Compositio Mathematica **117:** 77–98, 1999.

Linearization of Nambu Structures

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(Received: 11 December 1997; accepted in final form: 29 January 1998)

Abstract. Nambu structures are a generalization of Poisson structures in Hamiltonian dynamics, and it has been shown recently by several authors that, outside singular points, these structures are locally an exterior product of commuting vector fields. Nambu structures also give rise to co-Nambu differential forms, which are a natural generalization of integrable 1-forms to higher orders. This work is devoted to the study of Nambu tensors and co-Nambu forms near singular points. In particular, we give a classification of linear Nambu structures (integrable finite-dimensional Nambu-Lie algebras), and a linearization of Nambu tensors and co-Nambu forms, under the nondegeneracy condition.

Mathematics Subject Classification (1991): 58-XX.

Key words: generalized Poisson structures, singular foliations, integrable differential forms, normal forms

1. Introduction

Let V be an n-dimensional smooth manifold and $C = C^\infty(V)$ the space of smooth functions on V. A Nambu structure of order q on V is a multi-linear anti-symmetric application Π from the direct product of q samples of C to C, and denoted by the bracket { }:

$$\Pi: C \times C \times \cdots \times C \to C, \quad (f_1, f_2, \ldots, f_q) \mapsto \{f_1, f_2, \ldots, f_q\}$$

which satisfies the following two conditions

(i) Leibnitz condition:

$$\Pi_{f_1,\ldots,f_{q-1}}(fg) = f\Pi_{f_1,\ldots,f_{q-1}}(g) + g\Pi_{f_1,\ldots,f_{q-1}}(f) \tag{1}$$

(ii) Jacobi condition:

$$\Pi_{f_1,\ldots,f_{q-1}}(\{g_1, \ldots, g_q\}) = \sum_{i=1}^{q} \{g_1, \ldots, g_{i-1}, \Pi_{f_1,\ldots,f_{q-1}}(g_i), \ldots, g_q\} \tag{2}$$

for any $f_1, \ldots, f_{q-1}, g_1, \ldots, g_{q-1}, f, g \in C$, where $\Pi_{f_1,\ldots,f_{q-1}}$ denotes the contraction of Π by $f_1, \ldots, f_{q-1}$: $\Pi_{f_1,\ldots,f_{q-1}}(f) := \{f_1, \ldots, f_{q-1}, f\}$.

INTERPRINT(typeset) J.N.B. comp 4372 (compkap:mathfam) v.1.15
165881.tex; 4/05/1999; 11:41; p.1

The Leibnitz condition (together with the antisymmetricity of Π) means that Π is given by an (anti-symmetric) q-vector field on V, which we will also denote by Π. When $q = 1$ the Jacobi condition is empty and we simply have a vector field on V. When $q = 2$ the Jacobi condition is the usual condition for a 2-vector field to be a Poisson structure in Hamiltonian dynamics. Thus, Nambu structures, which are also called Nambu–Poisson structures, are a kind of generalization of Poisson structures when the order q is different from 2. They were introduced by Nambu [14] in an attempt to generalize Hamiltonian mechanics.

Given a Nambu structure of order q and a $(q-1)$-tuple of functions $(f_1, \ldots, f_{q-1})$ on V, one can associate to it a *Hamiltonian* vector field, which is the vector field corresponding to the derivation $\Pi_{f_1,\ldots,f_{q-1}}: C \to C$. The Jacobi condition means that this Hamiltonian vector field preserves the Nambu structure, like in usual Hamiltonian dynamics. From the definition it is evident that the contraction $\Pi_{f_1,\ldots,f_{q-r}}$ of a Nambu structure Π of order q with arbitrary $q-r$ smooth functions $f_1, \ldots, f_r (0 < r < q)$, $\Pi_{f_1,\ldots,f_{q-r}}(g_1, \ldots, g_r) := \Pi_{f_1,\ldots,f_{q-1}}(g_1, \ldots, g_r)$ is again a Nambu structure of order r. In particular, when $q \geqslant 3$ and $r = 2$, we get an infinite family of Poisson structures.

Nambu structures were studied by many people in recent years, and one can imagine various algebraic structures associated to them ([6, 17]). The most significant result obtained, which is in fact also quite simple to prove, is the following local normal form theorem, which was proved by Gautheron [6] and independently by Nakanishi [13] and Alekseevsky and Guha [1]. Hereafter by a *Nambu tensor* of order q we will mean an q-vector field associated to a Nambu structure.

THEOREM (Gautheron *et al*.). *Let Π be a Nambu tensor of order $q \geqslant 3$ on an n-dimensional manifold V, and $O \in V$ a point in which $\Pi(O) \neq 0$. Then in a small neighborhood of O one can find a local system of coordinates $(x_1, \ldots, x_n)$ such that $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_q$ in this neighborhood.*

The above theorem is a kind of Darboux theorem for Nambu structures. It also shows a big difference between Nambu structures of order $\geqslant 3$ and Poisson structures: the former ones are decomposable at nonzero points while the later ones are not in general.

The above theorem prompts us to study singularities of Nambu structures. The first obvious thing that we observe here is that each Nambu structure gives rise to an associated singular foliation (in the sense of Stefan–Sussmann), whose distribution is spanned by the Hamiltonian vector fields $\Pi_{f_1,\ldots,f_{q-1}}$. When $q \geqslant 3$ the leaves of this singular foliation is of dimension either 0 or q, while in case of Poisson structures ($q = 2$) they may have any even dimension (see, e.g., [18, 19] for the case of Poisson structures). These singular foliations give a geometric picture about the Nambu structures themselves.

By a *singularity of a Nambu structure* Π, or a *Nambu singularity* we mean a small neighborhood of a point O at which $\Pi(O) = 0$. When $\Pi(O) = 0$ at some point O, then its linearization at O is well-defined and gives us a linear Nambu structures. Thus the study of linear Nambu structures is a natural first step in the study of singularities of general Nambu structures. We have the following result (cf. Corollary 3.3).

THEOREM 1.1. *Every linear Nambu tensor Π of order $q = n - p \geqslant 3$ on an n-dimensional linear space V belongs to one of the following two types:*

Type 1: $\Pi = \sum_{j=1}^{r+1} \pm x_j \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{j-1} \wedge \partial/\partial x_{j+1} \wedge \cdots \wedge \partial/\partial x_{q+1} + \sum_{j=1}^{s} \pm x_{q+1+j} \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{r+j} \wedge \partial/\partial x_{r+j+2} \wedge \partial/\partial x_{q+1}$ *(with* $-1 \leqslant r \leqslant q, 0 \leqslant s \leqslant \min(p-1, q-r)$*).*

Type 2: $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge (\sum_{i,j=q}^{n} b_j^i x_i \partial/\partial x_j)$

We will call a Nambu singularity *of Type 1* if its linear part is of Type 1, and *of Type 2* in the other case. The singularties of Type 1 and Type 2 are very different geometrically, their corresponding foliations look very different, though they are in some natural sense *dual* to each other (cf. Section 3). We have the following result about the linearization of Nambu tensors near singular points (see Theorem 5.1, Theorem 5.2 and Theorem 6.2 for the precise formulations):

THEOREM 1.2. *Nondegenerate singularities of Type 1 of Nambu tensors of order $q \geqslant 3$ are formally linearizable. They are, up to multiplication by a function, C^∞-linearizable if they are analytic, and C^w-linearizable in the analytic (real or complex) case. Nondegenerate singularities of Type 2 of Nambu tensors of order $q \geqslant 3$ are C^∞-linearizable under some nonresonance condition, and analytically linearizable in the analytic case under some Diophantine condition.*

For nonelliptic singularities of type 1 of class C^∞, we have (see Section 5): In the case of signature $q-3$ they are not continuously linearizable in general. If the signature is different from $q-3$ then they are conjectured to be C^∞-linearizable. What we know is that in this case their associated singular foliations are homeomorphic to the ones given by the linear Nambu structures.

An important object which arises in the study of Nambu tensors are the so-called *co-Nambu forms*, which are obtained by the contraction of Nambu tensors with volume forms. For them we have some results analogous to the above theorem, which complement the ones obtained by Medeiros [10], and are similar to some results obtained before by Kupka [7], Reeb [20], Moussu [11, 12] and others for integrable 1-forms. Thus one can think of co-Nambu forms as integrable differential forms of higher orders. In fact, they are called *integrable p-forms* in [10]. In particular, we suspect that many results obtained by various authors for degenerate singularities of integrable 1-forms can be also generalized to the case of co-Nambu forms.

The rest of this paper is organized as follows: In Section 2 we give some preliminary results concerning Nambu structures, most notably about co-Nambu forms. In Section 3 we give a classification of linear Nambu structures, where we show that they can be divided in two types. In Section 4 we prove a theorem about the

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decomposition of Nambu structures near nondegenerate singularities. Section 5 and Section 6 contain our main results concerning the linearization problem.

