THE a-LOCAL OPERATOR PROBLEM

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Let $f(t) = \int e^{itx} d\mu(x)$ be the Fourier transform of a Borel measure of finite total variation. The formula

$$f'(t) = \int e^{itx} ix \, d\mu(x)$$

can be justified if the integral on the right converges absolutely. For

$$\frac{f(t+h) - f(t)}{h} = \int e^{itx} \frac{e^{ixh} - 1}{ixh} ix \, d\mu(x)$$

where

$$\left| \frac{e^{ixh} - 1}{ixh} \right| = \left| \frac{2}{xh} \sin \frac{xh}{2} \right| \leqslant 1.$$

Now let $h \to 0$ in both sides of this equation and use the Lebesgue dominated convergence theorem.

The formula suggests a concept of "derivative" for such absolutely convergent Fourier transforms. Notationally it is easier to drop a factor of i. Define an operator H to act on $f(t)$ by

$$H \cdot f(t) = \int e^{itx} x \, d\mu(x)$$

whenever

$$\int |x \, d\mu(x)| < \infty.$$

The operator H corresponds to $-i$ times differentiation and we have the formula

$$H \cdot f(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{ih}$$

whenever $f(t)$ is in the domain of H. In **(2)**, this formula was shown to have interesting consequences from the point of view of the operational calculus.

Let $K(x)$ be a complex valued, Borel measurable function of real x. Define a corresponding operator $K(H)$ on Fourier transforms

$$f(t) = \int e^{itx} d\mu(x)$$

of Borel measures of finite total variation by

$$K(H) \cdot f(t) = \int e^{itx} K(x) d\mu(x)$$

whenever $\int |K(x) d\mu(x)| < \infty$.

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For example, if $K(H) = H^2$,

$$\begin{aligned} H^2 \cdot f(t) &= \int e^{itx} x^2 \, d\mu(x) \\ &= \int e^{itx} x (x \, d\mu(x)) \\ &= H \cdot H f(t) \end{aligned}$$

and H^2 is the double iteration of H. Similarly, if $K(H)$ is any polynomial in H, the operator $K(H)$ may be thought of as a finite linear combination of iterations of H. The functional notation $K(H)$ which is so appropriate in these simple cases is also to be used when $K(x)$ is only restricted to be Borel measurable.

Let $a > 0$. The operator $K(H)$ is said to be a-local if whenever $f(t)$ in the domain of $K(H)$ vanishes in $[-a, a]$,

$$K(H) \cdot f(0) = 0.$$

The problem is to find conditions on the defining function $K(x)$ that the associated operator $K(H)$ be a-local.

The problem here is related to the problem in **(2)** of finding local operators on Fourier transforms, and was proposed by Pollard at the same time as the other problem. We continue with the ideas and notation of the previous paper, except that Fourier transforms of not absolutely continuous measures are admitted in the domain of the operator. The defining function $K(x)$ is assumed to have a finite value for each real x, and there are no a.e. identifications.

Recall the lemma of **(2)** which is the basis of our approach.

LEMMA 1. *If $K(z)$ is an entire function of exponential type such that*

(1) $$\overline{\lim_{r}}\, r^{-1} \log |K(re^{i\theta})| \leqslant a |\sin \theta|,$$

and

(2) $$e^{-a|y|} K(iy) \qquad (z = x + iy)$$

is bounded, and if

$$f(t) = \int e^{itx} d\mu(x)$$

is an absolutely convergent Fourier transform in the domain of $K(H)$ which vanishes in $[-a, a]$, then for all complex z

$$\int \frac{K(t) - K(z)}{t - z} \, d\mu(t) = 0.$$

Theorems I and II are the appropriate restatements of Theorem I of **(2)**.

