Formulae connected with the Radii of the Incircle and the Excircles of a Triangle.

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The notation employed in the following pages is that recommended in a paper of mine on "The Triangle and its Six Scribed Circles"* printed in the first volume of the *Proceedings of the Edinburgh Mathematical Society.* It may be convenient to repeat all that is necessary for the present purpose.

$a, \; b, \; c$	= the sides BC, CA, AB of triangle ABC.
$h_1, \; h_2, \; h_3$	= the perpendiculars from A, B, C on BC, CA, AB.
m_1, m_2, m_3	= the medians from A, B, C.
r	= the radius of the incircle.
$r_1, \; r_2, \; r_3$	= the radii of the 1st, 2nd, 3rd excircles. These are frequently denoted by $r_a, \; r_b, \; r_c$.
s	= semiperimeter † of ABC.
$s_1, \; s_2, \; s_3$	$= s - a, \; s - b, \; s - c$. ‡
Δ	= area of ABC.

When an equation expresses a property of a triangle relating to one of the excircles it is easily enough transformed into the corresponding equation for either of the other excircles. It is not however so easy at first sight to transform an equation relating to the incircle into the corresponding one relating to an excircle. The following table gives the substitutions that are necessary to effect the required transformation.

* The title is somewhat of a misnomer. Five only of these circles are treated of. The sixth (the nine-point circle) is discussed in the eleventh volume of the *Proceedings.*

† On the Continent of Europe p is generally employed instead of s.

‡ This notation was suggested by Thomas Weddle in 1842. See *Lady's and Gentleman's Diary* for 1843, p. 78. Professor Neuberg proposes $p_1, \; p_2, \; p_3$ instead of $p-a, \; p-b, \; p-c$ in *Mathesis,* III. 167 (1883).

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When r is changed into r_1

a	b	c	s	s_1	s_2	s_3	become
a	$-b$	$-c$	$-s_1$	$-s$	s_3	s_2	
r_1	r_2	r_3	h_1	h_2	h_3		become
r	$-r_3$	$-r_2$	$-h_1$	h_2	h_3		

A	B	C	R	Δ	become
$-$A	180° $-$ B	180° $-$ C	$-$R	$-\Delta$	

The greater part of this table is given in the *Lady's and Gentleman's Diary* for 1871, p. 93, and it is due either to the editor of the *Diary*, W. S. B. Woolhouse, or to one of his correspondents, W. B. G. (William Bywater Grove?). No demonstration however is offered of the law of transformation thus enunciated.

A discussion of this law by Mr E. Lemoine will be found in the *Bulletin de la Société Mathématique de France*, XIX. 133–141 (1891), in Mr De Longchamps' *Journal de Mathématiques Élémentaires*, 4th series, I. 62–69, 91–93, 103–106 (1892), and in *Mathesis*, 2nd series, II. 58–64, 81–92 (1892). Two articles on the same subject by Edouard Lucas will be found in *Nouvelle Correspondance Mathématique*, II. 384–391 (1876), III. 1–5 (1877).

The following algebraical identities will be found useful.

$$\left.\begin{aligned} s - s_1 &= s_2 + s_3 = a \\ s - s_2 &= s_3 + s_1 = b \\ s - s_3 &= s_1 + s_2 = c \end{aligned}\right\} \qquad \text{I.}$$

$$\left.\begin{aligned} s + s_1 &= b + c \qquad & s_2 - s_3 &= c - b \\ s + s_2 &= c + a \qquad & s_3 - s_1 &= a - c \\ s + s_3 &= a + b \qquad & s_1 - s_2 &= b - a \end{aligned}\right\} \qquad \text{II.}$$

$$\left.\begin{aligned} s + s_1 + s_2 + s_3 &= 2s \\ s - s_1 + s_2 + s_3 &= 2a \\ s + s_1 - s_2 + s_3 &= 2b \\ s + s_1 + s_2 - s_3 &= 2c \end{aligned}\right\} \qquad \text{III.}$$

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$$\left.\begin{aligned} s_1 + s_2 + s_3 &= s \\ s - s_3 - s_2 &= s_1 \\ s - s_1 - s_3 &= s_2 \\ s - s_2 - s_1 &= s_3 \end{aligned}\right\} \qquad \text{IV.}$$

$$s^2 + s_1^2 + s_2^2 + s_3^2 = a^2 + b^2 + c^2 \qquad \text{V.}$$

$$\left.\begin{aligned} ss_2 - s_1s_3 - s_2s_1 + s_3s &= a^2 \\ ss_3 - s_2s_1 - s_3s_2 + s_1s &= b^2 \\ ss_1 - s_3s_2 - s_1s_3 + s_2s &= c^2 \end{aligned}\right\} \qquad \text{VI.}$$

$$\left.\begin{aligned} s\,s_1 + s\,s_2 + s\,s_3 &= s^2 \\ s_1s - s_1s_3 - s_1s_2 &= s_1^2 \\ s_2s - s_2s_1 - s_2s_3 &= s_2^2 \\ s_3s - s_3s_2 - s_3s_1 &= s_3^2 \end{aligned}\right\} \qquad \text{VII.}$$

$$ss_1 + s_2s_3 = bc, \quad ss_2 + s_3s_1 = ca, \quad ss_3 + s_1s_2 = ab \qquad \text{VIII.}$$

$$\left.\begin{aligned} 2(ss_1 - s_2s_3) &= -a^2 + b^2 + c^2 \\ 2(ss_2 - s_3s_1) &= a^2 - b^2 + c^2 \\ 2(ss_3 - s_1s_2) &= a^2 + b^2 - c^2 \end{aligned}\right\} \qquad \text{IX.}$$

$$\left.\begin{aligned} 4(s_2s_3 + s_3s_1 + s_1s_2) &= 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\ 4(s_3s_2 - s_2s - s\,s_3) &= 2(bc - ca - ab) - (a^2 + b^2 + c^2) \\ 4(s_1s_3 - s_3s - s\,s_1) &= 2(ca - ab - bc) - (a^2 + b^2 + c^2) \\ 4(s_2s_1 - s_1s - s\,s_2) &= 2(ab - bc - ca) - (a^2 + b^2 + c^2) \end{aligned}\right\} \qquad \text{X.}$$

$$\left.\begin{aligned} &2(as_1 + b\,s_2 + c\,s_3) = 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\ &2(as_2 + bs_3 + c\,s_1) = 2(as_3 + b\,s_1 + c\,s_2) = a^2 + b^2 + c^2 \\ &-as_1 + bs_2 + c\,s_3 = 2s_2s_3 \qquad -as_3 + bs + cs_1 = 2ss_1 \\ &as_1 - bs_2 + c\,s_3 = 2s_3s_1 \qquad as_2 - bs_1 + cs = 2ss_2 \\ &as_1 + bs_2 - c\,s_3 = 2s_1s_2 \qquad as + bs_3 - cs_2 = 2ss_3 \\ &s_1(b - c) + s_2(c - a) + s_3(a - b) = 0 \end{aligned}\right\} \qquad \text{XI.}$$

