Bull. Aust. Math. Soc. **84** (2011), 387–392
doi:10.1017/S0004972711002437

PRODUCTS OF BASE-κ-PARACOMPACT SPACES AND COMPACT SPACES

LEI MOU

(Received 15 November 2010)

Abstract

Let λ be a regular ordinal with $\lambda \geq \omega_1$. Then we prove that $(\lambda + 1) \times \lambda$ is not base-countably metacompact. This implies that base-κ-paracompactness is not an inverse invariant of perfect mappings, which answers a question asked by Yamazaki.

2010 *Mathematics subject classification*: primary 54B10; secondary 54D20, 54D70.

Keywords and phrases: base, paracompact, product, ordinal.

1. Introduction

Throughout this paper, all spaces are assumed to be T_1 topological spaces. For a space X, $w(X)$ stands for the weight of X. For a subset A of a space X, $\mathrm{cl}_X A$ denotes the closure of A in X. As usual, an ordinal is the set of all smaller ordinals. The symbol ω denotes the first infinite ordinal and ω_1 is the first uncountable ordinal. Ordinals are considered as spaces with the usual order topology. Let κ denote an infinite cardinal.

Porter [8] called a space X *base-paracompact* if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$. Yamazaki [9] called a space X *base-κ-paracompact* if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X of cardinality at most κ has a locally finite refinement by members of $\mathcal{B}$. In particular, a space X is said to be base-countably paracompact if X is base-ω-paracompact. Note that X is base-paracompact if and only if X is base-κ-paracompact for every cardinal κ.

Yamazaki proved that the product of a base-κ-paracompact space X and a compact space Y with $w(Y) \leq \kappa$ is base-κ-paracompact [9, Proposition 6.4]. Our examples show that the condition '$w(Y) \leq \kappa$' above plays an important role. It is known that base-paracompactness is an inverse invariant of perfect mappings [8]. Yamazaki asked if base-κ-paracompactness is an inverse invariant of perfect mappings [9]. Our examples give a negative answer.

Supported by the Foundation for Excellent Scholars, Beijing Organization Ministry (No. 095402800) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry (No. 08221010086).

We call a space X *base-metacompact* (respectively, *base-κ metacompact*) if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover (respectively, open cover of cardinality at most κ) of X has a point-finite refinement by members of $\mathcal{B}$. Note that each paracompact subspace of products of two ordinals is base-paracompact [6], and each metacompact subspace of products of two ordinals is base-metacompact [7]. Theorem 2.4 below shows that κ-paracompact subspaces of products of two ordinals need not be base-κ-paracompact.

Yamazaki [9] defined a space X to be *base-normal* if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every binary open cover $\{U_1, U_2\}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that $\{\mathrm{cl}_X B : B \in \mathcal{B}\}$ refines $\{U_1, U_2\}$.

A subset S of a regular uncountable ordinal μ is said to be *stationary* in μ if it intersects all cub (that is, closed and unbounded) sets in μ. For a subset A of an ordinal μ, let $\mathrm{Lim}_\mu A$ denotes the set of all limit points of A in μ. Clearly, if A is unbounded in a regular uncountable ordinal μ, then $\mathrm{Lim}_\mu A$ is a cub set in μ.

Let $\mathrm{cf}(\mu)$ denote the cofinality of an ordinal μ. For a limit ordinal μ, a strictly increasing function $M : \mathrm{cf}(\mu) \to \mu$ is said to be normal if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma < \mathrm{cf}(\mu)$ and $\mu = \sup\{M(\gamma) : \gamma < \mathrm{cf}(\mu)\}$. Clearly, M carries $\mathrm{cf}(\mu)$ homeomorphically to the range $\mathrm{ran}(M)$ of M and $\mathrm{ran}(M)$ is closed and unbounded in μ.

Lemma 1.1 (The Pressing Down Lemma (PDL)). *Let $\mu > \omega$ be regular, S a stationary subset in μ, and $f : S \to \mu$ such that $f(\gamma) < \gamma$ for each $\gamma \in S$. Then for some $\alpha < \mu$, $f^{-1}(\alpha)$ is stationary in μ.*

2. Main results

Lemma 2.1 [9]. *For a space X, the following statements are equivalent:*

(1) *X is base-normal and base-κ-paracompact;*
(2) *X is base-normal and κ-paracompact;*
(3) *X is normal and base-κ-paracompact.*

Lemma 2.2 [6]. *Each subspace of any ordinal is base-normal.*

Proposition 2.3. *Let λ be an ordinal with $\mathrm{cf}(\lambda) \geq \omega_1$. Then for each cardinal κ with $\kappa < \mathrm{cf}(\lambda)$, λ is base-κ-paracompact.*