2. Preliminaries

Let Ω be a volume form on an n-dimensional manifold V, and Π a q-vector field on V, with $n > q > 2$. Put $p = n - q$ and denote by ω the p-form obtained by contracting Π and Ω: $\omega = i_\Pi \Omega$. Then the condition for Π to be a Nambu tensor can be rewritten in terms of ω:

PROPOSITION 2.1. *With the above notations,* Π *is Nambu if and only if* ω *satisfies the following two conditions:*

$$i_A \omega \wedge \omega = 0, \tag{3}$$

$$i_A \omega \wedge \mathrm{d}\omega = 0, \tag{4}$$

for any $(p-1)$*- vector* A.

In case $p = 1$ the above conditions simply mean that $\mathrm{d}\omega \wedge \omega = 0$, i.e. ω is an integrable 1-form.

The proof of the above proposition is based on the following two lemmas, which follow directly from the Leibnitz and Jacobi conditions (1), (2) and the normal form theorem of Gautheron *et al*.

LEMMA 2.2. Π *is a Nambu tensor if and only if it is so on the open set* $U = \{x \in V, \Pi(x) \neq 0\}$ *of points where it does not vanish.*

LEMMA 2.3. *Suppose that* $q \geqslant 3$*. Then a* q*-vector field* Π *is Nambu if and only if in a neighborhood of each point* O *where* $\Pi(O) \neq 0$*, we can find a local system of coordinates* $(x_1, \ldots, x_n)$ *in which* Π *can be written as* $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_q$.

Proof of Proposition 2.1. Let Π be a Nambu q-tensor with $q \geqslant 3$. In a neighborhood of a point O such that $\Pi(O) \neq 0$ we have $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_q$ in some system of coordinates, according to the theorem of Gautheron *et al*. Since $\Omega = f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ (with some nonzero function f), we have

$$\omega = \pm f\, \mathrm{d}x_{q+1} \wedge \cdots \wedge \mathrm{d}x_n \quad \text{and} \quad \mathrm{d}\omega = \pm\, \mathrm{d}f \wedge \mathrm{d}x_{q+1} \wedge \cdots \wedge \mathrm{d}x_n.$$

From here it is easy to verify that the Equation (3) and Equation (4) are satisfied for any $(p-1)$-vector A, where $p = n - q$. (At least they are satisfied at any nonzero point of Π, but then at any point, since zero points of Π are also zero points of ω.)

Conversely, let Π be a q-vector such that $\omega = i_\Pi \Omega$ satisfies the Equations (3) and (4). Fix a point $O \in V$ such that $\Pi(O) \neq 0$ (hence $\omega(O) \neq 0$). Then Equation (3) implies that ω is decomposable in a neighborhood of O: $\omega = \alpha_1 \wedge \cdots \wedge \alpha_p$,

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where α_i are independent 1-forms. One can find $(p-1)$-vectors $A_1, \ldots, A_p$ such that $i_{A_j}\omega = \alpha_j$, $j = \overline{1, p}$ in some neighborhood of O. Hereafter $\overline{1, p}$ means $1, 2, \ldots, p$. Then Equation (4) gives $\alpha_j \wedge \mathrm{d}\omega = 0$ for $j = \overline{1, p}$ in this neighborhood. But

$$\mathrm{d}\omega = \sum_{j=1}^{p} \alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge \mathrm{d}\alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_p.$$

Thus we have $\mathrm{d}\alpha_j \wedge \alpha_1 \wedge \cdots \wedge \alpha_p = 0$ for $j = \overline{1, p}$. In other words, α_j satisfy the conditions of Frobenius theorem (see e.g. [2]), which says that in this case there exists a local system of coordinates $(x_1, \ldots, x_n)$ such that $\alpha_j \wedge \mathrm{d}x_{q+1} \wedge \cdots \wedge \mathrm{d}x_n = 0$ for $j = \overline{1, p}$. It follows that $\omega = f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p$ for some nonzero function f, and $\Pi = g\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_q$ for some nonzero function g. Replacing x_1 by

$$x_1' = \int_{t=0}^{x_1} \frac{\mathrm{d}t}{g(t, x_2, \ldots, x_n)},$$

we have $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_q$. Applying Lemma 2.3, we obtain that Π is a Nambu structure $\square$

A simple corollary of Propositon 2.1 is that if Π is a Nambu tensor of order $q \geqslant 3$ and if f is a smooth function, then $f\Pi$ is again a Nambu structure.

DEFINITION 2.4. A differential p-form ω which satisfies the equations (3) and (4) in Proposition 2.1 will be called a *co-Nambu form* (of order p and co-order q).

We have a bijection $\Pi \leftrightarrow \omega$ between Nambu tensors and co-Nambu forms (if V is orientable). Of course, this bijection depends on the choice of a volume form on V, so it is not unique, but unique up to multiplication by a nonzero function. Thus the study of singularities of Π and that of ω are almost the same.

As a principle, when a structure vanishes at some point, then its linearization is well-defined, and if its linearization also vanishes, then its quadratization is well-defined, etc. It is also true for Nambu and co-Nambu structures. Let $O \in V$ be a point such that $\Pi(O) = 0$, and $(x_1, \ldots, x_n)$ a local system of coodinates in a neighborhood of O. Then we have a Taylor expansion of Π at O:

$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \Pi^{(3)} + \cdots$$

where

$$\Pi^{(i)} = \sum_{j_1 \leqslant \cdots \leqslant j_q} P^{(i)}_{j_1 \ldots j_q} \partial/\partial x_{j_1} \wedge \cdots \wedge \partial/\partial x_{j_q}$$

with $P^{(i)}_{j_1 \ldots j_q}$ being polynomials of order i in $x_1, \ldots, x_n$. It is easy to see from the definition that $\Pi^{(1)}$ is well-defined, and is also a Nambu structure. It is called the *linear part of* Π.

Similarly (by putting $\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$) we have $\omega = \omega^{(1)} + \omega^{(2)} + \omega^{(3)} + \cdots$, with $\omega^{(k)} = i_{\Pi^{(k)}}\Omega = \sum_{j_1 \leqslant \cdots \leqslant j_q} \pm P^{(k)}_{j_1 \ldots j_q}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}\hat{x}_{j_1} \wedge \cdots \wedge \mathrm{d}\hat{x}_{j_q} \wedge \cdots \wedge \mathrm{d}x_n$. In particular, the linear part $\omega^{(1)}$ of ω is well-defined by ω and is also a co-Nambu form. Note that $\omega^{(1)}$ is uniquely determined by $\Pi^{(1)}$, up to multiplication by a constant.

For co-Nambu 1-forms, Proposition 2.1 shows that they are nothing but integrable 1-forms. (This has been known to be true also for Poisson sturctures on 3-manifolds, cf. [5]). The singularities of integrable 1-forms have been extensively studied (see e.g. [7, 8, 10, 11, 20]). In particular, there is the following so-called Kupka's phenomenon (see [7, 10]): If O is a zero point of an integrable 1-form ω and $\mathrm{d}\omega(O) \neq 0$, then locally ω is a pull-back of an 1-form on a plane. In [10] a similar result is also proved for co-Nambu forms of higher orders.

3. Linear Nambu Structures

THEOREM 3.1. *If ω is a linear co-Nambu p-form of co-order $q = n - p \geqslant 3$ on a linear space V then there exist linear coordinates $(x_1, \ldots, x_n)$ such that ω belongs to one of the following two types:*

Type 1: $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha$ *where α is an exact 1-form of the type* $\alpha = \mathrm{d}[\sum_{j=p}^{p+r} \pm x_j^2/2 + \sum_{i=1}^{s} x_i x_{p+r+i}]$, *with* $-1 \leqslant r \leqslant q = n - p$, $0 \leqslant s \leqslant q - r$.

Type 2: $\omega = \sum_{i=1}^{p+1} a_i\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_{p+1}$ *with* $a_i = \sum_{j=1}^{p+1} a_i^j x_j$, *where a_j^i are constant. The matrix (a_i^j) can be chosen to be in Jordan form.*

Proof. Put $\omega = \sum_{j=1}^{n} x_j \omega_j$ where ω_j are constant p-forms. Then $\omega = \omega_j$ at points $(x_1 = 0, \ldots, x_j = 1, \ldots, x_n = 0)$. At any point ω is either decomposable (i.e. a wedge product of covectors) or zero, so does ω_j since it is constant. Denote by E_j the span of ω_j, i.e.

$$\begin{aligned} E_j &= \mathrm{Span}(\omega_j) \stackrel{\mathrm{def}}{=} \mathrm{Span}\{i_A \omega_j, \quad A \text{ is a } (p-1)-\text{vector}\} \\ &= \mathrm{Annulator}\{x \in V,\ i_x \omega_j = 0\} \subset V^* \end{aligned}$$

Then $\dim E_j = p$ if $\omega_j \neq 0$, because of decomposability. We have:

LEMMA 3.2. *If $\omega_i \neq 0$ and $\omega_j \neq 0$ for some indices i and j, then* $\dim(E_i \cap E_j) \geqslant p - 1$.