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$$\left.\begin{aligned} s_1^3+s_2^3+s_3^3+3abc&=s^3\\ ss_1s_2s_3\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}-\frac{1}{s}\right)&=abc\\ as_2s_3+bs_3s_1+cs_1s_2+2s_1s_2s_3&=abc\\ as_1^2+bs_2^2+cs_3^2+2s_1s_2s_3&=abc\\ 4(as_1^2+bs_2^2+cs_3^2)&\\ =a^3+b^3+c^3+6abc-b^2c-bc^2-c^2a-ca^2-a^2b-ab^2&\\ a(b-c)s_1^2+b(c-a)s_2^2+c(a-b)s_3^2&=0 \end{aligned}\right\} \quad \text{XII.}$$

$$r=\frac{\Delta}{s} \qquad r_1=\frac{\Delta}{s_1} \qquad r_2=\frac{\Delta}{s_2} \qquad r_3=\frac{\Delta}{s_3} \tag{1}$$

where $\Delta=\sqrt{ss_1s_2s_3}=\frac{1}{2}ah_1=\frac{1}{2}bh_2=\frac{1}{2}ch_3$

These results may be put into a variety of other forms, such as

$$\left.\begin{aligned} &sr=s_1r_1=s_2r_2=s_3r_3\\ &\frac{r}{r_1}=\frac{s_1}{s} \qquad \frac{r}{r_2}=\frac{s_2}{s} \qquad \frac{r}{r_3}=\frac{s_3}{s}\\ &\frac{r_2}{r_3}=\frac{s_3}{s_2} \qquad \frac{r_3}{r_1}=\frac{s_1}{s_3} \qquad \frac{r_1}{r_2}=\frac{s_2}{s_1} \end{aligned}\right\} \tag{2}$$

$$rr_1r_2r_3=\Delta^2=ss_1s_2s_3 \tag{3}$$

$$\left.\begin{aligned} \frac{r_1r_2r_3}{s}&=\Delta=\frac{s_1s_2s_3}{r}\\ \frac{r\,r_3r_2}{s_1}&=\Delta=\frac{s\,s_3s_2}{r_1}\\ \frac{r\,r_1r_3}{s_2}&=\Delta=\frac{s\,s_1s_3}{r_2}\\ \frac{r\,r_2r_1}{s_3}&=\Delta=\frac{s\,s_2s_1}{r_3} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} r_2r_3:s^2&=r^2:s_2s_3\\ r_3r_1:s^2&=r^2:s_3s_1\\ r_1r_2:s^2&=r^2:s_1s_2 \end{aligned}\right\} \tag{5}$$

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Other proportions may be obtained by putting for the extremes rr_1, ss_1, etc., and for the means r_1^2, s_1^2, etc.

$$\left.\begin{aligned} r_1r_2r_3 : s^3 &= r^3 : s_1s_2s_3 \\ r\,r_3r_2 : s_1^3 &= r_1^3 : s\,s_3s_2 \\ r\,r_1r_3 : s_2^3 &= r_2^3 : s\,s_1s_3 \\ r\,r_2r_1 : s_3^3 &= r_3^3 : s\,s_2s_1 \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} r^2 &= \frac{s_1s_2s_3}{s}, & r_1^2 &= \frac{ss_3s_2}{s_1}, & r_2^2 &= \frac{ss_1s_3}{s_2}, & r_3^2 &= \frac{ss_2s_1}{s_3} \\ s^2 &= \frac{r_1r_2r_3}{r}, & s_1^2 &= \frac{rr_3r_2}{r_1}, & s_2^2 &= \frac{rr_1r_3}{r_2}, & s_3^2 &= \frac{rr_2r_1}{r_3} \end{aligned}\right\} \qquad (7)$$

By means of (7) and I., II., III., IV., a large number of expressions may be obtained. The following is given as a specimen:

$$\sqrt{\frac{r_1r_2r_3}{r}} = \sqrt{\frac{r\,r_3r_2}{r_1}} + \sqrt{\frac{r\,r_1r_3}{r_2}} + \sqrt{\frac{r\,r_2r_1}{r_3}} \qquad (8)$$

$$\left.\begin{aligned} rr_1 &= s_2s_3 & rr_2 &= s_3s_1 & rr_3 &= s_1s_2 \\ ss_1 &= r_2r_3 & ss_2 &= r_3r_1 & ss_3 &= r_1r_2 \end{aligned}\right\} \qquad (9)$$

$$rr_1 + r_2r_3 = bc \qquad rr_2 + r_3r_1 = ca \qquad rr_3 + r_1r_2 = ab \qquad (10)$$

See VIII.

$$2(r_2r_3 + r_3r_1 + r_1r_2 + rr_1 + rr_2 + rr_3) = 2(bc + ca + ab) \qquad (11)$$

$$2(r_2r_3 + r_3r_1 + r_1r_2 - rr_1 - rr_2 - rr_3) = a^2 + b^2 + c^2 \qquad (12)$$

See IX.

$$\left.\begin{aligned} a^2 + 4r_2r_3 &= (b+c)^2 & a^2 - 4rr_1 &= (b-c)^2 \\ b^2 + 4r_3r_1 &= (c+a)^2 & b^2 - 4rr_2 &= (c-a)^2 \\ c^2 + 4r_1r_2 &= (a+b)^2 & c^2 - 4rr_3 &= (a-b)^2 \end{aligned}\right\} \qquad (13)$$

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$$\left.\begin{aligned} \frac{bc - r_2r_3}{r_1} = \frac{ca - r_3r_1}{r_2} = \frac{ab - r_1r_2}{r_3} = r \\ \frac{bc - s_2s_3}{s_1} = \frac{ca - s_3s_1}{s_2} = \frac{ab - s_1s_2}{s_3} = s \end{aligned}\right\} \quad (14)$$