Proof. By Lemmas 2.1 and 2.2, it is enough to show that λ is κ-paracompact. Let $f : \mathrm{cf}(\lambda) \to \lambda$ be a normal function. Let $\mathcal{U}$ be an open cover of λ with $|\mathcal{U}| \leq \kappa$. Assume that $\mathcal{U} = \{U_\beta : \beta < \delta\}$, where $\delta \leq \kappa$. Let $S = \{\alpha < \mathrm{cf}(\lambda) : \alpha \text{ is a limit ordinal}\}$. Then S is stationary in $\mathrm{cf}(\lambda)$. For each $\alpha \in S$, take an ordinal $\xi(\alpha) < \alpha$ and $\eta(\alpha) < \delta$ such that $f(\alpha) \in (f(\xi(\alpha)), f(\alpha)] \subseteq U_{\eta(\alpha)}$. For each $\beta < \delta$, let $S_\beta = \{\alpha \in S : \eta(\alpha) = \beta\}$. Then $S = \bigcup\{S_\beta : \beta < \delta\}$. Since $\delta \leq \kappa < \mathrm{cf}(\lambda)$, there exists $\beta_0 < \delta$ such that S_{β_0} is stationary in $\mathrm{cf}(\lambda)$. By the PDL, there exist $\gamma < \mathrm{cf}(\lambda)$ and a stationary set $T \subseteq S_{\beta_0}$ such that $\xi(\alpha) = \gamma$ for each $\alpha \in T$. Hence, $(f(\gamma), f(\alpha)] \subseteq U_{\beta_0}$ for each $\alpha \in T$. Thus, $(f(\gamma), \lambda) \subseteq U_{\beta_0}$. Since $[0, f(\gamma)]$ is compact, we can take a finite subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{U}'$ covers $[0, f(\gamma)]$. Then $\mathcal{U}' \cup \{U_{\beta_0}\}$ is a finite subcover of $\mathcal{U}$. This implies that λ is κ-paracompact. □

https://doi.org/10.1017/S0004972711002437 Published online by Cambridge University Press

It is known that each subspace of $\mu \times \nu$ is hereditarily countably metacompact for any ordinals μ and ν [3]. We will show that such spaces need not be base-countably metacompact. The proof of the following Theorem 2.4 is based on that of [6, Theorem 2.1]. For the reader's convenience, we give its proof in full.

THEOREM 2.4. *Let λ be a regular ordinal with $\lambda \geq \omega_1$. Then $(\lambda + 1) \times \lambda$ is not base-countably metacompact.*

PROOF. Obviously, $w(X) = \lambda$. Put $X = (\lambda + 1) \times \lambda$. Suppose that $\mathcal{B}$ is a base of X with $|\mathcal{B}| = \lambda$. We will show that $\mathcal{B}$ cannot satisfy base-countable metacompactness of X.

Claim 1. Let $B \in \mathcal{B}$. If $\{\delta < \lambda : \langle \gamma, \delta \rangle \in B\}$ is stationary in κ, then there exist a cub set $C(B)$ in κ, a function $f(B, \cdot) : C(B) \to \kappa$ and an ordinal $g(B) < \min(C(B))$ such that $(f(B, \gamma), \kappa] \times (g(B), \gamma] \subseteq B$ for each $\gamma \in C(B)$.

PROOF OF CLAIM 1. For each $\delta \in \lambda$ with $\langle \lambda, \delta \rangle \in B$, fix $p(B, \delta) < \lambda$ and $q(B, \delta) < \delta$ such that $(p(B, \delta), \lambda] \times (q(B, \delta), \delta] \subseteq B$. Applying the PDL, we can find an ordinal $g(B) < \lambda$ and a stationary set S in λ such that $S \subseteq \{\delta < \lambda : \langle \lambda, \delta \rangle \in B\}$ and $q(B, \delta) = g(B)$ for each $\delta \in S$. Let $C(B) = \{\gamma \in \lambda : \gamma > \min(S)\}$. For each $\gamma \in C(B)$, let $\psi(\gamma) = \min\{\delta \in S : \gamma \leq \delta\}$, and $f(B, \gamma) = p(B, \psi(\gamma))$. Then

$$(f(B, \gamma), \lambda] \times (g(B), \gamma] \subseteq (p(B, \psi(\gamma)), \lambda] \times (g(B), \psi(\gamma)] \subseteq B.$$

