FACTORIZATION AND BOUNDED APPROXIMATE IDENTITIES FOR A CLASS OF CONVOLUTION BANACH ALGEBRAS

by S. I. OUZOMGI

(Received 10 July, 1985)

An algebra A *factors* if, for each $a \in A$, there exist $b, c \in A$ with $a = bc$. A *bounded approximate identity* for a Banach algebra A is a net $(e_\alpha) \subset A$ such that $ae_\alpha \to a$ and $e_\alpha a \to a$ for each $a \in A$ and such that $\sup \|e_\alpha\| < \infty$. It is well known [**2**, 11.10] that if A has a bounded approximate identity, then A factors. But a Banach algebra may factor even if it does not have a bounded approximate identity: an example which is non-commutative and separable, and an example which is commutative and non-separable, are given in [**3**, §22]. However, we do not know an example of a commutative, separable Banach algebra which factors, but which does not have a bounded approximate identity. See [**4**] for related work.

In this note, we show that, for a certain class of commutative, separable Banach algebras, an algebra factors if and only if it has a bounded approximate identity.

A real-valued function ω defined on $\mathbb{R}^+$ is a *weight function* if ω is Lebesgue measurable, if $\omega(t) > 0\ (t \in \mathbb{R}^+)$, and if

$$\omega(s+t) \leqslant \omega(s)\omega(t) \qquad (s, t \in \mathbb{R}^+).$$

Let ω be a weight function on $\mathbb{R}^+$. We denote by $L^1(\omega)$ the set of complex-valued, measurable functions on $\mathbb{R}^+$ such that

$$\|f\| \equiv \int_0^\infty |f(t)|\, \omega(t)\, dt < \infty.$$

As usual, we equate functions which are equal almost everywhere. Then $L^1(\omega)$ is a Banach space with respect to pointwise addition and scalar multiplication. For $f, g \in L^1(\omega)$, we define $f * g$ by setting

$$(f * g)(t) = \int_0^t f(t-s)g(s)\, ds \qquad (t \in \mathbb{R}^+).$$

Then $f * g$ is finite almost everywhere and defines an element of $L^1(\omega)$. With respect to this convolution multiplication, $L^1(\omega)$ is a commutative Banach algebra, and clearly $L^1(\omega)$ is separable. The algebras $L^1(\omega)$ are discussed in [**1**], for example.

In the theorem below, we write m for Lebesgue measure on $\mathbb{R}^+$ and $\operatorname{supp} f$ for the support of a function f. If A is an algebra, then A^2 denotes the linear span of the set of products of two elements of A.

THEOREM. *Let ω be a weight function on $\mathbb{R}^+$. Then the following conditions on ω are* *equivalent*:

(1) *there exists* $M>0$ *such that, for each* $\delta>0$, $m\{t\in[0,\delta]:\omega(t)<M\}$ *is greater than* 0;

(2) $L^1(\omega)$ *has a bounded approximate identity*;

(3) $L^1(\omega)$ *factors*;

(4) $[L^1(\omega)]^2=L^1(\omega)$.

Proof. (1)$\Rightarrow$(2). Let $E_n=\{t\in(0,1/n]:\omega(t)\leq M\}$. By hypothesis, $m(E_n)>0$. Let χ_n be the characteristic function of E_n, and let $e_n=\chi_n/m(E_n)$. Clearly, $\|e_n\|\leq M$, and so (e_n) is a bounded sequence in $L^1(\omega)$.

A standard argument using the uniform continuity of a continuous function with compact support shows that (e_n) is a bounded approximate identity for $L^1(\omega)$.

(2)$\Rightarrow$(3). This is Cohen's factorization theorem [**2**, 11.10].

(3)$\Rightarrow$(4). Immediate.

(4)$\Rightarrow$(1). To obtain a contradiction, suppose that (4) holds but that (1) fails. Define a function $\bar{\omega}$ on $\mathbb{R}^+$ by setting

$$\bar{\omega}(t)=\operatorname{ess\,inf}\{\omega(s):0<s<t\}\qquad(t>0).$$

Then $\bar{\omega}$ is measurable on $\mathbb{R}^+$, and $\bar{\omega}(t)\leq\omega(t)$ for almost all $t>0$. Take $s,t>0$ and $\varepsilon>0$. Then there are sets $S\subset(0,s)$ and $T\subset(0,t)$ such that S and T have positive measure and such that

$$\omega(s')\leq\bar{\omega}(s)+\varepsilon\qquad(s'\in S),\qquad\omega(t')\leq\bar{\omega}(t)+\varepsilon\qquad(t'\in T).$$

Then $S+T$ is a subset of $(0,s+t)$ which has positive measure, and

$$\omega(s'+t')\leq\omega(s')\omega(t')\leq(\bar{\omega}(s)+\varepsilon)(\bar{\omega}(t)+\varepsilon)\qquad(s'\in S,t'\in T).$$

Thus $\bar{\omega}(s+t)\leq(\bar{\omega}(s)+\varepsilon)(\bar{\omega}(t)+\varepsilon)$. This is true for each $\varepsilon>0$, and so $\bar{\omega}(s+t)\leq\bar{\omega}(s)\bar{\omega}(t)$. Hence $\bar{\omega}$ is a weight function on $\mathbb{R}^+$, because $\bar{\omega}$ is measurable. Further $\bar{\omega}$ is decreasing.

Define a function Ω on $(0,\infty)$ by

$$\Omega(\delta)=\sup\left\{\frac{\bar{\omega}(s+t)}{\bar{\omega}(s)\bar{\omega}(t)}:s,t>0,s+t\leq\delta\right\}\qquad(\delta>0).$$

Clearly, Ω is monotonically increasing on $(0,\infty)$. Since (1) fails, $\bar{\omega}(t)\to\infty$ as $t\to0+$, and so $\Omega(\delta)\to0$ as $\delta\to0+$.