Proof. Putting $x_k = 0$ for every $k \neq i, j$, we obtain that $x_i \omega_i + x_j \omega_j = \omega$ is decomposable or null for any x_i, x_j. In particular, $\omega_i + \omega_j$ is decomposable. If $\dim(E_i \cap E_j) = d < p$ then there is a basis $(e_1, \ldots, e_d, f_1, \ldots, f_{p-d}, g_1, \ldots, g_{p-d})$ of $E_i + E_j$ such that $\omega_i = e_1 \wedge \cdots \wedge e_d \wedge f_1 \wedge \cdots \wedge f_{p-d}$, $\omega_j = e_1 \wedge \cdots \wedge e_d \wedge g_1 \wedge \cdots \wedge g_{p-d}$ and

$$\omega_i + w_j = e_1 \wedge \cdots \wedge e_d \wedge [f_1 \wedge \cdots \wedge f_{p-d} + \wedge g_1 \wedge \cdots \wedge g_{p-d}].$$

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It follows easily that if $p - d \geqslant 2$ then $\mathrm{Span}(\omega_i + \omega_j) = E_i + E_j$, $\dim \mathrm{Span}(\omega_i + \omega_j) > p$ and $\omega_i + \omega_j$ is not decomposable. □

Return now to Theorem 3.1. We can assume that $E_1, \ldots, E_h \neq 0$ and $E_{h+1}, \ldots, E_n = 0$ for some number h. Put $E = E_1 \cap E_2 \cap \ldots \cap E_h$. Then there are two alternative cases: $\dim E \geqslant p - 1$ and $\dim E < p - 1$.

Case 1. $\dim E \geqslant p-1$. Then denoting by $(x_1, \ldots, x_{p-1})$ a set of $p-1$ linearly independent covectors contained in E, and which are considered as linear functions on V, we have $\omega_i = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha_i$, $i = \overline{1, h}$ for some constant 1-forms α_i, and hence

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha, \tag{5}$$

where $\alpha = \Sigma x_i \alpha_i$ is a linear 1-form.

Case 2. In this case, without loss of generality, we can assume that $\dim(E_1 \cap E_2 \cap E_3) < p-1$. Then Lemma 3.2 implies that $\dim(E_1 \cap E_2 \cap E_3) = p-2$. For an arbitrary index i, $3 < i \leqslant h$, put $F_1 = E_1 \cap E_i$, $F_2 = E_2 \cap E_i$, $F_3 \cap E_i$. Recall that $\dim F_1$, $\dim F_2$, $\dim F_3 \geqslant p-1$ according to Lemma 3.2, but $\dim(F_1 \cap F_2 \cap F_3) = \dim(E_1 \cap E_2 \cap E_3 \cap E_i) < p-1$, hence we cannot have $F_1 = F_2 = F_3$. Thus we can assume that $F_1 \neq F_2$. Then either F_1 and F_2 are two different hyperplanes in E_i, or one of them coincides with E_i. In any case we have $E_i = F_1 + F_2 \subset E_1 + E_2 + E_3$. It follows that $\sum_1^n E_i = \sum_1^h E_i = E_1 + E_2 + E_3$. On the other hand, we have $\dim(E_1 + E_2 + E_3) = \dim E_1 + \dim E_2 + \dim E_3 - \dim(E_1 \cap E_2) - \dim(E_1 \cap E_3) - \dim(E_2 \cap E_3) + \dim(E_1 + E_2 + E_3) = 3p - 3(p-1) + (p-2) = p+1$. Thus

$$\dim(E_1 + E_2 + \cdots + E_n) = p + 1.$$

It follows that there is a system of linear coordinates $(x_1, \ldots, x_n)$ on V such that $(x_1, \ldots, x_{p+1})$ span $E_1 + \cdots + E_n$ and therefore

$$\omega_i = \sum_{j=1}^{p+1} \gamma_i^j \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p+1}$$

Hence we have

$$\omega = \sum x_i \omega_i = \sum_{j=1}^{p+1} a_j \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p+1} \tag{6}$$

where a_j are linear functions on V.

To finish the proof of Theorem 3.1, we still need to normalize further the obtained forms (5) and (6).

Return now to Case 1 and suppose that $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha$ where $\alpha = \Sigma\alpha_j\,\mathrm{d}x_j$ with α_j being linear functions. We can put $\alpha_j = 0$ for $j = \overline{1, p-1}$ since it will not affect ω. Then we have $\alpha = \Sigma_{j\geqslant p, i=\overline{1,n}}\alpha^i_j x_i\,\mathrm{d}x_j$. Equation (4) implies that $\alpha \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \mathrm{d}\alpha = 0$. If we consider $(x_1, \ldots, x_{p-1})$ as parameters and denote by d' the exterior derivation with respect to the variables $(x_p, \ldots, x_n)$, then the last equation means $\alpha \wedge d'\alpha = 0$. That is, α can be considered as an integrable 1-form in the space of variables $(x_p, \ldots, x_n)$, parametrized by $(x_1, \ldots, x_{p-1})$. We will distinguish two subcases: $d'\alpha = 0$ and $d'\alpha \neq 0$.

Subcase (a). Suppose that $d'\alpha = 0$. Then according to Poincaré Lemma we have $\alpha_j = \Sigma_{i=1}^{p-1}\alpha^i_j x_i + \partial/\partial x_j q^{(2)}$, where $q^{(2)}$ is a quadratic function in the variables $(x_p, \ldots, x_n)$. By a linear change of coordinates on $(x_p, \ldots, x_n)$, we have $q^{(2)} = \Sigma_{j=p}^{p+r} \pm x_j^2/2$, for some number $r \geqslant -1$, and accordingly

$$\alpha = \sum_{j=p}^{p+r}\left(\pm x_j + \sum_{i=1}^{p-1}\alpha^i_j x_i\right)\mathrm{d}x_j + \sum_{i=\overline{1,p-1},\, j=\overline{p+r+1,n}} \alpha^i_j x_i\,\mathrm{d}x_j.$$

By a linear change of coordinates $(x_1, \ldots, x_{p-1})$ on one hand, and $(x_{p+r+1}, \ldots, x_n)$ on the other hand, we can normalize the second part of the above expression to obtain

$$\alpha = \sum_{j=p}^{p+r}\left(\pm x_j + \sum_{i=1}^{p-1}\tilde{\alpha}^i_j x_i\right) + \sum_{j=1}^{s} x_j\,\mathrm{d}x_{p+r+j}$$

for some number $s (0 \leqslant s \leqslant \min(p-1, n-p-r))$.

Replacing $x_j (j = \overline{p, p+r})$ by new $x_j = x_j \mp \tilde{\alpha}^i_j x_i$ we have $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha$ where

$$\alpha = d\left[\sum_{j=p}^{p+r} \pm x_j^2/2 + \sum_{i=1}^{s} x_i x_{p+r+i}\right]$$

(with $-1 \leqslant r \leqslant q = n-p, 0 \leqslant s \leqslant q-r$). These are the linear co-Nambu forms of Type 1 in Theorem 3.1.

Subcase (b). Suppose that $d'\alpha \neq 0$. Then since $d'\alpha$ is a constant coefficients, we can change the coordinates $(x_p, \ldots, x_n)$ linearly so that $d'\alpha = \mathrm{d}x_p \wedge \mathrm{d}x_{p+1} + \cdots + \mathrm{d}x_{p+2r} \wedge \mathrm{d}x_{p+2r+1}$ in these new coordinates, for some $r \geqslant 0$.

If $r \geqslant 1$, then considering the coefficients of the term $\mathrm{d}x_p \wedge \mathrm{d}x_{p+1} \wedge \mathrm{d}x_i (i > p+1)$, $\mathrm{d}x_p \wedge \mathrm{d}x_{p+2} \wedge \mathrm{d}x_{p+3}$ and $\mathrm{d}x_{p+1} \wedge \mathrm{d}x_{p+2} \wedge \mathrm{d}x_{p+3}$ in $0 = \alpha \wedge d'\alpha$, we obtain that all the coefficients of α are zero, i.e. $\alpha = 0$, which is absurd. Thus $d'\alpha = \mathrm{d}x_p \wedge \mathrm{d}x_{p+1}$, and the condition $\alpha \wedge d'\alpha = 0$ implies that $\alpha = \alpha_1\,\mathrm{d}x_p + \alpha_2\,\mathrm{d}x_{p+1}$ with linear functions α_1 and α_2 depending only on $x_1, \ldots, x_{p-1}, x_p, x_{p+1}$. In this Subcase (b), $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha$ also has the form (6), as in Case 2.

Suppose now that ω has the form (6), as in Case 2 or Subcase (b) of Case 1:

$$\omega = \sum x_i \omega_i = \sum_{j=1}^{p+1} a_j \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j-1} \wedge x_{j+1} \wedge \ldots \wedge \mathrm{d}x_{p+1}.$$

There are also 2 subcases:

(a) $\partial a_j / \partial x_i = 0$ for $j = \overline{1, p+1}$, $i = \overline{p+2, n}$. In other words,

$$\omega = \sum_{i,j=1}^{p+1} a^i_j x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p+1}$$

with constant coefficients a^i_j.

