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Homotopical stability of pseudo-Anosov diffeomorphisms

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Abstract. We show that the pseudo-Anosov diffeomorphisms have a kind of stability even outside their own homotopy class, this generalizes some results of Lewowicz and Handel. As a corollary, we show that two pseudo-Anosov maps, with the same dilatation coefficient, which are semi-conjugate on the π_1 level are also semi-conjugate as dynamical systems by a map which is a ramified cover.

Our main interest in this work is to find when a dynamical system $g: N \to N$ on a compact connected space can be semi-conjugated onto a pseudo-Anosov map. This is reminiscent of J. Franks work [**F**], it is also related to work of J. Lewowicz [**L**] and M. Handel [**H1, H2**].

A pseudo-Anosov diffeomorphism f of a surface M is a homeomorphism, for which there exists a pair of transverse measured foliations $(\mathscr{F}^s, \mu^s)$, $(\mathscr{F}^u, \mu^u)$ and $\lambda > 1$ such that $f_*(\mathscr{F}^s, \mu^s) = (\mathscr{F}^s, \lambda^{-1}\mu^s)$ and $f_*(\mathscr{F}^u, \mu^u) = (\mathscr{F}^u, \lambda\mu^u)$ – see [**FLP**] for Thurston's theory of measured foliations and pseudo-Anosov diffeomorphisms. In the beginning, our work was motivated by an attempt to classify pseudo-Anosov maps up to ramified covers, because we wanted to find out the smallest surface on which a given pseudo-Anosov diffeomorphism was 'living' naturally. Our Theorem 2 below shows that this can be reduced to an algebraic semi-conjugacy problem on the fundamental group level. When M. Handel's paper [**H1**] appeared, it was clear that this problem was related to a form of dynamical 'stability' for pseudo-Anosov diffeomorphisms which is the content of our Theorem 1 below. In fact, the proof of Theorem 1 produces more than its statement, we find a natural hyperbolic extension of the pseudo-Anosov map in which the surface sits naturally as the smallest non-trivial invariant compact connected subset. The space on which the hyperbolic extension exists has a universal cover which is the product of the two trees obtained from the stable and unstable foliations.

The following theorem generalizes some work of J. Lewowicz [**L**] and M. Handel [**H1**].

THEOREM 1. *Let* $f: M \to M$ *be a pseudo-Anosov map of the closed connected surface*

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M. Let $g: N \to N$ be a homeomorphism of the compact connected space N. Suppose that $\alpha: N \to M$ is a continuous map such that the diagram:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow{f} & M \end{array}$$

commutes up to homotopy. If α is not homotopic to a constant, then there exists a closed subset $Y \subset N$, which is g invariant, and a continuous surjective map $\beta: Y \to M$ such that β is homotopic to $\alpha \mid Y$ and the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g \mid Y} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ M & \xrightarrow{f} & M. \end{array}$$

The second theorem is a generalization of the fact that two homotopic pseudo-Anosov maps are conjugate, it should also be compared with [**H2**].

THEOREM 2. *Let $f: M \to M$ and $g: N \to N$ be pseudo-Anosov maps, with the same dilatation coefficient on the closed connected surfaces M and N. Suppose that $\alpha_*: \pi_1(M) \to \pi_1(N)$ is a non constant algebraic homomorphism such that the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{g_*} & \pi_1(N) \\ {\scriptstyle\alpha*}\downarrow & & \downarrow{\scriptstyle\alpha*} \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(M), \end{array}$$

where f_* and g_* are respectively representatives of the action of f and g on the fundamental groups. Then there exists a ramified cover $\alpha: N \to M$ which semi-conjugates g to f, and whose action on fundamental groups can be represented by α_*.

