Can. J. Math. Vol. 47 (6), 1995 pp. 1240–1252

THE TOPOLOGICAL NATURE OF TWO NOGUCHI THEOREMS ON SEQUENCES OF HOLOMORPHIC MAPPINGS BETWEEN COMPLEX SPACES

In memory of Professor O. G. Smith

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ABSTRACT. Let C, D, D^* be, respectively, the complex plane, $\{z \in C : |z| < 1\}$, and $D - \{0\}$. If $P^1(C)$ is the Riemann sphere, the Big Picard theorem states that if $f: D^* \longrightarrow P^1(C)$ is holomorphic and $P^1(C) - f(D^*)$ has more than two elements, then f has a holomorphic extension $\tilde{f}: D \longrightarrow P^1(C)$. Under certain assumptions on M, A and $X \subset Y$, combined efforts of Kiernan, Kobayashi and Kwack extended the theorem to all holomorphic $f: M - A \longrightarrow X$. Relying on these results, measure theoretic theorems of Lelong and Wirtinger, and other properties of complex spaces, Noguchi proved in this context that if $f: M - A \longrightarrow X$ and $f_n: M - A \longrightarrow X$ are holomorphic for each n and $f_n \longrightarrow f$, then $\tilde{f}_n \longrightarrow \tilde{f}$. In this paper we show that all of these theorems may be significantly generalized and improved by purely topological methods. We also apply our results to present a topological generalization of a classical theorem of Vitali from one variable complex function theory.

1. **Introduction.** Let $H(X, Y)$ be the family of holomorphic mappings from the complex space X to the complex space Y. If $\mathbf{C}$ is the complex plane, $\mathbf{D}$ the unit disk, $\{z \in \mathbf{C} : |z| < 1\}$, and $\mathbf{D}^* = \mathbf{D} - \{0\}$, the punctured disk, the Big Picard theorem asserts:

(1°) If $f \in H(\mathbf{D}^*, \mathbf{C})$ has an essential singularity at 0, then $\mathbf{C} - f(\mathbf{D}^*)$ has at most one element.

If $P^1(\mathbf{C})$ is the Riemann sphere, the result in (1°) may be stated in terms of $P^1(\mathbf{C})$ as in (2°).

(2°) If $f \in H\big(\mathbf{D}^*, P^1(\mathbf{C})\big)$ and $P^1(\mathbf{C}) - f(\mathbf{D}^*)$ has more than two elements, then f has an extension $\tilde{f} \in H\big(\mathbf{D}, P^1(\mathbf{C})\big)$.

This theorem has been extended as below in (3°) to higher dimensional settings by the combined efforts of Kiernan, Kobayashi and Kwack (see pp. 39–60 of [L] culminating in the "K^3 Theorem").

Received by the editors February 2, 1994.

AMS subject classification: Primary: 32H15, 32H20, 54C35; secondary: 54C20, 54D50.

Key words and phrases: complex spaces, big Picard theorem, continuous extensions, holomorphic maps, k-spaces, function spaces, compact-open topology, even continuity, Alexandroff one-point compactification, Ascoli Theorem.

https://doi.org/10.4153/CJM-1995-063-6 Published online by Cambridge University Press

(3°) Let X be a relatively compact hyperbolically imbedded complex subspace of a complex space Y. Let M be a complex manifold and let A be a divisor on M with normal crossings. Then each $f \in H(M-A,X)$ extends to $\tilde{f} \in H(M,Y)$.

Noguchi proved the following theorem (4°) for $\mathbf{D}$ on preservation of uniform convergence on compact subsets by holomorphic extensions [No 1] and also established (5°), a higher dimensional version of (4°) (see [No 2] and Theorems 4.1 and 5.4 of Chapter II in [L]). All function spaces in this paper are endowed with the compact-open topology (see [Ke] for definition and particulars). The closure of a subset A of a topological space will be denoted by $\bar{A}$.

(4°) Let X be a relatively compact hyperbolically imbedded complex subspace of a complex space Y. Let $f \in H(\mathbf{D}^*,X)$ and let $\{f_n\}$ be a sequence in $H(\mathbf{D}^*,X)$ such that $f_n \longrightarrow f$. Then $\tilde{f}_n \longrightarrow \tilde{f}$.