To see that (a^i_j) can be put in Jordan form, notice that the linear Nambu tensor corresponding to ω is, up to multiplication by a constant:

$$\Pi = \left(\sum_{i,j=1}^{p+1} \pm a^i_j x_i \partial / \partial x_j \right) \wedge \partial / \partial x_{p+2} \wedge \cdots \wedge \partial / \partial x_n.$$

The first term in Π is a linear vector field, which is uniquely defined by a linear transformation $\mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ given by the matrix (a^i_j), so this matrix can be put in Jordan form.

(b) There is $j \leqslant p+1$ and $i \geqslant p+2$ such that $\partial a_j / \partial x_i \neq 0$. We can assume that $\partial a_1 / \partial x_n \neq 0$. Putting $A = \partial / \partial x_3 \wedge \cdots \wedge \partial / \partial x_{p+1}$ in $0 = i_A \omega \wedge \mathrm{d}\omega$ we obtain

$$\begin{aligned} 0 \;=\; & (a_1 \, \mathrm{d}x_2 + a_2 \, \mathrm{d}x_1) \wedge \sum_{i=\overline{1,n},\, j=\overline{1,p+1}} \mathrm{d}x_i \wedge \\ & \wedge \frac{\partial a_j}{\partial x_i} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p+1}. \end{aligned}$$

Considering the coefficient of $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p+1} \wedge \mathrm{d}x_n$ in the above equation, we have

$$a_1 \partial a_2 / \partial x_n - a_1 \partial a_1 / \partial x_n = 0.$$

Since $\partial a_1 / \partial x_n \neq 0$, it follows that a_2 is linearly dependent of a_1. Similarly, a_j is linearly dependent of a_1 for any $j = \overline{1, p+1}$. Thus $\omega = a_1 \omega_1$ where ω_1 is decomposable and constant: $\omega_1 = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p$ in some linear system of coordinates. If a_1 is linearly independently on $(x_1, \ldots, x_p)$ then we can also assume that $a_1 = x_{p+1}$. Thus also in this Subcase (b), ω is of Type 2 in Theorem 3.1. □

The form of ω gives us a clear picture about the singular foliations associated to linear Nambu structures: The foliation of a linear Nambu structure of Type 1 has

p first integrals, namely $x_1, \ldots, x_{p-1}$ and $\sum_{j=p}^{p+r} \pm x_j^2 + \sum_{j=1}^{s} x_j x_{p+r+j}$, and the leaves of the foliation are uniquely determined by these first integrals. The singular foliation of a linear Nambu structure of Type 2 is a Cartesian product of a foliation given by a linear vector field in a linear space with (an 1-leaf foliation on) another linear space.

Rewritting Theorem 3.1 in terms of Nambu tensors, we have:

COROLLARY 3.3. *Every linear Nambu tensor* Π *of order* $q = n - p \geqslant 3$ *on an* n*-dimensional linear space* V *belongs to one of the following two types:*

Type 1: $\Pi = \sum_{j=1}^{r+1} \pm x_j \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{j-1} \wedge \partial/\partial x_{j+1} \wedge \cdots \wedge \partial/\partial x_{q+1} + \sum_{j=1}^{s} \pm x_{q+1+j} \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{r+j} \wedge \partial/\partial x_{r+j+2} \wedge \partial/\partial x_{q+1}$ (with $-1 \leqslant r \leqslant q, 0 \leqslant s \leqslant \min(p-1, q-r)$).

Type 2: $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge (\sum_{i,j=q}^{n} b_j^i x_i \partial/\partial x_j)$

Remark. Linear Nambu tensors may be viewed as finite-dimensional Nambu–Lie algebras which satisfy some integrability conditions (cf. [6, 17]). The case of four-dimensional Nambu–Lie algebras of order 3 has been done in [6].

We notice here a very interesting duality between Type 1 and Type 2: The formula for Π of Type 1 looks similar to that for ω of Type 2, and vice versa. This duality will play an important role in the rest of this paper. We should notice also that if a differential form ω can be written in one of the two forms presented in Theorem 3.1, then it is obviously a linear co-Nambu form.

We have the following natural notion of nondegeneracy for linear Nambu structures:

DEFINITION 3.4. A linear co-Nambu p-form ω (and its corresponding linear Nambu q-tensor Π) of Type 1 is called *nondegenerate* if and only if it can be written in the form $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \mathrm{d}q^{(2)}$, where $q^{(2)} = \sum_p^n \pm x_j^2$ (is nondegenerate). In this case, ω and Π are called *elliptic* if $q^{(2)}$ is negative-definite or positive-definite. The absolute value of the signature of the quadratic function $q^{(2)}$ is called the *signature* of ω. The *index* of ω is the index of $q^{(2)}$, defined only up to the involution $m \mapsto q + 1 - m$.

A linear co-Nambu p-form ω (and its corresponding linear Nambu q-tensor Π) of Type 2 is called *nondegenerate* if and only if it can be written in the form $\omega = \sum_{i=1}^{p+1} \sum_{j=1}^{p+1} a_i^j x_j \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_{p+1}$, with (a_j^i) being nondegenerate, i.e. having nonzero determinant.

It is evident that a linear Nambu structure of Type 1 is nondegenerate if and only if all the other linear Nambu structures nearby it are equivalent to it in a natural sense, and there is only a finite number of equivalence classes in this case, which are classified by a signature of $q^{(2)}$. On the other hand, for nondegenerate linear Nambu structures of Type 2, there is a continuum of equivalence classes, which are classified by the Jordan form of (a_j^i), modulo multiplication by a nonzero number.

4. Decomposition of Nondegenerate Nambu Singularities

We will say that a singularity of a Nambu structure is *of Type 1* (*of Type 2, nondegenerate, elliptic, hyperbolic*) if its linear part is so. In this Section we will show that Nambu structures are decomposable also at nondegenerate singularities.

THEOREM 4.1. (a) *Let $O \in V$ be a nondegenerate singular point of Type* 1 *of a co-Nambu p-form (of co-order $q \geqslant 3$) ω. Then in a small neighborhood of O in V, ω is decomposable: it can be written as $\omega = \gamma_1 \wedge \cdots \wedge \gamma_{p-1} \wedge \alpha$, where γ_i are* 1*-forms which do not vanish at O, and α is an* 1*-form which vanishes at O.*

(b) *Let $O \in V$ be a nondegenerate singular point of Type* 2 *of a Nambu q-tensor (of order $q \geqslant 3$) Π. If $q = n - 1$ then we will also assume that in the normal form of its linear part $\Pi^{(1)} = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge (\sum_{i,j=q}^{n} b^i_j x_i \partial/\partial x_j)$ as given in Corollary 3.3, the (2×2) matrix (b^i_j) has a nonzero trace. Then in a small neighborhood of O in V, Π is decomposable: it can be written as $\Pi = V_1 \wedge \cdots \wedge V_{q-1} \wedge X$, where V_i are vector fields which do not vanish at O, and X is a vector field which vanishes at O.*

Proof. First we will prove (a). The proof will not make use of the integrability of Nambu tensors (or similar property of co-Nambu forms), so in fact the above theorem can be stated in a stronger form.

According to the definition of nondegenerate singularities of Type 1, we can suppose that ω has a Taylor expansion $\omega = \omega^{(1)} + \omega^{(2)} + \cdots$, with $\omega^{(1)} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \mathrm{d}q^{(2)}$, where $q^{(2)} = \sum_{j=p}^{n} \pm x_j^2/2$. Express ω as a polynomial in $\mathrm{d}x_1, \ldots, \mathrm{d}x_{p-1}$: $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha + \sum_{j=1}^{p-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \beta_j + \sum_{1 \leqslant i < j \leqslant p-1} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \gamma_{ij} + \cdots$ Here $\alpha, \beta_i, \gamma_{ij}, \ldots$ are differential forms which, when written in coordinates $(x_1, \ldots, x_n)$, do not contain the terms $\mathrm{d}x_1, \ldots, \mathrm{d}x_{p-1}$. Applying the equation $i_A\omega \wedge \omega = 0$ to $A = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{p-1}$, we have $\alpha \wedge \omega = 0$. It follows that $\alpha \wedge \beta_j = 0$, $\alpha \wedge \gamma_{ij} = 0$, etc. We can consider α and β_j as differential forms on the space of variables $\{x_p, \ldots, x_n\}$, parametrized by $x_1, \ldots, x_{p-1}$, and by our assumption of nondegeneracy, we can apply DeRham division theorem (cf. [4]), which says that, since the number of variables is $q + 1 > 2$ which is the order of β_j, β_j is divisible by α: $\beta_j = \alpha \wedge \theta_j$ where θ_j are smooth 1-forms.