The proof of Claim 1 is complete. □

Let $\mathcal{B}' = \{B \in \mathcal{B} : \{\delta < \lambda : \langle \lambda, \delta \rangle \in B\}$ is stationary in $\lambda\}$. Rewrite $\mathcal{B}' = \{B_\alpha : \alpha < \xi\}$, where ξ is a cardinal. By Claim 1, for each $\alpha < \xi$, there exist a cub set C_α in λ, a function $f(B_\alpha, \cdot) : C_\alpha \to \lambda$ and an ordinal $g(B_\alpha) < \min(C_\alpha)$ such that $(f(B_\alpha, \gamma), \lambda] \times (g(B_a), \gamma] \subseteq B_\alpha$ for each $\gamma \in C_\alpha$. If $\xi < \lambda$, let $C' = \bigcap_{\alpha<\xi} C_\alpha$. If $\xi = \lambda$, let $C' = \{\gamma \in \lambda : \text{for all } \alpha < \gamma (\gamma \in C_\alpha)\}$. In any case, C' is a cub set in λ [4, Ch. II, Lemmas 6.8 and 6.14]. Let $C = \mathrm{Lim}_\lambda(C')$. Then C is a cub set in λ and $C \subseteq C'$. For each $\gamma \in C$, take a limit ordinal $f(\gamma) < \lambda$ such that $f(\gamma) > \sup\{f(B_\alpha, \gamma) : \alpha < \gamma\}$. We may assume that $f(\gamma') < f(\gamma)$ if $\gamma' < \gamma$. Let $U_1 = \bigcup\{(f(\gamma), \lambda] \times [0, \gamma] : \gamma \in C\}$. Then $\{\lambda\} \times \lambda \subseteq U_1$. Let $U_2 = \lambda \times \lambda$. Then $\{U_1, U_2\}$ is an open cover of X. We will show that $\{U_1, U_2\}$ admits no point-finite refinement by members of $\mathcal{B}$. Suppose $\mathcal{B}^*$ is a refinement of $\{U_1, U_2\}$ by members of $\mathcal{B}$. To complete the proof, it suffices to show that $\mathcal{B}^*$ is not point-finite in X.

Claim 2. For each $\alpha < \xi$, $B_\alpha \setminus U_1 \neq \emptyset$.

PROOF OF CLAIM 2. Fix $\alpha < \xi$. Take $\gamma_1 \in C$ such that $\gamma_1 > \alpha$. Let $\gamma_2 = \min\{\gamma \in C : \gamma > \gamma_1\}$. By the definition of C, we have $\gamma_1 \in C_\alpha$ and $\gamma_2 \in C_\alpha$. Since $f(\gamma_2) > f(B_\alpha, \gamma_2)$ and $f(\gamma_2)$ is a limit ordinal, there exists an ordinal $\alpha' \in \lambda$ such that $f(B_\alpha, \gamma_2) < \alpha' < f(\gamma_2)$. Since $\gamma_2 > \gamma_1$ and γ_2 is a limit ordinal, there exists an ordinal $\beta' \in \lambda$ such that $\gamma_1 < \beta' < \gamma_2$. Since $g(B_\alpha) < \min(C_\alpha)$ and $\gamma_1 \in C_\alpha$, we have $\gamma_1 > g(B_\alpha)$. Hence,

$$\langle \alpha', \beta' \rangle \in (f(B_\alpha, \gamma_2), \lambda] \times (\gamma_1, \gamma_2] \subseteq (f(B_\alpha, \gamma_2), \lambda] \times (g(B_\alpha), \gamma_2] \subseteq B_\alpha.$$

https://doi.org/10.1017/S0004972711002437 Published online by Cambridge University Press

Since $\{f(\gamma) : \gamma \in C\}$ is strictly increasing and γ_2 is the successor of γ_1 in C, it follows from the definition of U_1 that $\langle \alpha', \beta' \rangle \notin U_1$. The proof of Claim 2 is complete. □