(5°) Let X be a relatively compact hyperbolically imbedded complex subspace of a complex space Y. Let M be a complex manifold and let A be a divisor on M with normal crossings. If $f \in H(M-A,X)$ and $\{f_n\}$ is a sequence in $H(M-A,X)$ such that $f_n \longrightarrow f$, then $\tilde{f}_n \longrightarrow \tilde{f}$.

It should be pointed out that Noguchi [No 2] has improved (5°) by allowing $f \in H(M-A,Y)$; in the process, he showed that (3°) is valid for $f \in \overline{H(M-A,X)}$ (the closure being taken in $H(M-A,Y)$).

The proofs of (4°) and (5°) as given in [L] rely on (3°), measure theoretic results of Wirtinger and Lelong for complex spaces and additional complex space arguments (see pp. 43–64 in [L]). In [J-K], proofs of extensions of (4°) and (5°) are given which escape the reliance upon measure theory and complex space arguments other than those needed to establish (3°). It is shown that extensions and generalizations of (3°), (4°) and (5°) may be based solely on the theory of function spaces and (6°).

(6°) The following statements are equivalent for a complex subspace X of a complex space Y:

(i) X is hyperbolically imbedded in Y.

(ii) If $\{f_n\}$ and $\{z_n\}$ are sequences in $H(\mathbf{D}^*,X)$ and $\mathbf{D}^*$ respectively such that $z_n \longrightarrow 0$ and $f_n(z_n) \longrightarrow p$, then $f_n(z_n') \longrightarrow p$ for each sequence $\{z_n'\}$ in $\mathbf{D}^*$ such that $z_n' \longrightarrow 0$.

(iii) For any complex manifold M and divisor A on M with normal crossings, if $\{f_n\}$ and $\{z_n\}$ are sequences in $H(M-A,X)$ and $M-A$ respectively such that $z_n \longrightarrow z \in M$ and $f_n(z_n) \longrightarrow p$, then $f_n(z_n') \longrightarrow p$ for each sequence $\{z_n'\}$ in $M-A$ such that $z_n' \longrightarrow z$.

If X and Y are topological spaces, we denote the set of functions from X to Y by $F(X,Y)$ and abstract the properties in (ii) and (iii) of (6°).

PROPERTY κ. If X and Y are topological spaces and $X_0 \subset X$ is dense, we say that $\Omega \subset F(X_0,Y)$ *satisfies property* κ *with respect to* (X_0,X,Y) if for each $x \in X$, $y \in Y$ and net $\{(f_\alpha,x_\alpha,v_\alpha)\}$ in $\Omega \times X_0 \times X_0$ such that $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x$ and $f_\alpha(x_\alpha) \longrightarrow y$ we have $f_\alpha(v_\alpha) \longrightarrow y$.

In this paper, we use property κ to extend and generalize (3°), (4°) and (5°) by purely topological methods. For facility in reaching these results we give next some definitions for topological spaces and notational conventions. The set of open subsets of a space having x as an element will be denoted by $\Sigma(x)$ and the Alexandroff one-point compactification of a space Y will be denoted by Y^+. The collection of continuous functions from a space X to a space Y will be denoted by $C(X, Y)$; if $X_0 \subset X$, $Y_0 \subset Y$ and $\Omega \subset C(X_0, Y_0)$, we use the notation $C[X, Y; \Omega]$ for the collection, of $\tilde{f} \in C(X, Y)$ such that $\tilde{f}$ is the unique extension of $f \in \Omega$. All spaces will be Hausdorff and, unless otherwise indicated, Y will always be a locally compact space, while X will always be a k-space. A space is a *k-space* if a subset C of the space is closed when $C \cap K$ is closed in K for each compact subset K of the space. See [Ke] and [K-T] as references for topological notions used in this paper.

We now give a sample of the results which are established in Section 2 of this paper.

THEOREM. *Let X_0 be a dense subset of X and let $\Omega \subset F(X_0, Y)$. If Ω satisfies property κ with respect to (X_0, X, Y), then*

(1) Each $f \in \bar{\Omega}$ has an extension $\tilde{f} \in C(X, Y^+)$.

(2) $C[X, Y^+; \Omega]$ is relatively compact in $C(X, Y^+)$.

(3) $C[X, Y^+; \bar{\Omega}]$ is compact in $C(X, Y^+)$.

(4) If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ and $f_\alpha \longrightarrow f$, then $\tilde{f}_\alpha \longrightarrow \tilde{f}$.

(5) Ω is relatively compact in $C(X_0, Y^+)$.

