueness of z follows from the proof of the theorem, since the continuity of S was not used to prove the uniqueness.

COROLLARY 2. *Let S and T be mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) \leq c \,.\, \max\{d(x, y), d(x, Ty), d(y, Sx)\}$$

for all x, y in X, where $0 \leq c < 1$. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and T.

Proof. Let x be arbitrary point in X. Then again the sequences $\{S^n x : n = 1, 2, \ldots\}$ and $\{T^n x : n = 1, 2, \ldots\}$ converge to a point z in X. Further

$$\begin{aligned} d(Sz, z) &\leq d(Sz, T^n x) + d(T^n x, z) \\ &\leq c \,.\, \max\{d(z, T^{n-1} x), d(z, T^n x), (T^{n-1} x, Sz)\} + d(T^n x, z). \end{aligned}$$

Letting n tend to infinity it follows that

$$d(Sz, z) \leq cd(Sz, z)$$

proving that $Sz = z$, since $c < 1$. Similarly, we can prove that z is also a fixed point of T. The uniqueness of z again follows from the proof of the theorem.

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REFERENCE

1. B. Fisher, Results on common fixed points on bounded metric spaces, *Math. Sem. Notes Kobe Univ.*, **7** (1979), 73–80.

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