Applying the equation $i_A\omega \wedge \omega = 0$ to $A = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{j-1} \wedge \partial/\partial x_{j+1} \wedge \cdots \wedge \partial/\partial x_{p-1} \wedge \partial/\partial x_p$, we get

$$0 = \omega \wedge [\langle \alpha, \partial/\partial x_p \rangle ((-1)^{p-j}\, \mathrm{d}x_j + \theta_j) - \langle \theta_j, \partial/\partial x_p \rangle \alpha].$$

Since $\langle \alpha, \partial/\partial x_p \rangle = \langle \alpha^{(1)}, \partial/\partial x_p \rangle + \cdots = \pm x_p + \cdots \neq 0$, and we already have $\omega \wedge \alpha = 0$, we get that $\omega \wedge \gamma_j = 0$ where $\gamma_j = \mathrm{d}x_j + (-1)^{p-j}\theta_j$. Since γ_j do not vanish and are linearly independent at O, it follows that ω is divisible by the product of γ_j: $\omega = \gamma_1 \wedge \cdots \wedge \gamma_{p-1} \wedge \alpha'$ for some 1-form α'. By adding a combination of γ_j to α', we assume that α' does not contain the terms $\mathrm{d}x_1, \ldots, \mathrm{d}x_{p-1}$ when written in the coordinates $(x_1, \ldots, x_n)$. Then considering the terms containing $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1}$

https://doi.org/10.1023/A:1001014910775 Published online by Cambridge University Press

on the two sides of the equation $\omega = \gamma_1 \wedge \cdots \wedge \gamma_{p-1} \wedge \alpha'$, it follows that in fact we have $\alpha' = \alpha$. Statement (a) is proved.

The proof of Statement (b) in case $q \geqslant n-2$ is the same as that of (a), by the duality *vector* $\leftrightarrow$ *covector*. We will now prove (b) for the case $q = n-1$. In this case we have $\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{n-2} \wedge (X_{n-1}\partial/\partial x_{n-1} + X_n \partial/\partial x_n) + (\sum_{i=1}^{n-2} B_i \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_{n-2}) \wedge \partial/\partial x_{n-1} \wedge \partial/\partial x_n$, where B_i contains only terms of degree $\geqslant 2$ in the Taylor expansion, and the linear part of the vector field $X = X_{n-1}\partial/\partial x_{n-1} + X_n \partial/\partial x_n$ has nonzero trace, that is $\partial X_{n-1}/\partial x_{n-1} + \partial X_n/\partial x_n \neq 0$. Notice that X is a Hamiltonian vector field of Π, given by the $(q-1)$-tuple of functions $(x_1, \ldots, x_{n-2})$ (here $n-2 = q-1$). Hence X preserves Π: $\mathcal{L}_X \Pi = 0$. Considering the coefficient of the term $\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_{n-2}$ in the equation $\mathcal{L}_X \Pi = 0$, we obtain a relation of the form

$$(X(B_i) - (\partial X_{n-1}/\partial x_{n-1} + \partial X_n/\partial x_n)B_i)\partial/\partial x_{n-1} \wedge \partial/\partial x_n = U \wedge X$$

for some $U = U_{n-1}\partial/\partial x_{n-1} + U_n \partial/\partial x_n$. Since $\partial X_{n-1}/\partial x_{n-1} + \partial X_n/\partial x_n \neq 0$, it follows that we have a relation of the form $B_i = V_{n-1}X_n - V_n X_{n-1}$ and, therefore, $B_i \partial/\partial x_{n-1} \wedge \partial/\partial x_n$ is divisible by X: $B_i \partial/\partial x_{n-1} \wedge \partial/\partial x_n = (V_{n-1}\partial/\partial x_{n-1} + V_n \partial/\partial x_n) \wedge X$. Thus in this case, by using the fact that X has nonzero trace, instead of its nondegeneracy, we also obtain the divisibility by X of the tems of degree $q-2$ in the expression of Π as a polynomial in $\partial/\partial x_1, \ldots, \partial/\partial x_{q-1}$. The rest of the proof is the same as for the case $q \leqslant n-2$. $\square$

The nondegeneracy implies that the 1-form α in the above theorem, considered as an 1-form on the space of the variables $(x_p, \ldots, x_n)$, will have exactly one (nondegenerate) zero point of each value of the parameters $(x_1, \ldots, x_{p-1})$, and of course this zero point will depend smoothly on the parameters $(x_1, \ldots, x_{p-1})$. A similar statement is true for the vector field X in the second case. Thus we have:

COROLLARY 4.2. *If O is a nondegenerate singular point of Type 1 of a Nambu tensor Π of order $q \geqslant 3$ in an n-dimensional manifold, then the set of zero points of Π near O forms a $(n-q-1)$-dimensional submanifold. If O is a nondegenerate singular point of Type 2 of a Nambu tensor Π of order $q \geqslant 3$ in an n-dimensional manifold (when $q = n-1$ we need the same additional assumption as in the previous theorem), then the set of zero points of Π near O forms a $(q-1)$-dimensional submanifold.*

5. Nondegenerate Singularities of Type 1

We have the following result about the linearization of co-Nambu forms of Type 1:

THEOREM 5.1. *Let O be a nondegenerate singular point of Type 1 of a smooth co-Nambu p-form ω of co-order $q > 2$.*

(a) *If the singular point O is of elliptic type then ω is linearizable in a neighborhood of O, up to multiplication by a nonzero smooth function. In other words, there is a local smooth system of coordinates $(x_1, \ldots, x_n)$ in a neighborhood of O such that we have $\omega = f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(1)}$, where f is a smooth function which does not vanish at O, and $\alpha^{(1)} = dq^{(2)}$ is a nondegenerate linear closed 1-form in the variables $(x_p, \ldots, x_n)$ (which does not depend on $(x_1, \ldots, x_{p-1})$).*

(b) *If ω is analytic (real or complex), then it is linearizable analytically in a neighborhood of O, up to multiplication by an analytic function which does not vanish at O.*

(c) *If ω is only C^∞ but not analytic, and O is not of elliptic type, then ω is still formally linearizable at O, up to multiplication by a formal function which does not vanish at O.*

Proof. Statement (a) and Statement (b) of the above theorem are absolutely similar to that of Reeb [20], as improved by Moussu [11], for the case of integrable 1-forms, and the proof is essentially the same except for some additional regular first intregrals. So we will only give a sketch of the proof here. The details of the steps can be found in [11, 20]. In the elliptic case, we can blow up along the local $(p-1)$-dimensional submanifold of elliptic singular points of ω (cf. Corollary 4.2), and then take a double covering of the blown-up manifold. In this double covering we have a regular foliation induced by the foliation associated to the Nambu structure. All the leaves of this foliation are diffeomorphic to S^q due to Reeb's stability theorem, and the foliation itself is a regular fibration of fiber S^q. On the p-dimensional base space of this fibration we have a smooth involution, whose fixed point set is a local $(p-1)$-dimensional manifold (which corresponds to the manifold of zero points of ω). It follows that there is a system of coordinates $(f_1, \ldots, f_p)$ on the base manifold of the fibration such that $(f_1, \ldots, f_{p-1})$ are invariant under the involution, $f_p = 0$ on the submanifold of fixed points and f_p^2 is invariant under the involution. These coordinates give rise to the first integrals of the singular foliations of the Nambu structure: the first $(p-1)$ first integrals are regular and functionally independent, the last one is zero on the submanifold of zero of the co-Nambu form ω and is nondegenerate positive-definite in the transversal direction to this submanifold. Taking the first $(p-1)$ first integrals as coordinates and applying the Morse's lemma to the last first integral, we get the linearization of ω up to multiplication by a nonzero smooth function. In the real analytic case, one can complexify the picture, then realify it back (in a different way) so that the singularity becomes elliptic, and then proceed as above. The complex analytic case is similar, without the step of complexifying.

Let us now prove Statement (c) of the theorem. Since in this case the blowing-up argument does not work so easily, we adopt a different strategy. By induction we assume that we have found a new system of local coordinates $(x_1, \dots, x_n)$ such that the Taylor expansion of ω in these coordinates have 'good' $(r-1)$ first terms:

$$\begin{aligned}
\omega^{(1)} &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(1)} \\
\omega^{(2)} &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(2)} \\
&\dots\dots \\
\omega^{(r-1)} &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(r-1)},
\end{aligned}$$

where $\alpha^{(1)} = \sum_p^n \pm x_i \,\mathrm{d}x_i$. When $r = 2$, this assumption follows from the definition of nondegenerate singularities of Type 2. We will show that we can make so that $\omega^{(r)} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(r)}$.

Let us use the following notations:

$$\begin{aligned}
x &= (x_1, \dots, x_{p-1}), \\
y &= (x_p, \dots, x_n) = (y_1, \dots, y_{q+1}), \\
\mathrm{d}x &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1}, \\
\mathrm{d}\hat{x}_i &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_{p-1}, \\
\partial x &= \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{p-1}, \\
\partial\hat{x}_i &= \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \partial/\partial x_{p-1}.
\end{aligned} \tag{7}$$

Decompose $\omega^{(r)}$ into $\omega^{(r)} = \mathrm{d}x \wedge \alpha^{(r)} + \omega'$, where ω' consists of the terms which are not divisible by $\mathrm{d}x$. Put $A = \partial\hat{x}_k \wedge \partial/\partial y_1$ for some index $k < q$. We have that $i_A\omega^{(r)} = \sum_j \nu_j \,\mathrm{d}y_j \pm \alpha_1^{(r)} \,\mathrm{d}x_k$ for some functions ν_j. The terms of degree $r+1$ in the relation $i_A\omega \wedge \omega = 0$ give:

$$\pm y_1 \,\mathrm{d}x_k \wedge \omega' + \left(\sum_j \nu_j \,\mathrm{d}y_j\right) \wedge \mathrm{d}x \wedge \alpha^{(1)} = 0, \tag{8}$$

which implies that $\pm y_1 \,\mathrm{d}x_k \wedge \omega' = \mathrm{d}x \wedge \gamma_k$ for some γ_k, and $x_k \wedge \omega' = \mathrm{d}x \wedge \gamma_k'$ for some γ_k'. By varying k from 1 to $p-1$, we obtain that

$$\omega' = \sum_{k=1}^{p-1} \mathrm{d}\hat{x}_k \wedge \omega^k \tag{9}$$

with $\omega^k = \sum_{i<j} \omega_{ij}^k \,\mathrm{d}y_i \wedge \mathrm{d}y_j$.