THEOREM I. *If $K(z)$ is an entire function of exponential type satisfying* (1) *and* (2), *then the operator $K(H)$ is a-local.*

THEOREM II. *If $K(H)$ is an a-local operator on Fourier transforms and if $K(H)$ has in its domain a function which vanishes in $[-a, a]$ and does not*

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vanish identically, then $K(x)$ *is the restriction to the real axis of an entire function of exponential type satisfying* (1) *and*

$$\int \frac{\log^+|K(t)|}{1+t^2}\, dt < \infty. \tag{3}$$

The gap between Theorem I and Theorem II arising from (2) has caused difficulty. The hypothesis (2) cannot be entirely removed from Theorem I. In fact, the function $K(z) = z \sin az$ does not define an a-local operator on Fourier transforms. This follows from the following uniqueness theorem.

THEOREM III. *If* $K(H)$ *is a-local and if*

$$\sum_{-\infty}^{+\infty} (1+n^2)^{-1}|K(n\pi/a)| < \infty,$$

then $K(x)$ *is the restriction to the real axis of an entire function* $K(z)$ *of exponential type satisfying* (1) *and*

$$e^{-a|y|}K(iy) = o(y) \tag{4}$$

as $|y| \to \infty$.

Although Theorem III shows that the hypothesis (2) cannot be entirely removed from Theorem I, we conjecture that it can be removed if some hypothesis is made which keeps the modulus of $K(z)$ from being too bumpy. We have been able to obtain the conclusion of Lemma 1 from a hypothesis of this nature.

LEMMA 2. *If* $K(z)$ *is an entire function of exponential type satisfying* (1) *and* (3) *and*

$$\frac{\partial}{\partial y}\frac{y}{\pi}\int \frac{\log^+|K(t)|}{(t-x)^2+y^2}\, dt \leqslant c \tag{5}$$

is bounded above in the upper half plane $y > 0$, *and if*

$$f(t) = \int e^{itx} d\mu(x)$$

in the domain of $K(H)$ *vanishes in* $[-a, a]$, *then for all complex* z

$$\int \frac{K(t) - K(z)}{t-z}\, d\mu(t) = 0.$$

We have not been able to show that under the hypotheses of Lemma 2, the operator $K(H)$ is a-local. The trouble lies in the following property of a-local operators which we have been unable to relate to any explicit description of $K(z)$.

THEOREM IV. *If* $K(z)$ *is an entire function of exponential type such that*

$$\overline{\lim_r}\, r^{-1}\log K(re^{i\theta}) = a|\sin\theta|, \tag{6}$$

and if the operator $K(H)$ *is a-local, and if* μ *is a Borel measure on the real line, of finite total variation, such that*

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$$\int |K(t) d\mu(t)| < \infty$$

and for all complex z

$$\int \frac{K(t) - K(z)}{t - z} d\mu(t) = 0,$$

then

$$\int K(t) d\mu(t) = 0.$$

It is interesting to compare the hypotheses of Lemma 2 with the following alternative hypotheses for Levinson's theorem (Theorem II of **(2)**).

THEOREM V. *If* $K(x) \geqslant 1$ *is a continuous function of real x such that* $\log K(x)$ *is uniformly continuous and*

$$\int \frac{\log K(x)}{1 + x^2} dx = \infty, \tag{7}$$

then there is no non-zero Borel measure μ *such that*

$$\int |K(x) d\mu(x)| < \infty, \quad \text{and} \quad \int e^{itx} d\mu(x)$$

vanishes in an interval.

Proof of Theorem I. Let $f(t) = \int e^{itx} d\mu(x)$ be in the domain of $K(H)$ and vanish in $[-a, a]$. By Lemma 1,

$$\begin{aligned}\int K(t) d\mu(t) &= \int \frac{tK(t) - zK(t)}{t - z} d\mu(t) \\ &= \int \frac{tK(t) - zK(z)}{t - z} d\mu(t) \\ &= \int \frac{tK(t) - ze^{iaz}K(z)e^{-iat}}{t - z} d\mu(t).\end{aligned}$$