Similar expressions may be obtained for r_1, r_2, r_3 and for s_1, s_2, s_3.

$$\left.\begin{aligned} r_2r_3 + r_3r_1 + r_1r_2 &= s^2 \\ r_3r_2 - r_2r - r\,r_3 &= s_1^2 \\ r_1r_3 - r_3r - r\,r_1 &= s_2^2 \\ r_2r_1 - r_1r - r\,r_2 &= s_3^2 \end{aligned}\right\} \quad (15)$$

See VII.

$$\left.\begin{aligned} -r_2r_3 + r_3r_1 + r_1r_2 &= s\,(2a - s) \\ r_2r_3 + r_2r + r\,r_3 &= s_1(2a + s_1) \\ r_3r_1 + r_3r + r\,r_1 &= s_2(2b + s_2) \\ r_1r_2 + r_1r + r\,r_2 &= s_3(2c + s_3) \end{aligned}\right\} \quad (16)$$

See III.

$$\left.\begin{aligned} 4r\,(r_1 + r_2 + r_3) &= 2(bc + ca + ab) - (a^2 + b^2 + c^2) \\ 4r_1(r - r_3 - r_2) &= 2(bc - ca - ab) - (a^2 + b^2 + c^2) \\ 4r_2(r - r_1 - r_3) &= 2(ca - ab - bc) - (a^2 + b^2 + c^2) \\ 4r_3(r - r_2 - r_1) &= 2(ab - bc - ca) - (a^2 + b^2 + c^2) \end{aligned}\right\} \quad (17)$$

See X.

$$\left.\begin{aligned} r\,(r_1 + r_2 + r_3) &= bc - s_1^2 = ca - s_2^2 = ab - s_3^2 \\ r_1(r - r_3 - r_2) &= bc - s^2 = -ca - s_3^2 = -ab - s_2^2 \\ r_2(r - r_1 - r_3) &= ca - s^2 = -ab - s_1^2 = -bc - s_3^2 \\ r_3(r - r_2 - r_1) &= ab - s^2 = -bc - s_2^2 = -ca - s_1^2 \end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned} 4r_2r_3 + r_3r_1 + r_1r_2 - 4rr_1 - rr_2 - rr_3 &= 4m_1^2 \\ 4r_3r_1 + r_1r_2 + r_2r_3 - 4rr_2 - rr_3 - rr_1 &= 4m_2^2 \\ 4r_1r_2 + r_2r_3 + r_3r_1 - 4rr_3 - rr_1 - rr_2 &= 4m_3^2 \end{aligned}\right\} \quad (19)$$

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$$\left.\begin{array}{llll} \dfrac{h_1}{2r}=\dfrac{s}{a} & \dfrac{h_1}{2r_1}=\dfrac{s_1}{a} & \dfrac{h_1}{2r_2}=\dfrac{s_2}{a} & \dfrac{h_1}{2r_3}=\dfrac{s_3}{a} \\ \dfrac{h_2}{2r}=\dfrac{s}{b} & \dfrac{h_2}{2r_1}=\dfrac{s_1}{b} & \dfrac{h_2}{2r_2}=\dfrac{s_2}{b} & \dfrac{h_2}{2r_3}=\dfrac{s_3}{b} \\ \dfrac{h_3}{2r}=\dfrac{s}{c} & \dfrac{h_3}{2r_1}=\dfrac{s_1}{c} & \dfrac{h_3}{2r_2}=\dfrac{s_2}{c} & \dfrac{h_3}{2r_3}=\dfrac{s_3}{c} \end{array}\right\} \quad (20)$$

$$\frac{h_2h_3}{4r_2r_3}=\frac{rr_1}{bc} \qquad \frac{h_3h_1}{4r_3r_1}=\frac{rr_2}{ca} \qquad \frac{h_1h_2}{4r_1r_2}=\frac{rr_3}{ab} \quad (21)$$

$$\frac{8s^3}{abc}r^3=\frac{8s_1^3}{abc}r_1^3=\ldots \quad =h_1h_2h_3 \quad (22)$$

$$\triangle = sr_1\left(1-\frac{2r}{h_1}\right)=s_1r\left(1+\frac{2r_1}{h_1}\right)=\ldots \quad (23)$$

$$\left.\begin{array}{l} \dfrac{1}{r}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}= \quad \dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3} \\ \dfrac{1}{r_1}=\dfrac{1}{r}-\dfrac{1}{r_3}-\dfrac{1}{r_2}=-\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3} \\ \dfrac{1}{r_2}=\dfrac{1}{r}-\dfrac{1}{r_1}-\dfrac{1}{r_3}=-\dfrac{1}{h_2}+\dfrac{1}{h_3}+\dfrac{1}{h_1} \\ \dfrac{1}{r_3}=\dfrac{1}{r}-\dfrac{1}{r_2}-\dfrac{1}{r_1}=-\dfrac{1}{h_3}+\dfrac{1}{h_1}+\dfrac{1}{h_2} \end{array}\right\} \quad (24)$$

These may be put into many other forms; for example

$$\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$$

$$\left.\begin{array}{ll} \dfrac{1}{r}-\dfrac{1}{r_1}=\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{2}{h_1} & \dfrac{2rr_1}{r_1-r}=\dfrac{2r_2r_3}{r_2+r_3}=h_1 \\ \dfrac{1}{r}-\dfrac{1}{r_2}=\dfrac{1}{r_3}+\dfrac{1}{r_1}=\dfrac{2}{h_2} & \dfrac{2rr_2}{r_2-r}=\dfrac{2r_3r_1}{r_3+r_1}=h_2 \\ \dfrac{1}{r}-\dfrac{1}{r_3}=\dfrac{1}{r_1}+\dfrac{1}{r_2}=\dfrac{2}{h_3} & \dfrac{2rr_3}{r_3-r}=\dfrac{2r_1r_2}{r_1+r_2}=h_3 \end{array}\right\} \quad (25)$$