Putting Equation (9) into the left-hand side of Equation (8) we get

$$\pm y_1 \omega^k \wedge \mathrm{d}x = \left(\sum_j \nu_j \,\mathrm{d}y_j\right) \wedge \alpha^{(1)} \wedge \mathrm{d}x,$$

which implies

$$\omega^k = \alpha^{(1)} \wedge \beta^k \tag{10}$$

with $\beta^k = \sum_l \beta_l^k \,\mathrm{d}y_l$ for some β_l^k.

The term of degree r in $i_A\omega \wedge \mathrm{d}\omega = 0$ gives

$$i_A\omega^{(1)} \wedge \mathrm{d}\omega^{(r)} + i_A\omega^{(2)} \wedge \mathrm{d}\omega^{(r-1)} + \cdots + i_A\omega^{(r)} \wedge \mathrm{d}\omega^{(1)} = 0,$$

which implies $\pm y_1 \,\mathrm{d}x \wedge \mathrm{d}\omega^k = 0$, and hence $\mathrm{d}x \wedge \mathrm{d}\omega^k = 0$.

Thus the derivation of ω^k with respect to the variables y is zero: $\mathrm{d}_y\omega^k = 0$. Putting the relation (10) in this equation we get

$$\alpha^{(1)} \wedge \mathrm{d}_y\beta^k = 0. \tag{11}$$

Now we will use the nondegeneracy of $\alpha^{(1)}$. The division theorem of DeRham (cf. [4]) says that in this case we can divide $\mathrm{d}_y\beta^k$ by $\alpha^{(1)}$:

$$\mathrm{d}_y\beta = \alpha^{(1)} \wedge \beta^{(r-2)}, \tag{12}$$

where $\beta^{(r-2)}$ is a homogeneous 1-form of degree $r-2$. Differentiating (12) with respect to the variables y, we get $\alpha^{(1)} \wedge \mathrm{d}_y\beta^{(r-2)} = 0$, which implies

$$\mathrm{d}_y\beta^{(r-2)} = \alpha^{(1)} \wedge \beta^{(r-4)}$$

for some $\beta^{(r-4)}$. Repeat the above process until we get a form $\beta^{(r-2h)}$ with d_y $\beta^{(r-2h)} = 0$. Then we go back: $\beta^{(r-2h)} = \mathrm{d}_y\phi^{(r-2h+1)}$ and the equation d_y $\beta^{(r-2h+2)} = \alpha^{(1)} \wedge \beta^{(r-2h)}$ gives $\beta^{(r-2h+2)} = -\phi^{(r-2h+1)}\alpha^{(1)} + \mathrm{d}_y\phi^{(r-2h+3)}$. Keep going back until we refined β^k in the form

$$\beta^k = -\phi_k^{(r-1)}\alpha^{(1)} + \mathrm{d}_y\phi_k^{(r+1)}.$$

It follows that (10) we can change β_k by an exact 1-form $\omega^k = \alpha^{(1)} \wedge \mathrm{d}_y\phi_k^{(r+1)}$.

Consider now the following new system of coordinates

$$\begin{aligned} x_1' &= x_1 \pm \phi_1^{(r+1)} \\ \cdots &\quad \cdots \\ x_{p-1}' &= x_{p-1} \pm \pi\phi_{p-1}^{(r+1)} \\ y' &= y. \end{aligned} \tag{13}$$

In these new coordinates, $\omega^{(1)}$ becomes $\mathrm{d}x_1' \wedge \cdots \wedge \mathrm{d}x_{p-1}' \wedge (\sum_{j=1}^{q+1} \pm y_j \,\mathrm{d}y_j) = \omega^{(1)} + \sum_{k=1}^{p-1} \pm \mathrm{d}\hat{x}_k \wedge \alpha^{(1)} \wedge \mathrm{d}_y\phi^{(r+1)k} + (\textit{terms of degree} > q)$.

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Thus, by choosing appropriate signs in the above change of variables, we can kill the term $\omega' = \sum \mathrm{d}\hat{x}_k \wedge \omega^k = \sum_{k=1}^{p-1} \pm \mathrm{d}\hat{x}_k \wedge \alpha^{(1)} \wedge \mathrm{d}_y \phi_k^{(r+1)}$ in the expression $\omega^{(r)} = \mathrm{d}x \wedge \alpha^{(r)} + \omega'$.

Repeating the above procedure for r going from 2 to infinity, we find a formal system of coordinates $(x_1, \ldots, x_n)$ in which $\omega = \sum_{r=1}^{\infty} \omega^{(r)}$ with $\omega^{(r)} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1} \wedge \alpha^{(r)}$ for every r. In particular, $\omega = \mathrm{d}x \wedge \alpha$, where $\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p-1}$ and $\alpha = \sum \alpha^{(k)}$. The relation $i_{\partial_x} \omega \wedge \mathrm{d}\omega = 0$ implies that $\alpha \wedge \mathrm{d}_y \alpha = 0$, that is α can be considered as an integrable 1-form in the variables y, with a nondegenerate closed linear part $\alpha^{(1)}$, and parametrized by O. It is well known that in this case α is formally linearizable up to multiplication by a formal function f (see, e.g., [11]). Theorem 5.1 is proved. $\square$

Remarks. In the above theorem, we have linearization only up to multiplication by a function, because ω is not a closed form in general. It is closed (outside singular points) only up to multiplication by a function. There is another simple proof of the analytic (and formal) case of the above theorem, which uses Theorem 4.1 and Malgrange's Frobenius theorem with singularities [8].

The above theorem implies that a nondegenerate Nambu tensor of Type 1 is (maybe formally) linearizable up to multiplication by a function. In fact, at least formally, we can linearize it without the need of multiplication by a function:

THEOREM 5.2. *Let O be a nondegenerate singular point of Type* 1 *of a smooth Nambu q-tensor Π, $q > 2$. The Π is formally linearizable at O: there is a formal system of coordinates $(x_1, \ldots, x_n)$ such that*

$$\Pi \overset{\text{formally}}{=} \sum_{i=1}^{q+1} \pm x_i \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_n.$$

Proof. According to Theorem 5.1, we can write $\Pi = f\Pi_1$ where $\Pi_1 = \sum_{i=1}^{q+1} \pm x_i \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{i-1} \wedge \partial/\partial x_{i+1} \wedge \cdots \wedge \partial/\partial x_n$

We want to change the coordinates $(x_1, \ldots, x_{q+1})$ (and leave $x_{q+2}, \ldots, x_n$ unchanged) so that to make $f = 1$. We will forget about the parameters $(x_{q+2}, \ldots, x_n)$ and will assume for simplicity that $n = q + 1$.

Write $f = \sum f^{(r)}$ where $f^{(r)}$ is homogeneous of order r in $(x_1, \ldots, x_{q+1})$. By a change of coordinates of the type $x_1' = gx_1, \ldots, x_{q+1}' = gx_{q+1}$, we can make $f^{(0)} = 1$. We assume now that we already have $f^{(1)} = \cdots = f^{(r-1)} = 0$ for some $r \geqslant 1$. We will show that there is a change of coordinates which changes x_i by terms of degree $\geqslant r$, and which kills $f^{(r)}$. It amounts to find a vector field X such that $\mathcal{L}_X \Pi_1 = f^{(r)} \Pi_1$, where $\mathcal{L}$ denotes the Lie derivative. Consider the volume form $\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{q+1}$. Then it is easy to see, by contracting Π_1 with Ω, that the equation $\mathcal{L}_X \Pi_1 = f^{(r)} \Pi_1$ is equivalent to the equation $\mathrm{d}X(Q) = (f^{(r)} + \mathrm{div}_\Omega X)\mathrm{d}Q$, where $Q = 1/2\Sigma \epsilon_i x_i^2$ with $\epsilon_i = \pm 1$. In turn, this equation is equivalent to the following system of equations:

$$\mathrm{div}_\Omega X + f^{(r)} = \mathrm{d}[2QF(Q)]/\mathrm{d}Q \qquad X(Q) = 2QF(Q),$$

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where F is an unknown function. Write $X = A + Y$, where $A = F(Q)\sum x_i\partial/\partial x_i$, and Y is a vector field such that $Y(Q) = 0$. Then the above system of equations is equivalent to a system of the type $Y(Q) = 0, \beta(Q) + \mathrm{div}_\Omega \mathrm{Y} = \mathrm{f}^{(\mathrm{r})}$, where β is an unknown function. The equation $Y(Q) = 0$ is equivalent to the fact that $Y = \sum_{i<j} f_{ij}Y_{ij}$ where $Y_{ij} = \epsilon_i x_j\partial/\partial x_i - \epsilon_j x_i\partial/\partial x_j$. For such an Y, we have $\mathrm{div}_\Omega \mathrm{Y} = \sum_{\mathrm{i<j}} \mathrm{Y_{ij}}(\mathrm{f_{ij}})$. Denote by J the set of homogeneous polynomials of degree r. The solvability of the above system of equation follows from the following facts, which can be verified easily by choosing appropriate f_{ij}:

(1) If a monomial $x^I = x_1^{I_1}\dots x_{q+1}^{I_{q+1}}$ has one of I_i to be an odd number, then it belongs to J.
(2) Q^s is equivalent to λx_1^{2s} modulo J for some nonzero number λ.
(3) Any monomial $x^I = x_1^{I_1}\dots x_{q+1}^{I_{q+1}}$, with all I_i even, is equivalent to $\lambda x_1^{\Sigma I_i}$ modulo J for some number λ.