1. *Some facts about trees and measured foliations*

Let M be a closed connected surface and let $\mathscr{F}$ be a minimal measured foliation on M. Call $\tilde{M}$ the universal cover of M. Denote by $\tilde{\mathscr{F}}$ the pullback of $\mathscr{F}$ to $\tilde{M}$. We call a leaf of $\tilde{\mathscr{F}}$ either a regular leaf which does not contain a separatrix or the union of a singularity and all separatrices ending in that singularity. Since $\mathscr{F}$ is minimal, any such leaf is a closed subset of $\tilde{M}$. We call $\mathscr{T}$ the set of leaves $\tilde{M}/\tilde{\mathscr{F}}$. If we define the distance between two leaves of $\tilde{\mathscr{F}}$ as the minimum of the transverse measures of arcs joining the two leaves, we obtain a distance on $\mathscr{T}$ which turns $\mathscr{T}$ into a tree - see Morgan and Shalen [**MS**, § 2]. If the genus of M is ≥ 2, then $\mathscr{T}$ is not complete for this distance. We will denote by $\hat{\mathscr{T}}$ its completion, it is also a tree, see [**MS**, proof of Theorem 2.1.9]. To simplify notations, we will denote by Γ the fundamental group of M the group Γ acts in a natural way as a group of isometries of $\mathcal{T}$ and hence it also acts as a group of isometries of $\hat{\mathcal{T}}$.

LEMMA 1.1. *If a is a closed arc contained in $\hat{\mathcal{T}}$ then a minus its endpoints is contained in $\mathcal{T}$. If $f \in \mathcal{T}$, the set $\hat{\mathcal{T}} \setminus \Gamma f$ is totally disconnected.*

In order to prove 1.1, we need a couple of sublemmas.

SUBLEMMA 1.2. *Let α (resp. α') be an arc between x and y (resp. x' and y') in a tree. The intersection $\alpha \cap \alpha'$ is an arc of length $l(\alpha \cap \alpha')$ satisfying:*

$$l(\alpha \cap \alpha') \geq l(\alpha) - d(x, x') - d(y, y').$$

The proof of this sublemma can be found in [**MS**, proof of Corollary 2.1.7].

SUBLEMMA 1.3. *Let $\hat{\mathcal{T}}$ be the completion of the tree $\mathcal{T}$. If α is an arc in $\hat{\mathcal{T}}$, then α minus its extremities is contained in $\mathcal{T}$.*

Proof. Call x and y the extremities of α. Choose $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in \mathcal{T}$. Call $\alpha_n \subset \mathcal{T}$ the arc between x_n and y_n. We have $\alpha \cap \alpha_n \subset \mathcal{T}$ and $l(\alpha \cap \alpha_n) \to l(\alpha)$ by 1.2. The sublemma follows easily. □

Proof of 1.1. Since the foliation $\mathcal{F}$ is minimal and any non trivial arc in $\mathcal{T}$ contains the image under the map $\tilde{M} \to \mathcal{T}$ of a non trivial arc transverse to $\tilde{\mathcal{F}}$, we obtain that any non trivial arc of $\mathcal{T}$ contains a point of Γf. By 1.2, the same is true for any non trivial arc in $\hat{\mathcal{T}}$. Lemma 1.1 follows from this last fact. □

2. *Embedding a pseudo-Anosov map in a hyperbolic dynamical system*

Let $f: M \to M$ be a pseudo-Anosov map. Let $\lambda > 1$ be its dilatation coefficient and let $\mathcal{F}^s$ and $\mathcal{F}^u$ be its stable and unstable foliations. Call $(\mathcal{T}^u, d^u)$ and $(\mathcal{T}^s, d^s)$ the trees $\tilde{M}/\tilde{\mathcal{F}}^s$ and $\tilde{M}/\tilde{\mathcal{F}}^u$ with their respective metrics. Fix a lift $\tilde{f}$ of f to $\tilde{M}$. This $\tilde{f}$ induces actions $\tilde{F}^u$ and $\tilde{F}^s$ on the trees $\mathcal{T}^u$ and $\mathcal{T}^s$ which satisfy:

$$\forall a, a' \in \mathcal{T}^u, \qquad d^u(\tilde{F}^u(a), \tilde{F}^u(a')) = \lambda d^u(a, a')$$

$$\forall b, b' \in \mathcal{T}^s, \qquad d^s(\tilde{F}^s(b), \tilde{F}^s(b')) = \lambda^{-1} d^s(b, b').$$

This implies that these actions extend to the completions $\hat{\mathcal{T}}^u$ and $\hat{\mathcal{T}}^s$ and that these extensions verify the same equalities.