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$$\left.\begin{array}{ll} \dfrac{1}{r}+\dfrac{1}{r_1}=\dfrac{2}{h_2}+\dfrac{2}{h_3} & \dfrac{2rr_1}{r+r_1}=\dfrac{h_2h_3}{h_2+h_3} \\[2ex] \dfrac{1}{r}+\dfrac{1}{r_2}=\dfrac{2}{h_3}+\dfrac{2}{h_1} & \dfrac{2rr_2}{r+r_2}=\dfrac{h_3h_1}{h_3+h_1} \\[2ex] \dfrac{1}{r}+\dfrac{1}{r_3}=\dfrac{2}{h_1}+\dfrac{2}{h_2} & \dfrac{2rr_3}{r+r_3}=\dfrac{h_1h_2}{h_1+h_2} \end{array}\right\} \quad (26)$$

$$\left.\begin{array}{ll} \dfrac{1}{r_3}-\dfrac{1}{r_2}=\dfrac{2}{h_2}-\dfrac{2}{h_3} & \dfrac{2r_2r_3}{r_3-r_2}=\dfrac{h_2h_3}{h_2-h_3} \\[2ex] \dfrac{1}{r_1}-\dfrac{1}{r_3}=\dfrac{2}{h_3}-\dfrac{2}{h_1} & \dfrac{2r_3r_1}{r_1-r_3}=\dfrac{h_3h_1}{h_3-h_1} \\[2ex] \dfrac{1}{r_2}-\dfrac{1}{r_1}=\dfrac{2}{h_1}-\dfrac{2}{h_2} & \dfrac{2r_1r_2}{r_2-r_1}=\dfrac{h_1h_2}{h_1-h_2} \end{array}\right\} \quad (27)$$

$$\left.\begin{array}{l} r=\dfrac{r_1r_2r_3}{r_2r_3+r_3r_1+r_1r_2}=\dfrac{h_1h_2h_3}{h_2h_3+h_3h_1+h_1h_2} \\[2ex] r_1=\dfrac{r\,r_3r_2}{r_3r_2-r_2r\;-r\,r_3}=\dfrac{h_1h_2h_3}{-h_2h_3+h_3h_1+h_1h_2} \\[2ex] r_2=\dfrac{r\,r_1r_3}{r_1r_3-r_3r\;-r\,r_1}=\dfrac{h_1h_2h_3}{-h_3h_1+h_1h_2+h_2h_3} \\[2ex] r_3=\dfrac{r\,r_2r_1}{r_2r_1-r_1r\;-r\,r_2}=\dfrac{h_1h_2h_3}{-h_1h_2+h_2h_3+h_3h_1} \end{array}\right\} \quad (28)$$

$$\left.\begin{array}{l} \dfrac{r_1^2r_2^2r_3^2}{r_2r_3+r_3r_1+r_1r_2}=\dfrac{r^2r_3^2r_2^2}{r_3r_2-r_2r\;-r\,r_3} \\[2ex] =\dfrac{r^2r_1^2r_3^2}{r_1r_3-r_3r\;-r\,r_1}=\dfrac{r^2r_2^2r_1^2}{r_2r_1-r_1r\;-r\,r_2} \\[2ex] =\Delta^2 \\[1ex] =\text{the reciprocal of} \\[1ex] \left(\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3}\right)\left(-\dfrac{1}{h_1}+\dfrac{1}{h_2}+\dfrac{1}{h_3}\right)\left(-\dfrac{1}{h_2}+\dfrac{1}{h_3}+\dfrac{1}{h_1}\right)\left(-\dfrac{1}{h_3}+\dfrac{1}{h_1}+\dfrac{1}{h_2}\right) \end{array}\right\} \quad (29)$$

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$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{r_2r_3}&=\frac{1}{s_2s_3}+\frac{1}{ss_1}=\frac{4}{h_2h_3}=\frac{bc}{\Delta^2}\\
\frac{1}{rr_2}+\frac{1}{r_3r_1}&=\frac{1}{s_3s_1}+\frac{1}{ss_2}=\frac{4}{h_3h_1}=\frac{ca}{\Delta^2}\\
\frac{1}{rr_3}+\frac{1}{r_1r_2}&=\frac{1}{s_1s_2}+\frac{1}{ss_3}=\frac{4}{h_1h_2}=\frac{ab}{\Delta^2}
\end{aligned}\right\} \quad (30)$$

$$\left.\begin{aligned}
\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)&=\left(\frac{1}{s_1}-\frac{1}{s}\right)\left(\frac{1}{s_2}+\frac{1}{s_3}\right)=\frac{4}{h_1^2}=\frac{a^2}{\Delta^2}\\
\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)&=\left(\frac{1}{s_2}-\frac{1}{s}\right)\left(\frac{1}{s_3}+\frac{1}{s_1}\right)=\frac{4}{h_2^2}=\frac{b^2}{\Delta^2}\\
\left(\frac{1}{r}-\frac{1}{r_3}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)&=\left(\frac{1}{s_3}-\frac{1}{s}\right)\left(\frac{1}{s_1}+\frac{1}{s_2}\right)=\frac{4}{h_3^2}=\frac{c^2}{\Delta^2}
\end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{rr_2}+\frac{1}{rr_3}+\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}&=\frac{4}{h_2h_3}+\frac{4}{h_3h_1}+\frac{4}{h_1h_2}\\
&=\frac{bc+ca+ab}{\Delta^2}
\end{aligned}\right\} \quad (32)$$

$$\left.\begin{aligned}
\frac{1}{rr_1}+\frac{1}{rr_2}+\frac{1}{rr_3}-\frac{1}{r_2r_3}-\frac{1}{r_3r_1}-\frac{1}{r_1r_2}&=\frac{2}{h_1^2}+\frac{2}{h_2^2}+\frac{2}{h^2}\\
&=\frac{a^2+b^2+c^2}{2\Delta^2}
\end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned}
\frac{1}{r^2}&=\frac{1}{s_2s_3}+\frac{1}{s_3s_1}+\frac{1}{s_1s_2}\\
\frac{1}{r_1^2}&=\frac{1}{s_3s_2}-\frac{1}{s_2s}-\frac{1}{s\,s_3}
\end{aligned}\right\} \quad (34)$$

and so on.

$$\left.\begin{aligned}
\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}&=\frac{4}{h_1^2}+\frac{4}{h_2^2}+\frac{4}{h_3^2}\\
&=\frac{a^2+b^2+c^2}{\Delta^2}
\end{aligned}\right\} \quad (35)$$