Let $\mathcal{B}'' = \mathcal{B} \setminus \mathcal{B}'$. For each $\alpha < \lambda$, there exist $s(\alpha) < \lambda$, $t(\alpha) < \lambda$ and $V_\alpha \in \mathcal{B}$ such that $\langle \lambda, \alpha \rangle \in V_\alpha \subseteq (s(\alpha), \lambda] \times (t(\alpha), \alpha] \subseteq U_1$. By Claim 1, we have $V_\alpha \in \mathcal{B}''$. Obviously, $V_\alpha \neq V_\beta$ whenever $\alpha \neq \beta$. Hence, $|\mathcal{B}''| = \lambda$. Rewrite $\mathcal{B}'' = \{B^\beta : \beta < \lambda\}$. For each $\beta < \lambda$, since $\{\delta < \lambda : \langle \lambda, \delta \rangle \in B^\beta\}$ is not stationary in λ, there exists a cub set D_β in λ such that $D_\beta \cap \{\delta < \lambda : \langle \lambda, \delta \rangle \in B^\beta\} = \emptyset$. Let $D = \{\sigma \in \lambda : \text{for all } \beta < \sigma (\sigma \in D_\beta)\}$. Then D is a cub set in λ. Since $\mathcal{B}^*$ is a refinement of $\{U_1, U_2\}$, we can take $\sigma_0 \in D$ and $W_0 \in \mathcal{B}^*$ such that $\langle \lambda, \sigma_0 \rangle \in W_0 \subseteq U_1$. By Claim 2, we have $W_0 \in \mathcal{B}''$. Hence, $W_0 = B^{\beta(0)}$ for some $\beta(0) \in \lambda$. Since D is unbounded in λ, we can chose $\sigma_1 \in D$ such that $\sigma_1 > \sigma_0$ and $\sigma_1 > \beta(0)$. Take $W_1 \in \mathcal{B}^*$ such that $\langle \lambda, \sigma_1 \rangle \in W_1 \subseteq U_1$. By Claim 2, $W_1 \in \mathcal{B}''$. Take $B^{\beta(1)} \in \mathcal{B}''$ such that $B^{\beta(1)} = W_1$. By the definition of D, we have $\sigma_1 \in D_\beta$ for each $\beta < \sigma_1$. Hence, $\langle \lambda, \sigma_1 \rangle \notin B^\beta$ for each $\beta < \sigma_1$. Since $\langle \lambda, \sigma_1 \rangle \in B^{\beta(1)}$, we have $\beta(1) \geq \sigma_1$. Thus, $\beta(1) > \beta(0)$ since $\sigma_1 > \beta(0)$. Proceeding by induction, we can choose a strictly increasing sequence $\{\sigma_\alpha : \alpha \in \lambda\}$ in D and a strictly increasing sequence $\{\beta(\alpha) : \alpha \in \lambda\}$ in λ such that:

(1) for each $\alpha < \lambda$, $\langle \lambda, \sigma_\alpha \rangle \in B^{\beta(\alpha)} \in \mathcal{B}^* \cap \mathcal{B}''$;
(2) for each $\alpha < \lambda$, $\beta(\alpha) < \sigma_{\alpha+1}$;
(3) for each limit ordinal $\alpha < \lambda$, $\sigma_\alpha = \sup\{\sigma_\gamma : \gamma < \alpha\}$.

By condition (2), for any $\alpha_1, \alpha_2 < \lambda$ with $\alpha_1 < \alpha_2$, we have $\beta(\alpha_1) < \sigma_{\alpha_2}$. Clearly, $\{\sigma_\alpha : \alpha \in \lambda\}$ is a cub set in λ. For each σ_α, take $\mu(\sigma_\alpha) < \lambda$ and $\nu(\sigma_\alpha) < \sigma_\alpha$ such that $(\mu(\sigma_\alpha), \lambda] \times (\nu(\sigma_\alpha), \sigma_\alpha] \subseteq B^{\beta(\alpha)}$. By the PDL, there exist an ordinal η and a stationary set $T \subseteq \{\sigma_\alpha : \alpha \in \lambda\}$ such that $\nu(\sigma_\alpha) = \eta$ for each $\sigma_\alpha \in T$. Then $\{B^{\beta(\alpha)} : \alpha \in \lambda\}$ is not point-finite at the point $\langle \lambda, \eta + 1 \rangle$. Hence, $\mathcal{B}^*$ is not point-finite in X. The proof is complete. □

The following result solves an open problem mentioned by Yamazaki in [9, p. 139].

Theorem 2.5. *For each infinite cardinal κ, base-κ-paracompactness is not an inverse invariant of perfect mappings.*

Proof. Take an uncountable regular ordinal λ such that $\lambda > \kappa$. Let $f : (\lambda + 1) \times \lambda \to \lambda$ be the projection. Then f is a perfect mapping. By Proposition 2.3, λ is base-κ-paracompact. By Theorem 2.4, $(\lambda + 1) \times \lambda$ is not base-κ-paracompact. □

Theorem 2.6. *Let λ be a regular ordinal and κ an infinite cardinal with $\lambda > \kappa$. Then $(\lambda + 1) \times \lambda$ is κ-paracompact and not base-κ-paracompact.*

Proof. By Proposition 2.3, λ is κ-paracompact. We know that the product of a κ-paracompact space and a compact space is κ-paracompact [5, Theorem 2.1]. Hence, $(\lambda + 1) \times \lambda$ is κ-paracompact. By Theorem 2.4, $(\lambda + 1) \times \lambda$ is not base-κ-paracompact. □