If we look at the product action $\tilde{F} = \tilde{F}^s \times \tilde{F}^u$ on $\tilde{Z} = \hat{\mathcal{T}}^s \times \hat{\mathcal{T}}^u$ with the product metric $d = d^s + d^u$ we obtain what we can call a metrically split hyperbolic homeomorphism – generalizing to arbitrary metric spaces [**F**, Definition on p. 67].

There is an inclusion $\tilde{M} \hookrightarrow \tilde{Z}$, which is obtained by sending a point in M to the pair of leaves – one from $\tilde{\mathcal{F}}^s$ and the other from $\tilde{\mathcal{F}}^u$ – that contain it. It is well known that the metric d induces on $\tilde{M}$ the usual topology.

The action of $\Gamma = \pi_1(M)$ on $\tilde{M}$ induces isometries on $\tilde{Z}$ in a natural way. The inclusion $\tilde{M} \hookrightarrow \tilde{Z}$ is equivariant with respect to these actions of Γ.

If we write $\tilde{f}\gamma = f_{\#}(\gamma)\tilde{f}$ for $\gamma \in \Gamma$, we also have $\tilde{F}\gamma = f_{\#}(\gamma)\tilde{F}$.

LEMMA 2.1. *There exists $\varepsilon > 0$ such that*:

$$\forall \gamma \in \Gamma \backslash \{\mathrm{Id}\}, \quad \forall z \in \tilde{Z}, d(\gamma z, z) \geq \varepsilon.$$

Proof. If $z = (a^u, a^s) \in \tilde{\mathcal{F}}^u \times \tilde{\mathcal{F}}^s$ then $\gamma z = (\gamma a^u, \gamma a^s)$. This implies:

$$\begin{aligned} d(\gamma z, z) &= d^u(\gamma a^u, a^u) + d^s(\gamma a^s, a^s) \\ &\geq i(\gamma, \mathcal{F}^u) + i(\gamma, \mathcal{F}^s). \end{aligned}$$

It is well known that there exists $\varepsilon > 0$ such that:

$$\forall \gamma \in \Gamma \backslash \{\mathrm{Id}\}, i(\gamma, \mathcal{F}^u) + i(\gamma, \mathcal{F}^s) \geq \varepsilon. \qquad \square$$

COROLLARY 2.2. *The action of Γ on $\tilde{Z}$ is properly discontinuous.*

If we call $Z = \tilde{Z}/\Gamma$, we obtain a metric space since d is Γ equivariant. The map $\tilde{Z} \to Z$ appears as the universal cover of Z since $\tilde{Z}$ is contractible and locally contractible. The inclusion $\tilde{M} \hookrightarrow \tilde{Z}$ gives an inclusion $M \hookrightarrow Z$ which is an isomorphism on the fundamental group. The map $\tilde{F}$ gives a map $F: Z \to Z$ which is a hyperbolic homeomorphism whose universal cover is metrically split. The images of the foliations $\{\hat{\mathcal{T}}^s \times b \mid b \in \hat{\mathcal{T}}^u\}$ and $\{a \times \hat{\mathcal{T}}^u \mid a \in \hat{\mathcal{T}}^s\}$ under the map $\tilde{Z} \to Z$ give the stable and unstable foliations of F. The restriction of F to M is f.