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$$\left.\begin{aligned}\frac{1}{r^2}+\frac{1}{r_1^{\,2}}-\frac{1}{r_2^{\,2}}-\frac{1}{r_3^{\,2}}&=\frac{8}{h_2h_3}\\ \frac{1}{r^2}-\frac{1}{r_1^{\,2}}+\frac{1}{r_2^{\,2}}-\frac{1}{r_3^{\,2}}&=\frac{8}{h_3h_1}\\ \frac{1}{r^2}-\frac{1}{r_1^{\,2}}-\frac{1}{r_2^{\,2}}+\frac{1}{r_3^{\,2}}&=\frac{8}{h_1h_2}\end{aligned}\right\}\qquad(36)$$

$$\left.\begin{aligned}\frac{a}{h_1}+\frac{b}{h_2}+\frac{c}{h_3}+\tfrac{1}{2}\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)&=\frac{s}{r}\\ \frac{a}{h_1}+\frac{b}{h_2}+\frac{c}{h_3}-\tfrac{1}{2}\left(\frac{a}{r}+\frac{b}{r_3}+\frac{c}{r_2}\right)&=\frac{s_1}{r_1}\end{aligned}\right\}\qquad(37)$$

and so on.

$$\left.\begin{aligned}\frac{h_1+h_2+h_3-(r_1+r_2+r_3)}{s}-3\left(\frac{r}{a}+\frac{r}{b}+\frac{r}{c}\right)+\frac{r_1}{a}+\frac{r_2}{b}+\frac{r_3}{c}&=0\\ \frac{h_1-h_2-h_3+(r\;-r_3-r_2)}{s_1}-3\left(\frac{r_1}{a}-\frac{r_1}{b}-\frac{r_1}{c}\right)+\frac{r}{a}+\frac{r_3}{b}+\frac{r_2}{c}&=0\end{aligned}\right\}\qquad(38)$$

and so on.

$$\left.\begin{aligned}\frac{h_2+h_3}{r_1}+\frac{h_3+h_1}{r_2}+\frac{h_1+h_2}{r_3}&=6\\ \frac{h_2+h_3}{r}-\frac{h_3-h_1}{r_3}+\frac{h_1-h_2}{r_2}&=6\end{aligned}\right\}\qquad(39)$$

and so on.

$$\left.\begin{aligned}\frac{h_1+h_2+h_3}{r}-\left(\frac{h_1}{r_1}+\frac{h_2}{r_2}+\frac{h_3}{r_3}\right)&=6\\ \frac{-h_1+h_2+h_3}{r_1}+\left(\frac{h_1}{r}+\frac{h_2}{r_3}+\frac{h_3}{r_2}\right)&=6\end{aligned}\right\}\qquad(40)$$

and so on

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$$\left.\begin{array}{ll} r\,a=s_1(r_1-r) & r_1a=s\,(r_1-r) \\ r\,b=s_2(r_2-r) & r_1b=s_3(r_3+r_1) \\ r\,c=s_3(r_3-r) & r_1c=s_2(r_2+r_1) \\ r_2a=s_3(r_3+r_2) & r_3a=s_2(r_2+r_3) \\ r_2b=s\,(r_2-r) & r_3b=s_1(r_1+r_3) \\ r_2c=s_1(r_1+r_2) & r_3c=s\,(r_3-r) \end{array}\right\} \quad (41)$$

$$\left.\begin{array}{ll} s\,a=r_1(r_2+r_3) & s_1a=r\,(r_3+r_2) \\ s\,b=r_2(r_3+r_1) & s_1b=r_3(r_2-r) \\ s\,c=r_3(r_1+r_2) & s_1c=r_2(r_3-r) \\ s_2a=r_3(r_1-r) & s_3a=r_2(r_1-r) \\ s_2b=r\,(r_1+r_3) & s_3b=r_1(r_2-r) \\ s_2c=r_1(r_3-r) & s_3c=r\,(r_2+r_1) \end{array}\right\} \quad (42)$$

Many formulae may be obtained from (41) and (42) by appropriate grouping.

$$\left.\begin{array}{c} a(br_3-cr_2)=r\,(r_3^2-r_2^2) \qquad b(cr_1-ar_3)=r\,(r_1^2-r_3^2) \\ c(ar_2-br_1)=r\,(r_2^2-r_1^2) \\ a(br_2-cr_3)=r_1(r_2^2-r_3^2) \qquad b(ar_2-cr)=r_1(r_2^2-r^2) \\ c(ar_3-br)=r_1(r_3^2-r^2) \\ a(br_1-cr)=r_2(r_1^2-r^2) \qquad b(cr_3-ar_1)=r_2(r_3^2-r_1^2) \\ c(br_3-ar)=r_2(r_3^2-r^2) \\ a(cr_1-br)=r_3(r_1^2-r^2) \qquad b(cr_2-ar)=r_3(r_2^2-r^2) \\ c(ar_1-br_2)=r_3(r_1^2-r_2^2) \end{array}\right\} \quad (43)$$

$$\left.\begin{array}{c} \dfrac{a}{r_1}+\dfrac{b}{r_2}+\dfrac{c}{r_3}=\dfrac{2(r_1+r_2+r_3)}{s} \\[2ex] \dfrac{a}{r}+\dfrac{b}{r_3}+\dfrac{c}{r_2}=\dfrac{2(-r+r_3+r_2)}{s_1} \end{array}\right\} \quad (44)$$

and so on.

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$$\left.\begin{aligned}\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)\frac{a+b+c}{r_1+r_2+r_3}&=4\\ \left(\frac{a}{r}+\frac{b}{r_3}+\frac{c}{r_2}\right)\frac{-a+b+c}{-r+r_3+r_2}&=4\end{aligned}\right\}\quad(45)$$

and so on.

$$\left.\begin{aligned}r_1-r&=\frac{arr_1}{\Delta}=\frac{a\Delta}{r_2r_3} & r_2+r_3&=\frac{ar_2r_3}{\Delta}=\frac{a\Delta}{rr_1}\\ r_2-r&=\frac{brr_2}{\Delta}=\frac{b\Delta}{r_3r_1} & r_3+r_1&=\frac{br_3r_1}{\Delta}=\frac{b\Delta}{rr_2}\\ r_3-r&=\frac{crr_3}{\Delta}=\frac{c\Delta}{r_1r_2} & r_1+r_2&=\frac{cr_1r_2}{\Delta}=\frac{c\Delta}{rr_3}\end{aligned}\right\}\quad(46)$$

$$\frac{s^2-r_2r_3}{s}=a\qquad\frac{s^2-r_3r_1}{s}=b\qquad\frac{s^2-r_1r_2}{s}=c\qquad(47)$$

$$\left.\begin{aligned}r_1(r_2+r)&=s_3(c+a) & r_1(r_3+r)&=s_2(a+b)\\ r_2(r_3+r)&=s_1(a+b) & r_2(r_1+r)&=s_3(b+c)\\ r_3(r_1+r)&=s_2(b+c) & r_3(r_2+r)&=s_1(c+a)\end{aligned}\right\}\quad(48)$$

See II.

$$\left.\begin{aligned}r(r_2-r_3)&=s_1(b-c) & r_1(r_2-r_3)&=s(b-c)\\ r(r_3-r_1)&=s_2(c-a) & r_2(r_3-r_1)&=s(c-a)\\ r(r_1-r_2)&=s_3(a-b) & r_3(r_1-r_2)&=s(a-b)\end{aligned}\right\}\quad(49)$$

See II.