Thus the above system of equations can always be solved. The theorem is proved. □

Suppose now that ω is of class C^∞, is nondegenerate of Type 1 at a zero point O, is not elliptic at O but has an index different from 2 and $q-1$ (i.e. signature different from $q-3$, cf. Definition 3.4). Then the regular local leaves of the fibration associated to the linear part $\omega^{(1)}$ of ω are simply-connected (they are diffeomorphic to a direct product of a disk with a sphere of dimension different from 1). It follows from Reeb's stability theorem that the local regular leaves of ω are diffeomorphic to that of $\omega^{(1)}$. One can show easily in this case that the singular foliation associated to ω is homeomorphic to the one associated to the linear part of ω (see [10] for the case $p = 1$). According to Moussu [11, 12], if ω is the order $p = 1$ (i.e. is an 1-form) and its index is different from 2 and $q-1$, or if its index is 2 but all of its leaves are closed except for a finite number of leaves which contain the origin in the limit, then it admits a smooth first integral, which means that ω is smoothly linearizable up to multiplication by a smooth function which does not vanish at O. We suspect that it is also true for the case $p > 1$. If ω is of index 2 at O and without the condition on the closedness of the leaves, then it may have no local first integral, (which implies in particular that it may not be linearizable up to multiplication by a function), as the following example shows:

Example. Consider the 2-form $\omega = [\mathrm{d}q + l(q)\alpha] \wedge [\mathrm{d}x_3 + h(q)\alpha]$ near 0 in $\mathbb{R}^{3+k}$, where

$$q = x_1^2 + x_2^2 - y_1^2 - \cdots - y_k^2, \qquad \alpha = \frac{x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1}{x_1^2 + x_2^2}$$

is a singular closed 1-form, $l(q)$ and $h(q)$ are two flat functions in q at 0 such that $l(q) = h(q) = 0$ when $q \leqslant 0$. The conditions on $l(q)$ and $h(q)$ assure that ω is a smooth two-form whose linear part at 0 is $\mathrm{d}q \wedge \mathrm{d}x_4$. Near a point P such that

$x_1(P)^2 + x_2(P)^2 \neq 0$, we can write $\alpha = \mathrm{d}f$ for some function f. Thus near this point ω can be considered as a pull-back of a two-form on a three-dimensional space via the map $(x_1, x_2, x_3, y_1, \ldots, y_k) \mapsto (q, f, x_3)$. Since any two-form on a three-dimensional space is a co-Nambu form and a pull-back of a co-Nambu form is also a co-Nambu form, it follows that ω is a co-Nambu form. When $h(q) = 0$ and $l(q) > 0$ for $q > 0$, the leaves of the singular foliation associated to ω will spiral toward the cones ($q = 0, x_3 =$ constant). In this case the foliation has only one local first integral (up to functional dependence), which is x_3. If $l(q) \geqslant 0$ and $h(q)$ is not identically 0 when $q > 0$, then the leaves of the singular foliation associated to ω also drift in x_3, and if we chose l, h well enough this phenomenon will prevent the foliation from having a nontrivial first integral. For example, we can make $l(q)$ and $h(q)$ vanish together at a series of points q_i which tend to 0. Near each point q_i we make $h(q)$ vary from positive to negative an infinite number of times and chose $l(q)$ and $h(q)$ so that the drift in terms of x_3 of a leaf passing via some point $x \in \mathbb{R}^4$ with $q_i < q(x) < q_{i-1}$ and spiraling inwards or outwards (i.e. the curve drawn by the value of x_3 of a point on this leaf while this point is moving inwards or outwards), is contained in a small interval $[-\epsilon_i, +\epsilon_i]$ ($\lim \epsilon_i = 0$) and spans this interval an infinite number of times. It follows that this leaf contains the leaves ($q = q_i, x_3 =$ constant $\in [-\epsilon_i, +\epsilon_i]$) or the leaves ($q = q_i, x_3 =$ constant $\in [-\epsilon_i, +\epsilon_i]$) in its limit. By invariance with respect to $\partial/\partial x_3$ and $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$ of our construction, any other leaf nearby this leaf will have the same property (with the interval $[-\epsilon_i, +\epsilon_i]$ replaced by an interval $[-\epsilon_i, +\delta, +\epsilon_i + \delta]$). It follows that for any local continuous function f which is invariant on the leaves of the foliation, there is an open set containing 0 in the boundary, in which f is constant.

6. Nondegenerate Singularities of Type 2

THEOREM 6.1. *Let O be a nondegenerate singular point of Type 2 of a Nambu tensor Π of order $q \geqslant 3$ on an n-manifold V, whose linear part has the form $\Pi^{(1)} = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge (\sum_{i,j=q}^n b^i_j x_i \partial/\partial x_j)$. If $q = n - 1$ then we will also assume that the matrix (b^i_j) has a nonzero trace. Then there is a local system of coordinates $(x_1, \ldots, x_n)$ in which Π can be written as*

$$\Pi = f\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge X,$$

where f is a function with $f(O) \neq 0$ and $X = \sum_{i=q}^n c_i(x_q, \ldots, x_n)\partial/\partial x_i$ is a vector field which does not depend on $(x_1, \ldots, x_{q-1})$.

Proof. Write $\Pi = V_1 \wedge \cdots \wedge V_{q-1} \wedge X$ as in Theorem 4.1, where in some local system of coordinates $(x_1, \ldots, x_n)$ we have $V_i = \partial/\partial x_i + \sum_{k=q,\ldots,n} v^k_i \partial/\partial x_k$ and X does not contain terms in $\partial/\partial x_i, i < q$ and has $\sum_{i,j=q}^n b^i_j x_i \partial/\partial x_j$ as its linear part.

Using Corollary 4.2, we can, and will, make so that $X = 0$ on the submanifold $(x_q = \cdots = x_n = 0)$.

The integrability of Π implies that for any pair of indices $i, j < q$, $[V_i, V_j] \wedge V_1 \wedge \cdots \wedge V_{q-1} \wedge X = 0$. Considering the terms containing $\partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1}$ in this equation, we get that $[V_i, V_j] \wedge X = 0$. Notice that $[V_i, V_j]$ may be considered as a vector field on the space of the variables $(x_q, \ldots, x_n)$, parametrized by the parameters $(x_1, \ldots, x_{q-1})$. The nondegeneracy of the linear part of X allows us to use DeRham division theorem (cf. [4]), which says that $[V_i, V_j]$ is divisible by X: $[V_i, V_j] = g_{ij}X$ where f_i is some smooth function. Using these properties, we will change Π, where g_{ij} is some smooth function. Similarly, we have that $[V_i, X] = f_i X$, V_i and X so that Π is only changed by multiplication by a nonzero function, the above relations still hold, but in addition V_i commutes with $X, V_2, \ldots, V_{q-1}$.

The equation $[V_1, X] = f_1 X$ implies that $[V_1, gX] = (V_1(g) + f_1 g)X$ for any function g. The equation $V_1(g) + f_1 g = 0$ can be solved locally because V_1 is nonzero at O. Replacing X by gX and Π by $g\Pi$, we still have $\Pi = V_1 \wedge \cdots \wedge V_{q-1} \wedge X$, but with $[V_1, X] = 0$. Assume now that we already have $[V_1, X] = 0$. For $i > 1$, $i < q$ we have $[V_1, V_i + \gamma_i X] = (g_{1i} + V_1(\gamma_i))X$. One can easily solve the equation $(g_{1i} + V_1(\gamma_i)) = 0$ to find a γ_i such that $[V_1, V_i + \gamma_i X] = 0$. Replacing V_i by $V_i + \gamma_i X$, we get that V_i commutes with $V_2, \ldots, V_{q-1}, X$.