LEMMA 2.3. *In the situation described above, two continuous maps $g_1, g_2: X \to M$ are homotopic as maps with values in M if and only if they are homotopic as maps with values in Z.*

Proof. Since the genus of the surface M is ≥ 1, we can endow M with a Riemannian metric such that any pair of points in the universal cover $\tilde{M}$ can be connected by a unique geodesic. Given any path $\alpha \subset Z$ connecting the two points $x, y \in M$, a lift $\tilde{\alpha}$ to $\tilde{Z}$ connects two points $\tilde{x}, \tilde{y} \in \tilde{M}$, call α' the image in M of the unique geodesic connecting $\tilde{x}, \tilde{y}$ in $\tilde{M}$. The map $\alpha \mapsto \alpha'$ is well defined and continuous in the compact open topology. The lemma follows routinely from this fact. $\square$

LEMMA 2.4. *If $x \in M$, then $Z \backslash (W^s(x, F) \cup W^u(x, F))$ is totally disconnected.*

Proof. Choose a point $(f, f') \in \tilde{Z}$ above x. We have $f \in \mathcal{T}^s$, $f' \in \mathcal{T}^u$. It follows from 1.1 that $(\hat{\mathcal{T}}^s \backslash \Gamma f) \times (\hat{\mathcal{T}}^u \backslash \Gamma f')$ is totally disconnected. But this product is precisely the inverse image of $Z \backslash (W^s(x, F) \cup W^u(x, F))$ under the covering map $\tilde{Z} \to Z$. $\square$

PROPOSITION 2.5. *If X is a closed connected non empty subset of Z which is F invariant then either it is reduced to a point or it contains M.*

Proof. Suppose that X is not reduced to a point. Since the periodic points of $f = F \mid M$ are dense in M and X is closed, it suffices to show that X contains these periodic points. Fix such a periodic point p. Since X is closed and invariant under F, it suffices to show that either $W^s(p, F)$ or $W^u(p, F)$ intersects X. But this follows clearly from 2.4, since X is connected and not reduced to a point. $\square$

The next theorem is a generalization of Theorem 1.

THEOREM 2.6. *Suppose that we have a diagram*:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Z & \xrightarrow{F} & Z \end{array}$$

commutative up to homotopy, where g is a homeomorphism of the compact connected space N. Then there exists a continuous map $\beta : N \to Z$, *homotopic to* α, *and such that the diagram*:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ Z & \xrightarrow{F} & Z \end{array}$$

commutes. Moreover, if α *is not homotopic to a constant, then the image* $\beta(N)$ *contains M. It follows that* $Y = \beta^{-1}(M)$ *is invariant under g and that* $\beta \mid Y$ *is a continuous surjection onto M, which is a semi-conjugacy between* $g \mid Y$ *and f. Moreover, if* $\alpha(Y) \subset M$ *then* $\alpha \mid Y$ *and* $\beta \mid Y$ *are homotopic as maps with values in M.*

Proof. Since Z has a universal cover on which F has a hyperbolic metrically split lift with a complete metric. It is easy to see that the machinery developed by J. Franks in [**F**, § 4] can be applied to give $\beta : N \to Z$, homotopic to α which gives a semi-conjugacy:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ Z & \xrightarrow{F} & Z. \end{array}$$

If α is not homotopic to a constant, by 2.3, the image $\beta(N)$ is a compact connected F invariant subset of Z which is not reduced to a point. By 2.5, we have $M \subset \beta(N)$. The last assertion follows from 2.3. □

3. *Pseudo-Anosov diffeomorphisms with the same dilatation coefficient*

The following lemma is certainly well known.

LEMMA 3.1. *Let* $g : N \to N$ *be a pseudo-Anosov diffeomorphism on a closed connected surface, with dilatation coefficient* $\lambda > 1$. *Suppose that* $Y \subset N$ *is a compact g-invariant subset, if the topological entropy of* $g \mid Y$ *is* $\log \lambda$, *then* $Y = N$.

Proof. From [**FLP**, Exposé 10 § IV and § VI], there exists an irreducible subshift of finite type (Σ_A, σ_A), whose topological entropy is $\log \lambda$, and a surjective semi-conjugacy $\theta : \Sigma_A \to N$ between σ_A and g. The closed subset $\theta^{-1}(Y)$ is σ_A-invariant and the topological entropy of σ_A restricted to that subset is $\log \lambda$, it follows from [**CP**, Theorem 3.3] that $\theta^{-1}(Y) = \Sigma_A$. The surjectivity of θ implies that $Y = N$. □

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Let $f: M \to M$ and $g: N \to N$ be pseudo-Anosov maps, with the same dilatation coefficient λ on the closed connected surfaces M and N. Suppose that $\alpha_*: \pi_1(N) \to \pi_1(M)$ is a non constant algebraic homomorphism such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{g_*} & \pi_1(N) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \pi_1(M) & \xrightarrow{f_*} & \pi_1(M), \end{array}$$

where f_* and g_* respectively represent f and g on the fundamental group.