Many formulae may be obtained from (48) and (49) by appropriate grouping.

$$\left.\begin{aligned}ar_3-br&=\frac{rr_3}{r_2}(b+a)\\ ar_2-br_1&=\frac{r_1r_2}{r_3}(b-a)\end{aligned}\right\}\quad(50)$$

and so on.

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$$\left.\begin{aligned} r_1+r &: r_1-r = b+c : a \\ r_2+r &: r_2-r = c+a : b \\ r_3+r &: r_3-r = a+b : c \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} r_2-r_3 &: r_2+r_3 = b-c : a \\ r_3-r_1 &: r_3+r_1 = c-a : b \\ r_1-r_2 &: r_1+r_2 = a-b : c \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} (r_1+r)(r_2-r_3) &= (b+c)(b-c) \\ (r_2+r)(r_3-r_1) &= (c+a)(c-a) \\ (r_3+r)(r_1-r_2) &= (a+b)(a-b) \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} & s^2 : r_1^2 = r_2+r_3 : r_1-r = s_1^2 : r^2 \\ & s^2 : r_2^2 = r_3+r_1 : r_2-r = s_2^2 : r^2 \\ & s^2 : r_3^2 = r_1+r_2 : r_3-r = s_3^2 : r^2 \\ & s_1^2 : r_2^2 = r_3-r : r_1+r_2 \qquad s_1^2 : r_3^2 = r_2-r : r_3+r_1 \\ & s_2^2 : r_3^2 = r_1-r : r_2+r_3 \qquad s_2^2 : r_1^2 = r_3-r : r_1+r_2 \\ & s_3^2 : r_1^2 = r_2-r : r_3+r_1 \qquad s_3^2 : r_2^2 = r_1-r : r_2+r_3 \end{aligned}\right\} \quad (54)$$

$$\left.\begin{aligned} (r_2+r_3)(r_1-r) &= a^2 \\ (r_3+r_1)(r_2-r) &= b^2 \\ (r_1+r_2)(r_3-r) &= c^2 \end{aligned}\right\} \quad (55)$$

See I.

$$\left.\begin{aligned} \frac{r_1^2(r_2+r_3)^2}{r_2r_3+r_3r_1+r_1r_2} &= a^2 \\ \frac{r_2^2(r_3+r_1)^2}{r_2r_3+r_3r_1+r_1r_2} &= b^2 \\ \frac{r_3^2(r_1+r_2)^2}{r_2r_3+r_3r_1+r_1r_2} &= c^2 \end{aligned}\right\} \quad (56)$$

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$$\left.\begin{aligned}
\frac{r_1(r_2+r_3)}{a} &= \frac{r_2(r_3+r_1)}{b} = \frac{r_3(r_1+r_2)}{c}\\
\frac{r\,(r_3+r_2)}{a} &= \frac{r_3(r_2-r\,)}{b} = \frac{r_2(r_3-r\,)}{c}\\
\frac{r_3(r_1-r\,)}{a} &= \frac{r\,(r_1+r_3)}{b} = \frac{r_1(r_3-r\,)}{c}\\
\frac{r_2(r_1-r\,)}{a} &= \frac{r_1(r_2-r\,)}{b} = \frac{r\,(r_2+r_1)}{c}
\end{aligned}\right\} \quad (57)$$

$$\left.\begin{aligned}
\frac{r_2r_3(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2} &= bc\\
\frac{r_3r_1(r_1+r_2)(r_2+r_3)}{r_2r_3+r_3r_1+r_1r_2} &= ca\\
\frac{r_1r_2(r_2+r_3)(r_3+r_1)}{r_2r_3+r_3r_1+r_1r_2} &= ab
\end{aligned}\right\} \quad (58)$$

$$\left.\begin{aligned}
(r_2+r_3)(r_3+r_1)(r_1+r_2) : abc &= s : r\\
(r_1-r\,)(r_2-r\,)(r_3-r\,) : abc &= r : s
\end{aligned}\right\} \quad (59)$$

$$\left.\begin{aligned}
ss_1 : s_2s_3 &= a^2 : (r_1-r)^2 = (r_2+r_3)^2 : a^2\\
ss_2 : s_3s_1 &= b^2 : (r_2-r)^2 = (r_3+r_1)^2 : b^2\\
ss_3 : s_1s_2 &= c^2 : (r_3-r)^2 = (r_1+r_2)^2 : c^2
\end{aligned}\right\} \quad (60)$$

$$\left.\begin{aligned}
2(r_2r_3-rr_1) &= -a^2+b^2+c^2\\
2(r_3r_1-rr_2) &= a^2-b^2+c^2\\
2(r_1r_2-rr_3) &= a^2+b^2-c^2
\end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned}
\frac{r_2r_3-rr_1}{r_2r_3+rr_1} &= \frac{-a^2+b^2+c^2}{2bc}\\
\frac{r_3r_1-rr_2}{r_3r_1+rr_2} &= \frac{a^2-b^2+c^2}{2ca}\\
\frac{r_1r_2-rr_3}{r_1r_2+rr_3} &= \frac{a^2+b^2-c^2}{2ab}
\end{aligned}\right\} \quad (62)$$

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$$\left.\begin{aligned}
&\frac{r_1 - r}{rr_1}\Delta = (r_1 - r)\sqrt{\frac{r_2 r_3}{rr_1}} \\
&= \frac{r_2 + r_3}{r_2 r_3}\Delta = (r_2 + r_3)\sqrt{\frac{rr_1}{r_2 r_3}} = a \\
&\frac{r_2 - r}{rr_2}\Delta = (r_2 - r)\sqrt{\frac{r_3 r_1}{rr_2}} \\
&= \frac{r_3 + r_1}{r_3 r_1}\Delta = (r_3 + r_1)\sqrt{\frac{rr_2}{r_3 r_1}} = b \\
&\frac{r_3 - r}{rr_3}\Delta = (r_3 - r)\sqrt{\frac{r_1 r_2}{rr_3}} \\
&= \frac{r_1 + r_2}{r_1 r_2}\Delta = (r_1 + r_2)\sqrt{\frac{rr_3}{r_1 r_2}} = c
\end{aligned}\right\} \quad (63)$$

$$\left.\begin{aligned}
2r\, r_1 &= -as_1 + bs_2 + cs_3 \\
2r_2 r_3 &= -as_3 + bs + cs_1
\end{aligned}\right\} \quad (64)$$

and so on.