Assume now that we already have that V_1 commutes with $V_2, \ldots, V_{q-1}, X$. In other words, everything is invariant with respect to V_1. Make the same process as above but with V_2, in a way which is invariant with respect to V_1, we get that $V_2, \ldots, V_{q-1}, X$ can be changed so that Π remains the same but V_2 becomes commuting with $V_3, \ldots, V_{q-1}, X$. Repeating the above process with $V_3, V_4, \ldots$. In the end we get a new family of vector fields V_i and X whose product is Π and which commute pairwise.

Since V_i commute pairwise and are linearly independent, there is a new local system of coordinates $(x_1, \ldots, x_n)$ such that in these coordinates we have $V_i = \partial/\partial x_i$ for $i = 1 \leqslant q - 1$. The fact that X commutes with V_i means that the cofficients of X is these coordinates will not depend on $(x_1, \ldots, x_{q-1})$. Of course, we can also assume that X does not contain the terms $\partial/\partial x_i$, $i = 1, \ldots, q-1$, since subtracting these terms form X will not change Π. Thus X can be considered as vector field on the space of the variables $(x_q, \ldots, x_n)$, which vanishes at the origin (and which does not depend on the parameters $(x_1, \ldots, x_{q-1})$). □

THEOREM 6.2. *Let O be a nondegenerate singular point of Type 2 of a Nambu tensor Π of order $q \geqslant 3$ on an n-manifold V, whose linear part has the form $\Pi^{(1)} = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge (\sum_{i,j=q}^{n} b^i_j x_i \partial/\partial x_j)$. If the matrix (b^i_j) is nonresonant, i.e. if its eigenvalues $(\lambda_1, \ldots, \lambda_{p+1})$ do not satisfy any relation of the form $\lambda_i = \sum_{j=1}^{p+1} m_j \lambda_j$ with m_j being nonnegative integers and $\sum m_i \geqslant 2$, then Π is smoothly* *linearizable, i.e. there is a local smooth system of coordinates* $(x_1, \ldots, x_n)$ *in a neighborhood of* O, *in which* Π *coincides with its linear part:*

$$\Pi = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_{q-1} \wedge \left(\sum_{i,j=q}^{n} b_j^i x_i \partial/\partial x_j \right).$$

The above linearization can be made analytic if Π *is analytic and the eigenvalues* $\lambda_1, \ldots, \lambda_{p+1}$ *of* (b_j^i) *satisfy the Bryuno's incommensurability condition: there exist positive constants* C, ϵ *such that for any* $(p+1)$*-tuple of nonnegative integers* $(m_1, \ldots, m_{p+1})$ *with* $\sum m_i \geqslant 2$ *and any index* $k \leqslant p+1$ *we have* $|(\sum \lambda_i m_i) - \lambda_k| > C \exp(-(\sum m_i)^{1-\epsilon})$.

Proof. Using Theorem 6.1, we can write $\Pi = V_1 \wedge \cdots \wedge V_{q-1} \wedge Y$, where $Y = fX = f \sum_{i=q}^{n} c_i(x_q, \ldots, x_n) \partial/\partial x_i$. (We will forget about the fact that $V_i = \partial/\partial x_i$). If the linear part of X satisfies the nonresonance condition then we can apply Sternberg's theorem [16] to linearize X smoothly, and if it satisfies the Bryuno's incommensurability condition then we can apply Bryuno's theorem (see e.g. [3, 9]) to linearize X analytically in the analytic case. Thus in both case we can assume that X is already linearized and normalized: $X = \sum_{i=p}^{n} \lambda_{i-p+1} x_i \partial/\partial x_i$. We want now to change V_i and Y so that they become commuting and the relation $\Pi = V_1 \wedge \cdots \wedge V_{q-1} \wedge Y$ still hold (without multiplying Π by the nonzero function).

Similarly to the proof of Theorem 6.1, we have $[V_1, Y] = [V_1, fX] = V_1(f) X = f_1 Y$ with $f_1 = V_1(f)/f$. The equation $V_1(g) + f_1 g = 0$ can be solved locally because $V_1(O) \neq 0$. This time we will solve it on the submanifold $(x_q = \cdots = x_n = 0)$ of zero points of Π. So let g be a nonzero function which does not depend on $(x_q, \ldots, x_n)$ and which satisfies $V_1(g) + f_1 g = 0$ on the submanifold $(x_q = \cdots = x_n = 0)$. Then $[V_1/g, gY] = h/gY$ where h is a function which vanishes on the submanifold $(x_q = \cdots = x_n = 0)$. Under the nonresonance condition, a theorem of Roussarie [15] says that the equation $Y(\gamma) = h$, or equivalently, $X(\gamma) = h/f$, has a smooth solution h. (Notice here an important fact that X does not depend on $(x_q, \ldots, x_n)$, which allows us to use Roussarie's theorem). In the analytic case, the equation $X(\gamma) = (\sum_{i=1}^{p+1} \lambda_i x_{q-1+i} \partial/\partial x_{q-1+i})(\gamma) = h/f = \sum_{s_q,\ldots,s_n} (h/f)_{s_q,\ldots,s_n} (x_1, \ldots, x_{q-1}) x_q^{s_q} \ldots x_n^{s_n}$ has the formal solution

$$\gamma = \sum_{s_q,\ldots,s_n} \frac{1}{\sum_{i=1}^{p+1} \gamma_i s_{q-1+i}} (h/f)_{s_q,\ldots,s_n} x_q^{s_q} \ldots x_n^{s_n},$$

which can be verified easily to converge near O, under the incommensurability condition of Bryuno.

With a smooth or analytic function γ such that $Y(\gamma) = h$, we have $[V_1/g + \gamma Y, gY] = 0$. Thus we can change V_1 by $V_1/g + \gamma Y$ and Y by gY to obtain $[V_1, Y] = 0$. Of course, this change does not affect Π. After that, we can change $V_2, \ldots, V_{q-1}$ so that they commute with V_1, in the same way as in the proof of Theorem 6.1.

Thus we can make V_1 commute with $V_2, \dots, V_{q-1}, Y$, without affecting Π. Just as in the proof of Theorem 6.1, by induction we can make $V_1, \dots, V_{q-1}, Y$ commute pairwise. Then we can put $V_i = \partial/\partial x_i$ in some new local system of coordinates, and can assume that Y does not contain the terms $\partial/\partial x_1, \dots, \partial/\partial x_{q-1}$. The we can linearize Y, using Bryuno's or Sternberg's theorem, to finish the linearization of Π. □

Remark. The Bryuno's incommensurability condition in the above theorem can indeed be replaced by a weaker so-called (Ω)-condition plus the nonresonance condition (see e.g. [3, 9] for the (Ω)-condition).

Talking about co-Nambu forms of Type 2, the above theorems show that such co-Nambu forms can be written locally as $\omega = f\omega_1$ where f is some nonzero function and ω_1 is a p-form which does not contain the terms $dx_1, \dots, dx_{q-1}$ and does not depend on the variables $x_1, \dots, x_{q-1}$, in some local system of coordinates $(x_1, \dots, x_n)$. In other words, ω_1 is a pull-back of a p-form on a $(p+1)$-dimensional space under a projection $\mathbb{R}^n \to \mathbb{R}^{p+1}$. Furthermore, ω_1 can be made linear if ω satisfies some nonresonance of incommensurability condition. If $d\omega(O) \neq 0$, then a result of Medeiros [10] (called fundamental lemma for integrable p-forms) says that ω itself is the pull-back of a p-form on a $(p+1)$-dimensional space under a projection $\mathbb{R}^n \to \mathbb{R}^{p+1}$. Let us give a proof of this fact, which is a slight simplification of the one given in [10].

First of all, notice that if ω is a co-Nambu p-form, then $d\omega$ is a co-Nambu $(p+1)$-form. Indeed, the condition (4) in Definition 2.4 is trivial for $d\omega$, and the condition (3) about the decomposability is easily verified: near a nonzero point of ω we can write $\omega = f\, dx_1 \wedge \cdots \wedge dx_p$, which implies $d\omega = df \wedge dx_1 \wedge \cdots \wedge dx_p$. If $d\omega(0) \neq 0$ then a Nambu tensor dual to it is regular at O and gives rise to a local regular foliation, denoted by $\mathcal{F}$. The tangent spaces of $\mathcal{F}$ are nothing but the spaces of vectors whose contraction with $d\omega$ is zero. Therefore if Z is a vector tangent to $\mathcal{F}$ at a point x near O we have $i_Z\, d\omega(x) = 0$. If $\omega(x) \neq 0$ then we also obtain that $i_Z\omega(x) = 0$, by using again the presentation $\omega = f\, dx_1 \wedge \cdots \wedge dx_p$. Since the set of nonzero points of ω is dense near O (because $d\omega(0) \neq 0$), by continuity we get that for any Z tangent to $\mathcal{F}$, $i_Z\omega(x) = 0$ and $i_Z\, d\omega(x) = 0$. It means that ω is locally a pull-back of a form on the local base space of $\mathcal{F}$. □

Acknowledgements

We thank N. Nakanishi whose visit to Montpellier in May 1997 attracted our attention to this problem.

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