The first remark is that there exists a map $\bar{\alpha}: N \to M$, which can be represented by α_* on π_1 and such that the diagram:

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \bar{\alpha}} \\ M & \xrightarrow{f} & M \end{array}$$

commutes up to homotopy. These are standard facts from algebraic topology. By Theorem 2.6, there exists $Y \subset N$ a g-invariant subset and a surjective semi-conjugacy $\alpha: Y \to M$, between $g \mid Y$ and f, with α homotopic to $\bar{\alpha} \mid Y$. Since the topological entropy of both f and g is $\log \lambda$, the topological entropy of $g \mid Y$ is also $\log \lambda$. By 3.1, we have $Y = N$ and, in fact, the map α is a surjective semi-conjugacy between g and f, which is homotopic to $\bar{\alpha}$. The following lemma finishes the proof of Theorem 2.

LEMMA 3.2. *Any non constant semi-conjugacy between two pseudo-Anosov maps, with the same dilatation coefficient on closed connected surfaces is a ramified cover..*

Proof. Let us denote by $\varphi: N \to M$ a non constant semi-conjugacy between the two pseudo-Anosov diffeomorphisms $g: N \to N$ and $f: M \to M$ with the same dilatation coefficient λ. We denote by $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ the invariant stable and unstable measured foliations of f and by $(\mathcal{G}^s, \nu^s)$, $(\mathcal{G}^u, \nu^u)$ those of g. Since φ is a semi-conjugacy it takes a leaf of $\mathcal{G}^s$ (resp. $\mathcal{G}^u$) to a leaf of $\mathcal{F}^s$ (resp. $\mathcal{F}^u$).

What we mean by a segment contained in a $\mathcal{G}^u$ leaf is a subset of a $\mathcal{G}^u$ leaf which is homeomorphic to $[0, 1]$ and if it contains a singularity then it must be entirely contained in the union of the singularity and of two separatrices adjacent to the same sector. Remark that the holonomy along $\mathcal{G}^s$ is always defined on one side of such a segment. If α is a segment in a $\mathcal{G}^u$ leaf, we denote by $\nu^s(\alpha)$ its ν^s measure and by $\bar{\mu}^s(\alpha)$ the measure of the $\mathcal{F}^u$ segment which has the same extremities as $\varphi(\alpha)$. It is clear that $\bar{\mu}^s$ is continuous on its domain of definition endowed with the compact open topology. Since g and f have the same dilatation coefficient λ, we have:

$$\forall \alpha \, \mathcal{G}^u\text{-segment } \bar{\mu}^s(f(\alpha)) = \lambda \bar{\mu}^s(\alpha). \tag{1}$$

If γ is a segment of a $\mathcal{G}^u$ leaf which is the union of the two subsegments α and β

then we have:

$$\bar{\mu}^s(\gamma) \le \bar{\mu}^s(\alpha) + \bar{\mu}^s(\beta). \tag{2}$$

If we move a segment in a $\mathscr{G}^u$ leaf by holonomy along $\mathscr{G}^s$ leaves to another segment in a $\mathscr{G}^u$ leaf, the two segments will have the same value under $\bar{\mu}^s$.

Our goal is to show that there exists $\rho > 0$ such that for any segment α in a $\mathscr{G}^u$ leaf $\bar{\mu}^s(\alpha) = \rho\nu^s(\alpha)$. We will use the fact that g has a Markov partition $\mathscr{R} = \{R_1, \dots, R_n\}$ - see [**FLP**, Exposé 10 § IV, § V and § VI] for the definition and properties of Markov partitions.