See XI.

$$\left.\begin{aligned}
\frac{r_1 + r}{rr_1}\Delta &= (r_1 + r)\sqrt{\frac{r_2 r_3}{rr_1}} = b + c \\
\frac{r_2 + r}{rr_2}\Delta &= (r_2 + r)\sqrt{\frac{r_3 r_1}{rr_2}} = c + a \\
\frac{r_3 + r}{rr_3}\Delta &= (r_3 + r)\sqrt{\frac{r_1 r_2}{rr_3}} = a + b
\end{aligned}\right\} \quad (65)$$

$$\left.\begin{aligned}
\frac{s}{r}(r_1 + r)(r_2 + r)(r_3 + r) &= (b + c)(c + a)(a + b) \\
\frac{s_1}{r_1}(r_1 + r)(r_3 - r_1)(r_2 - r_1) &= (b + c)(a - c)(a - b)
\end{aligned}\right\} \quad (66)$$

and so on.

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$$\left.\begin{aligned}\frac{r_2-r_3}{r_2r_3}\Delta &= (r_2-r_3)\sqrt{\frac{rr_1}{r_2r_3}} = b-c\\ \frac{r_3-r_1}{r_3r_1}\Delta &= (r_3-r_1)\sqrt{\frac{rr_2}{r_3r_1}} = c-a\\ \frac{r_1-r_2}{r_1r_2}\Delta &= (r_1-r_2)\sqrt{\frac{rr_3}{r_1r_2}} = a-b\end{aligned}\right\} \quad (67)$$

$$\left.\begin{aligned}\frac{r}{s}(r_2-r_3)(r_3-r_1)(r_1-r_2) &= (b-c)(c-a)(a-b)\\ \frac{r_1}{s_1}(r_2-r_3)(r_2+r)(r+r_3) &= (b-c)(c+a)(a+b)\end{aligned}\right\} \quad (68)$$

and so on.

$$\left.\begin{aligned}\Delta^2\left(\frac{1}{r^2}-\frac{1}{r_1^2}\right) &= (r_2+r_3)(r+r_1) = a(b+c)\\ \Delta^2\left(\frac{1}{r^2}-\frac{1}{r_2^2}\right) &= (r_3+r_1)(r+r_2) = b(c+a)\\ \Delta^2\left(\frac{1}{r^2}-\frac{1}{r_3^2}\right) &= (r_1+r_2)(r+r_3) = c(a+b)\end{aligned}\right\} \quad (69)$$

$$\left.\begin{aligned}\Delta^2\left(\frac{1}{r_3^2}-\frac{1}{r_2^2}\right) &= (r_2-r_3)(r_1-r) = a(b-c)\\ \Delta^2\left(\frac{1}{r_1^2}-\frac{1}{r_3^2}\right) &= (r_3-r_1)(r_2-r) = b(c-a)\\ \Delta^2\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right) &= (r_1-r_2)(r_3-r) = c(a-b)\end{aligned}\right\} \quad (70)$$

$$\left.\begin{aligned}(r_1+r)(r_2+r)(r_3+r) : (b+c)(c+a)(a+b) &= r : s\\ (r_2-r_3)(r_3-r_1)(r_1-r_2) : (b-c)(c-a)(a-b) &= s : r\end{aligned}\right\} \quad (71)$$

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$$\left.\begin{aligned}
&\Delta \\
&=\frac{r_2r_3(r_1+r)}{b+c}=\frac{rr_1(b+c)}{r_1+r}=\frac{rr_1(r_2-r_3)}{b-c}=\frac{r_2r_3(b-c)}{r_2-r_3} \\
&=\frac{r_3r_1(r_2+r)}{c+a}=\frac{rr_2(c+a)}{r_2+r}=\frac{rr_2(r_3-r_1)}{c-a}=\frac{r_3r_1(c-a)}{r_3-r_1} \\
&=\frac{r_1r_2(r_3+r)}{a+b}=\frac{rr_3(a+b)}{r_3+r}=\frac{rr_3(r_1-r_2)}{a-b}=\frac{r_1r_2(a-b)}{r_1-r_2}
\end{aligned}\right\} \quad (72)$$

$$\left.\begin{aligned}
&\frac{(r_1-r)(r_2-r)(r_3-r)\Delta}{r^2}=abc \\
&\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)r^2}{\Delta}=abc
\end{aligned}\right\} \quad (73)$$

$$(r_1+r)(r_2+r)(r_3+r)(r_2-r_3)(r_3-r_1)(r_1-r_2)=(b^2-c^2)(c^2-a^2)(a^2-b^2) \quad (74)$$

$$(r_1-r)(r_2-r)(r_3-r)(r_2+r_3)(r_3+r_1)(r_1+r_2)=a^2b^2c^2 \quad (75)$$

$$(r_1+r_2+r_3-r)\Delta=abc \quad (76)$$

$$\Delta^3\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=abc \quad (77)$$

$$\Delta^3\left(\frac{1}{rr_3r_2}+\frac{1}{rr_1r_3}+\frac{1}{rr_2r_1}-\frac{1}{r_1r_2r_3}\right)=abc \quad (78)$$

$$\frac{1}{r_3}-\frac{1}{r_1^3}-\frac{1}{r_2^3}-\frac{1}{r_3^3}=\frac{24}{h_1h_2h_3} \quad (79)$$

$$\left.\begin{aligned}
&\frac{r_2r_3}{r_1^3}+\frac{r_3r_1}{r_2^3}+\frac{r_1r_2}{r_3^3}=s^2r\left(\frac{1}{r_1^4}+\frac{1}{r_2^4}+\frac{1}{r_3^4}\right) \\
&\frac{r_2r_3}{r^3}+\frac{r_2r}{r_3^3}+\frac{rr_3}{r_2^3}=s_1^2r_1\left(\frac{1}{r^4}+\frac{1}{r_3^4}+\frac{1}{r_2^4}\right)
\end{aligned}\right\} \quad (80)$$

and so on.