Since

$$0 = \int e^{-iat} d\mu(t) = \int \frac{te^{-iat} - ze^{-iat}}{t - z} d\mu(t),$$

$$\int K(t) d\mu(t) = \int \frac{e^{iat}K(t) - e^{iaz}K(z)}{t - z} te^{-iat} d\mu(t).$$

Let $z = iy$ where $y \to + \infty$. By the Lebesgue dominated convergence theorem,

$$\int (t - z)^{-1} e^{iat} K(t) te^{-iat} d\mu(t) \to 0$$
$$\int (t - z)^{-1} te^{-iat} d\mu(t) \to 0$$

and since by (2) $e^{iaz}K(z)$ remains bounded in the limit,

$$K(H) \cdot f(0) = \int K(t) d\mu(t) = 0. \qquad \text{q.e.d.}$$

Proof of Theorem II. By hypothesis there is a function

$$f_1(t) = \int e^{itx} d\mu_1(x)$$

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in the domain of $K(H)$ which vanishes in $[-a, a]$ and does not vanish identically. Write

$$f_1(t) = \int e^{itx} B_1(x) d\mu(x)$$

where μ is a non-negative Borel measure and $B_1(x)$ is a Borel measurable function of absolute value 1. The hypothesis that $f_1(t)$ vanishes in $[-a, a]$ implies that the closed span in $L^1(\mu)$ of the functions e^{itx}, $-a \leqslant t \leqslant a$, is not all of $L^1(\mu)$. Since $f_1(t)$ is in the domain of $K(H)$, $K(x)$ is in $L^1(\mu)$. The hypothesis that $K(H)$ is a-local implies that for every Borel measurable function $B(x)$ which is essentially bounded with respect to μ, such that $\int e^{itx} B(x) d\mu(x) = 0$ for $-a \leqslant t \leqslant a$, we have $\int K(x) B(x) d\mu(x) = 0$. Since every continuous linear functional on $L^1(\mu)$ is defined by a Borel measurable function which is essentially bounded with respect to μ, it follows by the Hahn-Banach theorem that $K(x)$ lies in the closed span in $L^1(\mu)$ of the functions e^{itx}, $-a \leqslant t \leqslant a$. Let

$$M(z) = \sup |L(z)|$$

where $L(z)$ ranges in the finite linear combinations of the functions e^{itz} where $-a \leqslant t \leqslant a$. Since $L(z) = (\int d\mu)^{-1}$ appears in the supremum, $M(z) \geqslant (\int d\mu)^{-1}$. Since the closed span of the functions e^{itx}, $-a \leqslant t \leqslant a$, is not all of $L^1(\mu)$, it follows as in the proof of necessity of Theorem I of **(2)** that

$$\int \frac{\log M(t)}{1 + t^2} dt < \infty \tag{8}$$

and

$$\log M(x + iy) \leqslant a|y| + \frac{y}{\pi} \int \frac{\log M(t)}{(t - x)^2 + y^2} dt \qquad (y \neq 0) \tag{9}$$

and that $K(x)$ is equal a.e. with respect to μ to the restriction to the real axis of a unique entire function $F(z)$ such that for all complex z

$$F(z) \leqslant M(z) \int |K(t)| d\mu(t). \tag{10}$$

Let μ' be any non-negative Borel measure of finite total variation such that $\mu \leqslant \mu'$ and $\int |K(t)| d\mu'(t) < \infty$, and let $M'(z)$ be the corresponding majorant for μ'. Since for any Borel function $L(t)$,

$$\int |L(t)| d\mu(t) \leqslant \int |L(t)| d\mu'(t),$$

we have for all complex z,

$$M'(z) \leqslant M(z).$$

Therefore, the closed span in $L^1(\mu')$ of the functions, e^{itx} $-a \leqslant t \leqslant a$, is not all of $L^1(\mu')$ and $K(x)$ is equal a.e. with respect to μ' to the restriction to the real axis of the entire function $F(z)$, and for all complex z

$$F(z) \leqslant M'(z) \int |K(t)| d\mu'(t).$$

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By the arbitrariness in the choice of μ', $K(x) = F(x)$ for all real x. (1) and (3) now follow from (8), (9), and (10), q.e.d.