Corollary 2.7. *The space $(\omega_1 + 1) \times \omega_1$ is countably paracompact and not base-countably paracompact.*

https://doi.org/10.1017/S0004972711002437 Published online by Cambridge University Press

LEMMA 2.8 [2]. *Let A and B be subspaces of an ordinal. If $A \times B$ is normal, then $A \times B$ is countably paracompact.*

LEMMA 2.9 [6]. *Let A and B be subspaces of an ordinal. Then $A \times B$ is normal if and only if it is base-normal.*

By Lemmas 2.1, 2.8 and 2.9, we have the following result.

PROPOSITION 2.10. *Let A and B be subspaces of an ordinal. If $A \times B$ is normal, then $A \times B$ is base-countably paracompact.*

LEMMA 2.11 [2]. *If A and B are subspaces of ω_1, then normality and countable paracompactness of $A \times B$ are equivalent.*

PROPOSITION 2.12. *If A and B are subspaces of ω_1, then $A \times B$ is countably paracompact if and only if $A \times B$ is base-countably paracompact.*

Note that $(\omega_1 + 1) \times \omega_1$ is not normal. In [1], Gruenhage constructed a countably compact linearly ordered topological space (LOTS) which is not base-normal. By Lemma 2.1, this example is not base-countably paracompact. It is known that each LOTS is countably paracompact. By Lemmas 2.1 and 2.2, each subspace of an ordinal is base-countably paracompact normal. The following result shows that countably paracompact normal subspaces of products of two ordinals need not be base-countably paracompact.

THEOREM 2.13. *Let*

$$X = \{\langle \alpha, \beta \rangle : \beta < \alpha < \omega_1, \alpha \text{ and } \beta \text{ are successor ordinals}\} \cup (\{\omega_1\} \times \omega_1).$$

Then X is a countably paracompact normal space which is not base-countably paracompact.

PROOF. We show that X is countably paracompact. Let $\mathcal{U} = \{U_i : i \in \omega\}$ be a countable open cover of X. Similar to the proof of Proposition 2.3, there exists a finite subcollection $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{U}'$ covers $\{\omega_1\} \times \omega_1$. Put $Y = X \setminus \bigcup \mathcal{U}'$. Let $\mathcal{V} = \mathcal{U}' \cup \{\{\langle \alpha, \beta \rangle\} : \langle \alpha, \beta \rangle \in Y\}$. Then $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. Hence, X is countably paracompact.

By [7, Theorem 2.1], X is normal and not base-normal. By Lemma 2.1, X is not base-countably paracompact. □

We conclude this paper with the following questions.

QUESTION 2.14. Is each subspace of ω_1^2 base-countably metacompact?

QUESTION 2.15. Is each countably paracompact subspace of ω_1^2 base-countably paracompact?

We know that the class of κ-paracompact normal spaces is invariant under closed mappings [5].

https://doi.org/10.1017/S0004972711002437 Published online by Cambridge University Press

QUESTION 2.16. Is the class of base-κ-paracompact normal spaces invariant under perfect mappings (respectively, closed mappings)?

References

[1] G. Gruenhage, 'Base-paracompactness and base-normality of GO-spaces', *Questions Answers Gen. Topology* **23** (2005), 137–141.

[2] N. Kemoto, H. Ohta and K. Tamano, 'Products of spaces of ordinal numbers', *Topology Appl.* **45** (1992), 245–260.

[3] N. Kemoto and K. D. Smith, 'Hereditary countable metacompactness in finite and infinite product spaces of ordinals', *Topology Appl.* **77** (1997), 57–63.

[4] K. Kunen, *Set Theory. An Introduction to Independence Proofs* (North-Holland, Amsterdam, 1980).

[5] K. Morita, 'Paracompactness and product spaces', *Fund. Math.* **50** (1961/62), 223–236.

[6] L. Mou, 'Base-normality and total paracompactness of subspaces of products of two ordinals', *Sci. Math. Jpn.* **63** (2006), 89–100.

[7] L. Mou, R. Wang and S. Wang, 'Base metacompact spaces', *Adv. Math.* **40** (2011), 111–117.

[8] J. E. Porter, 'Base-paracompact spaces', *Topology Appl.* **128** (2003), 145–156.

[9] K. Yamazaki, 'Base-normality and product spaces', *Topology Appl.* **148** (2005), 123–142.

LEI MOU, Mathematics Department, Capital Normal University,
Beijing 100048, PR China
e-mail: moulei@mail.cnu.edu.cn