Denote by $S_i^u \subset R_i$ any segment in a $\mathscr{G}^u$ leaf which goes across R_i. The values $a_i = \nu^s(S_i^u)$ and $b_i = \bar{\mu}^s(S_i^u)$ do not depend on the choice of S_i^u because any two such choices differ by holonomy along $\mathscr{G}^s$ leaves. Call x_{ij} the number of times $g^{-1}(R_j)$ crosses the interior of R_i. This gives a positive matrix $X = (x_{ij})_{1 \le i,j \le n}$. From [**FLP**, Exposé 10 Lemma 1 p. 205], the matrix X has power which is strictly positive. Moreover, if $X^l = (x_{ij}^{(l)})_{1 \le i,j \le n}$, then $(x_{ij}^{(l)})_{1 \le i,j \le n}$ is the number of times that $f^{-l}(R_j)$ crosses R_i. It follows from this remark and (2) above that, if we define A (resp. B) as the vector with components $a_1, \dots, a_n$ (resp. $b_1, \dots, b_n$), then we have:

$$\forall l \ge 1, \lambda^l A = X^l A \quad \text{and} \quad \lambda^l B \le X^l B. \tag{3}$$

Call $\tilde{a}_1, \dots, \tilde{a}_n$ the components of the eigenvector with eigenvalue λ of the transpose of X normalized by the relation $\sum_{i=1}^n \tilde{a}_i a_i = 1$. From Perron–Frobenius theory, it is well known that $\tilde{a}_i > 0$ and $\lim_{l \to \infty} \lambda^{-l} X^l B = \rho A$, with $\rho = \sum_{i=1}^n \tilde{a}_i b_i$. It follows from (3) that $B \le \rho A$. If one of the components of the last inequality was strict multiplying the i-component by the strictly positive $\tilde{a}_i$ and summing would give $\rho = \sum_{i=1}^n \tilde{a}_i b_i < \rho \sum_{i=1}^n \tilde{a}_i a_i = \rho$ which is impossible. So we have:

$$\forall i = 1, \dots, n, \qquad b_i = \rho a_i. \tag{4}$$

Suppose now that α is a segment in a $\mathscr{G}^u$ leaf. For each $j = 1, \dots, n$ and each $l \ge 1$, call α_j^l the number of $\mathscr{G}^u$ segments intersecting α which are contained in $f^{-l}(R_i)$ and go across it. Using (1) through (4), we obtain:

$$\bar{\mu}^s(\alpha) \le \sum_{i=1}^n \alpha_j^l \lambda^{-l} b_j = \sum_{i=1}^n \alpha_j^l \rho \lambda^{-l} a_j \to \rho\nu^s(\alpha).$$

This shows the inequality $\bar{\mu}^s(\alpha) \le \rho\nu^s(\alpha)$. Note now that ρ cannot be zero, because that would imply that the image of $\varphi(N)$ would be contained in one leaf of $\mathscr{F}^s$ and hence, by the g invariance, it would be reduced to one point. Suppose that some segment β in a $\mathscr{G}^u$ leaf verifies $\bar{\mu}^s(\beta) < \rho\nu^s(\beta)$. Choose a point x in the interior of β and call x_∞ an accumulation point of the sequence $g^{-l}(x)$, $l \ge 0$. It is clear that the unstable leaf through x_∞ contains arbitrarily small segments α, containing x_∞, and such that $\bar{\mu}^s(\alpha) < \rho\nu^s(\alpha)$. The density of the stable leaf through x_∞ and the invariance by holonomy show that the same strict inequality is true for any segment in a $\mathscr{G}^u$ leaf. This is absurd, since we have equality for any segment in a $\mathscr{G}^u$ leaf contained in a R_i and going across it.

By rescaling ν^s, we can assume that $\rho = 1$. This allows us to interpret what we obtained in the following way: the map φ is an isometry of any non singular leaf of $\mathscr{G}^u$ endowed with the metric defined by ν^s onto the corresponding leaf of $\mathscr{F}^u$ endowed with the metric defined by μ^s. Of course, the same result can be obtained with stable foliations. It is now easy to finish the proof of the lemma. $\square$

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