Proof of Theorem III. Let x_1 and x_2 be any two distinct real numbers, neither of which is an integral multiple of π/a. Let μ be the measure which has mass $(\sin ax_2)^{-1}$ at x_2, mass $-(\sin ax_1)^{-1}$ at x_1, and for integral n has mass

$$\frac{(-1)^n (ax_2 - ax_1)}{(ax_2 - n\pi)(ax_1 - n\pi)}$$

at $n\pi/a$. It is easy to verify that $\int |d\mu(x)| < \infty$ and $\int e^{itx} d\mu(x) = 0$ for $-a \leqslant t \leqslant a$. (Use the Fourier series for e^{itx} in $-a \leqslant t \leqslant a$:

$$e^{itx} \sim \Sigma(-1)^n (ax - n\pi)^{-1} \sin ax\, e^{int/a}.)$$

Let $K(x)$ satisfy the hypotheses of the theorem. They imply that $\int |K(t) d\mu(t)| < \infty$. Since the operator $K(H)$ is a-local, $\int K(x) d\mu(x) = 0$. Equivalently,

$$\frac{K(x_2)}{\sin ax_2} - \frac{K(x_1)}{\sin ax_1} = \sum \frac{(-1)^n (ax_1 - ax_2) K(n\pi/a)}{(ax_2 - n\pi)(ax_1 - n\pi)}.$$

Define $K(z)$ for complex z in the unique way such that the same formula holds with x_1 and x_2 replaced by complex variables z_1 and z_2. Obvious estimates from this formula show that $K(z)$ is an entire function of exponential type satisfying (1) and (4). (A similar formula is discussed in **(1**, pp. 220–1.**)**

Proof of Lemma 2. Let $f(t)$ satisfy the hypotheses of the lemma and let

$$L(z) = \int \frac{K(t) - K(z)}{t - z} d\mu(t). \tag{11}$$

As in the proof in **(2)** of Lemma 1, $L(z)$ is an entire function of minimal exponential type. Define

$$\log S(x) = \log^+ |K(x)|$$

$$\log S(x + iy) = \frac{|y|}{\pi} \int \frac{\log S(t)}{(t - x)^2 + y^2} dt \qquad (y \neq 0)$$

The convergence follows from (3) and by (5)

$$S(x + iy) \leqslant e^{c|y|} S(x). \tag{12}$$

Let the real number t be held fixed and consider

$$\frac{K(t) - K(z)}{t - z}$$

as a function of complex z. Then

$$\left| \frac{K(t) - K(z)}{t - z} \right| \leqslant |y|^{-1} (|K(t)| + |K(z)|)$$

where by Boas **(1**, p. 93**)**,

$$|K(z)| \leqslant S(z) e^{a|y|}.$$

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Since $S(t)$, $S(z) \geqslant 1$, we have

$$\left|\frac{K(t) - K(z)}{t - z}\right| \leqslant 2|y|^{-1} S(t)S(z)e^{a|y|}.$$

Since

$$\frac{K(t) - K(z)}{t - z}$$

has the same growth properties as $K(z)$, the same reference to Boas shows that

$$\begin{aligned}\left|\frac{K(t) - K(x)}{t - x}\right| &\leqslant 2|y|^{-1} S(t)S(x + 2iy)e^{2a|y|} \\ &\leqslant 2|y|^{-1} e^{2(c+a)|y|} S(t)S(x)\end{aligned}$$

by (12). Choosing $y = (2c + 2a)^{-1}$, we have

$$\left|\frac{K(t) - K(x)}{t - x}\right| \leqslant 4e(c + a)S(t)S(x) \tag{13}$$

and hence by (11)