In fact, we show that various combinations of these properties produce necessary and sufficient conditions for Ω as described in the theorem to satisfy property κ with respect to (X_0, X, Y).

In Section 3, guided by criteria for continuous extendability of functions from dense subsets, we offer other characterizations of property κ as well as two results of "Ascoli type". New characterizations of hyperbolic imbeddedness of complex subspaces are obtained as a consequence.

2. Topological generalizations of results of Kiernan, Kobayashi, Kwack, and Noguchi for complex spaces.

In this section, we offer our main results. To effect these results, we initially give two propositions to be used in the sequel. The first is an immediate consequence of the topological Ascoli Theorem of Bagley and Yang [B-Y], and the second is a proposition establishing a characterization of even continuity. If X and Y are any topological spaces, we say that $\Omega \subset F(X, Y)$ is *evenly continuous from $A \subset X$ to $B \subset Y$* if for each a $\in$ A, $b \in B$ and $U \in \Sigma(b)$ in B, there are $W \in \Sigma(b)$ in B and $V \in \Sigma(a)$ in A such that

$$f \in \Omega \quad \text{and} \quad f(a) \in W \Rightarrow f(V) \subset U.$$

If $\Omega \subset F(X, Y)$ is evenly continuous from X to Y, we say simply that Ω is *evenly continuous*. Proposition 1 follows from Theorem 4 in [B-Y] since if Ω is an evenly continuous family of functions into a regular space, $\bar{\Omega}$ must also be evenly continuous, and since, for each $x \in X$, $\Omega(x) = \{f(x) : f \in \Omega\}$ is relatively compact in Y iff $\{f(x) : f \in \bar{\Omega}\}$ is relatively compact in Y.

PROPOSITION 1. *$\Omega \subset C(X, Y)$ is relatively compact in $C(X, Y)$ iff*
(a) Ω is an evenly continuous subset of $C(X, Y)$, and
(b) $\Omega(x)$ is relatively compact in Y for each $x \in X$.

PROPOSITION 2. *Let X be a space and let $X_0 \subset X$ be dense in X. Then $\Omega \subset C(X, Y)$ is evenly continuous iff Ω is evenly continuous from $X_0 \cup \{v\}$ to Y for each $v \in X$.*

PROOF. The necessity is clear. For the sufficiency, let $x \in X, y \in Y$ and let $U \in \Sigma(y)$. Choose $A \in \Sigma(x)$, B, $W \in \Sigma(y)$, such that $\bar{W} \subset U$ and such that

$$f \in \Omega \quad \text{and} \quad f(x) \in B \Rightarrow f\big(A \cap (X_0 \cup \{x\})\big) \subset W.$$

We show that

$$f \in \Omega \quad \text{and} \quad f(x) \in B \Rightarrow f(A) \subset \bar{W}.$$

Let $f \in \Omega, z \in A$, and let $H \in \Sigma\big(f(z)\big)$. Since $f \in C(X_0 \cup \{z\}, Y)$, choose a $Q \in \Sigma(z)$ such that

$$f\big(Q \cap (X_0 \cup \{z\})\big) \subset H. Q \cap A \in \Sigma(z), \quad \text{so } Q \cap A \cap X_0 \neq \emptyset \quad \text{and}$$
$$\emptyset \neq f(Q \cap A \cap X_0) \subset f\big(Q \cap (X_0 \cup \{z\})\big) \cap f\big(A \cap (X_0 \cup \{x\})\big) \subset H \cap W.$$

So $f(z) \in \bar{W}$ and the proof is complete.

REMARK 1. It is apparent from [B-Y] that the requirement that Y be Hausdorff regular would have been sufficient in Proposition 1 and the requirement that Y be merely regular would have sufficed in Proposition 2. No requirement on the space X was necessary in Proposition 2.

We come to our first main result.

THEOREM 1. *Let X_0 be a dense subset of the space X. The following statements are equivalent for $\Omega \subset F(X_0, Y)$:*
(1) Ω satisfies property κ with respect to (X_0, X, Y).
(2) Ω satisfies the following two properties:
(a) Each $f \in \Omega$ extends to $\tilde{f} \in C(X, Y^+)$, and
(b) $C[X, Y^+; \Omega]$ is relatively compact in $C(X, Y^+)$.