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In the following notes an endeavour has been made to assign the various formulae to the authors who first published them more or less explicitly. But it would be presumptuous to suppose that this endeavour has met with more than a partial success. I shall be grateful to any one who will inform me of earlier sources than those I have been able to find.

(1) Both of the expressions $sr = \triangle$ and $ss_1s_2s_3 = \triangle^2$ are given by Heron of Alexandria in his treatise "On the Dioptra." See Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, pp. 235-7 (1864).

(2) Weddle in the *Lady's and Gentleman's Diary* for 1845, p. 82.

(3) It is stated in a note in Gergonne's *Annales*, I. 150 (1810-11), that Mahieu, professor of mathematics at the College of Alais, discovered the theorem $\triangle^2 = rr_1r_2r_3$ about 1807, and that Lhuilier gives it in his *Élémens d'Analyse*, p. 224 (1809).

(4) The first half of the first expression is given by Feuerbach, *Eigenschaften... des...Dreiecks*, §4 (1822). All the expressions which have r's in the numerator are given by L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(5) T. S. Davies in the *Philosophical Magazine*, II. 28 (1827). In the same place Davies gives also the first proportion of (6).

(7) The first expression of the first line is given by Euler in *Novi Commentarii Academiae...Petropolitanae*, for the years 1747-8, I. 54 (1750); the second by T. S. Davies in the *Ladies' Diary* for 1835, p. 56. The first expression of the second line is given, implicitly, by Feuerbach, §4 (1822); all the expressions in the second line are given in Gergonne's *Annales*, XIX. 214 (1829).

(9) The first three expressions are given by T. S. Davies in the *Philosophical Magazine*, II. 28 (1827); the second three by L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(10) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 86.

(11), (12) Feuerbach, §6 (1822).

(15) The first expression occurs in the *Ladies' Diary* for 1759; all four occur in Gergonne's *Annales*, XIX. 214 (1829).

(16) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 83 (1892).

(17) The first expression is given by C. J. Matthes in his *Commentatio de Proprietatibus Quinque Circulorum*, p. 9 (1831).

(18) The first of these expressions is given by Mr R. Knowles in *Mathematical Questions from the Educational Times*, XLI. 93 (1884).

(19), (21) Weddle in the *Lady's and Gentleman's Diary* for 1848, p. 76, and for 1845, p. 78.

(23) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892).

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(24) The relation between r and r_1, r_2, r_3 was given by Steiner and Bobillier in 1828. See Steiner's *Gesammelte Werke*, I. 214. The relation between r and h_1, h_2, h_3 was given by L.P.F.R. in Gergonne's ***Annales***, XIX. 212 (1829).

(25) Half of these expressions were given by Lowry and Rutherford in the *Ladies' Diary* for 1836, p. 54; the other half and also the whole of (26) and (27) by Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 90·1.

(28) The expression for r in terms of r_1, r_2, r_3 occurs in the *Ladies' Diary* for 1759. The expressions for r, r_1, r_2, r_3 in terms of the h's are given by Lowry and Rutherford in the *Ladies' Diary* for 1836, pp. 53, 55; the other expressions, by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 81.

(29) The expressions for Δ^2 in terms of the r's were given by Steiner in 1828. See his *Gesammelte Werke*, I. 214. The expression in terms of the h's was given by J. A. Grunert in *Supplemente zu Klügels Wörterbuche der reinen Mathematik*, I. 703 (1833).

(30)—(33) Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 91·2.

(34) First expression given by T. S. Davies in the *Philosophical Magazine*, II. 29 (1827).

(35) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 91.

(36) Mr David Trowbridge in Runkle's *Mathematical Monthly*, III. 188 (1861).

(37), (38), (40) The first expression in each of these is given by Thomas Dobson in the *Lady's and Gentleman's Diary* for 1862, pp. 95·6; the first expression in (39) is given by Dobson in *Mathematical Questions from the Educational Times*, III. 104 (1865).

(41) The values of r_3b, r_2c are given in the *Ladies' Diary* for 1759; those of ra, rb, rc, and of r_1a, r_2b, r_3c, in a slightly different form, were given by Steiner in 1828. See his *Gesammelte Werke*, I. 215.

(42) The values of sa, sb, sc are given in the *Ladies' Diary* for 1759. All the expressions in (41) and (42) were given by Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 84.

(43) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892).

(44), (45) Thomas Dobson in the *Lady's and Gentleman's Diary* for 1865, p. 53, and for 1864, p. 83.

(46) Mr Bernhard Möllmann in Grunert's *Archiv*, XVII. 380·1 (1851).

(47) L.P.F.R. in Gergonne's *Annales*, XIX. 214 (1829).

(48), (49) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 85.

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(50) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 81 (1892). The first proportion in (51) and the last in (52) are given by C. J. Matthes in his *Commentatio*, p. 52 (1831).

(53), (54), (55) Weddle in the *Lady's and Gentleman's Diary* for 1843, pp. 85, 87, 80.

(56) and the first line of (57) are given by L.P.F.R. in Gergonne's *Annales*, XIX. 214-5 (1829).

(58) T. S. Davies in the *Ladies' Diary* for 1836, p. 51.

(61) Mr C. Hellwig in Grunert's *Archiv*, XIX. 50 (1852).

(63) The expressions in which – occurs were given by Steiner in 1828. See his *Gesammelte Werke*, I. 215. C. J. Matthes in his *Commentatio*, p. 52 (1831), gives one of the others.

The first value of $b+c$ in (65) and the first of $a-b$ in (67) are given by C. J. Matthes in his *Commentatio*, p. 52 (1831).

(66), (68) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 82 (1892).

(69), (70) The expressions where $\triangle^2$ occurs are given by Mr Lemoine in *Mathesis*, 2nd series, II. 91 (1892); the others by Mr C. Hellwig in Grunert's *Archiv*, XIX. 50 (1852).

(71), (72) Weddle in the *Lady's and Gentleman's Diary* for 1843, p. 86.

(76)—(78) T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 90.

(80) Mr Émile Lemoine in *Mathesis*, 2nd series, II. 84 (1892).

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