$$|L(x)| \leqslant 4e(c + a)S(x)||S\mu|| \tag{14}$$

where

$$||S\mu|| = \int S(t)|d\mu(t)| < \infty.$$

We claim that for $n = 1, 2, 3, \ldots$

$$|L(x)|^n \leqslant [4e(c + a)||S\mu||]^n S(x) \tag{15}$$

and hence (by the Boas reference)

$$|L(z)|^n \leqslant [4e(c + a)||S\mu||]^n S(z). \tag{16}$$

We prove the inequality (15) by induction on n, starting with (14) when $n = 1$. Suppose that for some n, (15) holds, and we will prove the corresponding inequality with n replaced by $n + 1$. Since $L(z)$ has minimal exponential type, so has $L^n(z)$, and by (15), $\int|L^n(t)d\mu(t)| < \infty$. By Lemma 1,

$$\int \frac{L^n(t) - L^n(z)}{t - z}\, d\mu(t) = 0$$

for all complex z. Therefore,

$$\begin{aligned}L^{n+1}(z) &= \int \frac{L^n(z)K(t) - K(z)L^n(z)}{t - z}\, d\mu(t) \\ &= \int \frac{L^n(z)K(t) - K(z)L^n(t)}{t - z}\, d\mu(t).\end{aligned}$$

By (16),

$$\begin{aligned}|L^{n+1}(z)| &\leqslant |y|^{-1}[|L^n(z)|\int|K(t)d\mu(t)| + |K(z)|\int|L^n(t)d\mu(t)|] \\ &\leqslant 2|y|^{-1}[4e(c + a)||S\mu||]^n\, S(z)e^{a|y|}||S\mu||.\end{aligned}$$

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The conclusion

$$L^{n+1}(x) \leqslant [4e(c+a)||S\mu||]^{n+1}S(x)$$

now follows in the same way we established (13). This completes the inductive step.

Since in (16), n is arbitrary, for all complex z,

$$|L(z)| \leqslant 4e(c+a)||S\mu||.$$

By Liouville's theorem, $L(z) = L$ is a constant.

Consider various cases depending on the number and location of the zeros of $K(z)$. Discard the case that $K(z)$ has only a finite number of zeros, for it follows by Lemma 1. In this case, $K(z) = P(z)e^{ihz}$ for some polynomial $P(z)$ and $-a \leqslant h \leqslant a$. If $|h| < a$, Lemma 1 can be used directly. For the limiting cases, let $h \to a$ or $-a$ and use dominated convergence.

If $K(z)$ admits an infinite sequence of zeros $z_n = x_n + iy_n$ such that $|z_n| \to \infty$ and y_n is bounded away from 0, then

$$L = \lim_n \int \frac{K(t)}{t - z_n}\, d\mu(t) = 0$$

by dominated convergence.

We are left with the case that $K(z)$ admits an infinite set of zeros, but all so close to the real axis that the last argument does not apply. Let $h > 0$. By (12), $K(t+ih)$ is dominated on the real axis by $S(t)$ and the above argument with $K(z)$ replaced by $K(z+ih)$ shows that

$$\int \frac{K(t+ih) - K(z+ih)}{t-z}\, d\mu(t)$$

is a constant, independent of z. Since $K(z+ih)$ has an infinite set of zeros at a positive distance from the real axis, the constant is 0. On letting $h \to 0$ and using (12), we have by the Lebesgue dominated convergence theorem,

$$\int \frac{K(t) - K(z)}{t-z}\, d\mu(t) = 0.$$

Proof of Theorem IV. Let K and μ be as in the hypotheses of the theorem, and let $0 < h < a$ be held fixed. As in the proof of Lemma 1,

$$L(z) = \int \frac{e^{iht} - e^{ihz}}{t-z}\, d\mu(t)$$

is an entire function of exponential type. By the hypotheses on $K(z)$,

$$\begin{aligned} K(z)L(z) &= \int \frac{K(z)e^{iht} - e^{ihz}K(z)}{t-z}\, d\mu(t) \\ &= \int \frac{K(z)e^{iht} - e^{ihz}K(t)}{t-z}\, d\mu(t) \end{aligned}$$

and hence

$$|K(z)L(z)| \leqslant |y|^{-1}(|K(z)| + e^{-hy})||\mu||$$

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where $||\mu|| = \int |d\mu| < \infty$. The hypothesis (6) now implies that

$$\overline{\lim_r}\, r^{-1} \log|K(re^{i\theta})L(re^{i\theta})| \leqslant a|\sin\theta|.$$