For $t>0$, set

$$S_t=\{s\in(0,t):\omega(s)\leq2\bar{\omega}(t)\}.$$

Then S_t has positive measure, and $\omega(s)\leq2\bar{\omega}(s)$ $(s\in S_t)$. We can inductively define a sequence (δ_n) such that $0<\delta_{n+1}<\delta_n$, such that $\sum_{n=1}^{\infty}\Omega(\delta_n)<\infty$, and such that $m(A_n)>0$, where $A_n=S_{\delta_n}\cap(\delta_{n+1},\delta_n)$.

Set

$$f(t)=\sum_{n=1}^{\infty}\frac{\Omega(\delta_n)}{m(A_n)\omega(t)}\chi_{A_n}(t)\qquad(t>0).$$

Then $\int_0^\infty |f(t)|\,\omega(t)\,dt=\sum_{n=1}^{\infty}\Omega(\delta_n)<\infty$, and so $f\in L^1(\omega)$.

We shall show that $f\notin[L^1(\omega)]^2$. To obtain a contradiction, suppose that $f=\sum_{i=1}^{k} g_i * h_i$, where $g_1,\ldots,g_k,h_1,\ldots,h_k\in L^1(\omega)$. Then

$$f(t)\leq\sum_{i=1}^{k}\int_0^t |g_i(t-s)h_i(s)|\,ds\qquad(t\in\mathbb{R}^+).$$

Since $\tilde{\omega}(t)\leq\omega(t)$ for almost all t and $\omega(t)\leq 2\tilde{\omega}(t)$ for $t\in\operatorname{supp} f$, we have

$$\Omega(\delta_n)=\int_{A_n} f(t)\omega(t)\,dt\leq 2\Omega(\delta_n)K_n,$$

where

$$K_n=\sum_{i=1}^{k}\int_{A_n}\int_0^t |g_i(t-s)h_i(s)|\,\omega(t-s)\omega(s)\,ds\,dt.$$

Thus $K_n\geq 1/2$ $(n\in\mathbb{N})$. However,

$$\begin{aligned}\sum_{n=1}^{\infty}K_n&\leq\sum_{i=1}^{k}\int_0^\infty\int_0^t |g_i(t-s)|\,\omega(t-s)\,|h_i(s)|\,\omega(s)\,ds\,dt\\&\leq\sum_{i=1}^{k}\|g_i\|\,\|h_i\|<\infty,\end{aligned}$$

and so $K_n\to 0$ as $n\to\infty$. This is the required contradiction.

This completes the proof of the theorem.

REMARK. If ω is bounded in a neighborhood of 0, then clearly the conditions of the theorem are satisfied. However, it is easy to give a weight function ω for which $\operatorname{ess\,lim\,sup}_{t\to 0+}\omega(t)=\infty$, but which satisfies the conditions of the theorem.

In the above proof, we introduced a new weight function $\tilde{\omega}$. This was necessary because there are weight functions ω for which (1) fails, but which are such that

$$\inf_{\delta>0}\operatorname{ess\,sup}\left\{\frac{\omega(s+t)}{\omega(s)\omega(t)}: s,t>0,\ s+t\leq\delta\right\}>0.$$

To exemplify these two remarks, we give one construction.

Let (c_n), (δ_n) be sequences with $c_1=0$, $c_{n+1}>c_n$, $\delta_1=1$, and $0<\delta_{n+1}<\delta_n$ for $n\in\mathbb{N}$ and $\delta_n\to 0$ as $n\to\infty$. Let $\eta_n(t)=(c_{n+1}-c_n)t$ $(t\in[0,\delta_n])$ and let $\eta_n(t)=0$ $(t>\delta_n)$. Then

$\eta_n(s+t) \leqslant \eta_n(s) + \eta_n(t)$ for $s, t \in \mathbb{R}^+$. Let $\eta(t) = \sum \eta_n(t)$, and let $\omega(t) = \exp \eta(t)$ $(t \in \mathbb{R}^+)$. Then ω is a weight function on $\mathbb{R}^+$, and $\eta(t) = c_{n+1}t$ $(t \in (\delta_{n+1}, \delta_n])$. Suppose further that $\delta_{n+1} < \delta_n/n$ and that $c_{n+1} = n/\delta_n$ $(n \in \mathbb{N})$. On $[\delta_n/n, 2\delta_n/n]$, $\eta(t) \leqslant 2$, and so ω satisfies condition (1), above. However, on $[\delta_n/2, \delta_n]$, $\eta(t) \geqslant n/2$, and so $\operatorname{ess} \limsup_{t \to 0+} \omega(t) = \infty$.

Secondly, take ω as above, choosing $\delta_{n+1} < \delta_n/4$ and $c_n = n/\delta_n$ $(n \in \mathbb{N})$. Then $\omega(s+t) = \omega(s)\omega(t)$ for $s, t \in (\frac{1}{4}\delta_n, \frac{1}{2}\delta_n)$. However, $\eta(t) \geqslant c_{n+1}\delta_{n+1}$ for $t \in (0, \delta_n]$, and so (1) fails.

ACKNOWLEDGEMENT. This paper is a part of a Ph.D. thesis submitted to Leeds University. I should like to thank my supervisor H. G. Dales for his constant help and encouragement.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
CALIFORNIA STATE UNIVERSITY, LONG BEACH
LONG BEACH, CALIFORNIA 90840