By applying the theorem (**1**, p. 116) on the effective asymptotic behaviour of these functions and using (6), we see that $L(z)$ has minimal exponential type. The theorem is applicable since $L(z)$ is bounded on a line

$$|L(z)| \leqslant 2|y|^{-1}||\mu||$$

and $K(z)$ is the ratio of two functions bounded and analytic in a half plane. By (**1**, p. 83), the entire function $L(z)$, being of minimal exponential type and bounded on a line, is a constant. By dominated convergence

$$L(z) = L = \lim_{y\to+\infty} \int \frac{e^{iht} - e^{-hy}}{t - iy}\, d\mu(t) = 0.$$

Again by dominated convergence,

$$\begin{aligned}\int e^{iht} d\mu(t) &= \lim_{y\to+\infty} -iy \int \frac{e^{iht}}{t - iy}\, d\mu(t)\\ &= \lim_{y\to+\infty} -iy \int \frac{e^{-hy}}{t - iy}\, d\mu(t)\\ &= 0.\end{aligned}$$

A similar argument shows that

$$f(h) = \int e^{iht} d\mu(t)$$

vanishes when $-a < h < 0$. By the continuity of the Fourier transforms, this function vanishes in $-a \leqslant h \leqslant a$. Since the operator $K(H)$ is a-local,

$$\int K(t)d\mu(t) = K(H)\cdot f(0) = 0.$$

Proof of Theorem V. Let $a > 0$ and let $T(x) = T_a(x) = \sup |L(x)|$ where $L(z)$ ranges in the entire functions of exponential type, satisfying (1) and (2), and such that for all real x, $|L(x)| \leqslant K(x)$. By the proof of Theorem II of (**2**), the conclusion of the theorem follows if we can show that

$$\int \frac{\log T(x)}{1 + x^2}\, dx = \infty \tag{17}$$

(for every $a > 0$). By the uniform continuity of $\log K(x)$, there is some $\epsilon > 0$ such that

$$|\log K(x_1) - \log K(x_2)| \leqslant \epsilon$$

whenever $|x_2 - x_1| \leqslant \pi/2a$. Let the real number x_0 be held fixed. For every positive integer n,

$$\log K(x) \geqslant \log K(x_0) - n\epsilon$$

whenever $|x - x_0| \leqslant n\pi/2a$, and we will use this estimate with n so chosen that

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$$n\ (\log \pi/2 + \epsilon) \leqslant \log K(x_0)$$

(if such an n exists). For then the function

$$L(x) = \left[\frac{\pi n}{2a(x - x_0)} \sin \frac{a(x - x_0)}{n}\right]^n$$

is an entire function of exponential type, satisfying (1) and (2), and when $|x - x_0| \leqslant n\pi/2a$,

$$\log |L(x)| \leqslant n \log \pi/2 \leqslant \log K(x)$$

and when $|x - x_0| \geqslant n\pi/2a$,

$$\log |L(x)| \leqslant 0 \leqslant \log K(x).$$

So $L(x)$ is one of the test functions in the definition of $T(x)$. Since

$$\log L(x_0) = n \log \pi/2,$$

we have

$$\log T(x_0) \geqslant n \log \pi/2$$

whenever

$$n(\log \pi/2 + \epsilon) \leqslant \log K(x_0).$$

So, for all real x,

$$\log T(x) \geqslant \frac{\log \pi/2}{\log \pi/2 + \epsilon} \log K(x) - \log \pi/2.$$

The hypothesis (7) now implies (17).

References

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