PROOF. (2) $\Rightarrow$ (1). If (1) does not hold, we may assume $x \in X, p, q \in Y^+, p \neq q$ and a net $\{(f_\alpha, x_\alpha, v_\alpha)\}$ in $\Omega \times X_0 \times X_0$ such that $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x, f_\alpha(x_\alpha) \longrightarrow p, f_\alpha(v_\alpha) \longrightarrow q$. For each α, f_α extends to $\tilde{f}_\alpha$ and there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ and $g \in C(X, Y^+)$ such that $\tilde{f}_{\alpha_\mu} \longrightarrow g; \tilde{f}_{\alpha_\mu}(x) \longrightarrow g(x)$ and $C[X, Y^+; \Omega]$ is evenly continuous, so $f_{\alpha_\mu}(x_{\alpha_\mu}) = \tilde{f}_{\alpha_\mu}(x_{\alpha_\mu}) \longrightarrow g(x)$ and $f_{\alpha_\mu}(v_{\alpha_\mu}) = \tilde{f}_{\alpha_\mu}(v_{\alpha_\mu}) \longrightarrow g(x)$, a contradiction.

(1) $\Rightarrow$ (2). Obviously $\Omega \subset C(X_0, Y^+)$. To show that $f \in \Omega$ extends to $\tilde{f} \in C(X, Y^+)$, it will be sufficient to show that f extends to $\tilde{f} \in C(X_0 \cup \{v\}, Y^+)$ for each $v \in X - X_0$ [B-D]. By (1), for such a v there is a $p \in Y^+$ such that if $\{x_\alpha\}$ is a net in X_0 and $x_\alpha \longrightarrow v$, then $f(x_\alpha) \longrightarrow p$. So, we may define $\tilde{f}(v) = p$. We see that $\tilde{f} = f$ on X_0 and that $\tilde{f} \in C(X_0 \cup \{v\}, Y^+)$. Thus (2a) holds. Now suppose $v \in X$ and $C[X, Y^+; \Omega]$ is not evenly

continuous from $X_0 \cup \{v\}$ to Y^+. There exist $x \in X_0 \cup \{v\}, p \in Y^+$, $U \in \Sigma(p)$, such that for each pair $(V, W) \in \Sigma(x) \times \Sigma(p)$ satisfying $W \subset U$, some $f_{(V,W)} \in \Omega$ and $x_{(V,W)} \in X_0 \cup \{v\}$ satisfy $x_{(V,W)} \in V, \tilde{f}_{(V,W)}(x) \in W, \tilde{f}_{(V,W)}(x_{(V,W)}) \in Y^+ - U \subset Y^+ - W$. Order the set of such pairs by $(V_1, W_1) \leq (V_2, W_2)$ iff $V_2 \subset V_1$ and $W_2 \subset W_1$. For each such pair (V, W), $\tilde{f}_{V,W}$ is continuous and $\tilde{f}_{(V,W)}(x) \in W$, so there is an $H \in \Sigma(x)$ such that $H \subset V$ and $\tilde{f}_{(V,W)}(H) \subset W$. Choose such an H and $y_{(V,W)} \in H \cap X_0$. Then $\{y_{(V,W)}\}$ is a net in X_0; if $B \in \Sigma(p)$ with $B \subset U$ and $A \in \Sigma(x)$, we have $x_{(V,W)}, y_{(V,W)} \in A$ and $f_{(V,W)}(y_{(V,W)}) \in B$ for each $(V, W) \geq (A, B)$. Hence $y_{(V,W)} \longrightarrow x, x_{(V,W)} \longrightarrow x$, and $f_{(V,W)}(y_{(V,W)}) \longrightarrow p$. We see that $x_{(V,W)} \neq x$ is satisfied for each (V, W); hence $x_{(V,W)} \neq v$ is satisfied eventually and we reach a contradiction of (1) since we may assume that $f_{(V,W)}(x_{(V,W)}) \longrightarrow q \in Y^+$, $q \neq p$. Hence $C[X, Y^+; \Omega]$ is evenly continuous in view of Proposition 2 and, by Proposition 1, we finish.

Corollary 1 generalizes results of Kiernan, Kobayashi, Kwack and Noguchi. If $\Omega \subset C(X_0, Y)$, $\bar{\Omega}$ is the closure of Ω in $C(X_0, Y^+)$.

COROLLARY 1. *Let X_0 be a dense subset of the space X and let $\Omega \subset F(X_0, Y)$ satisfy property κ with respect to (X_0, X, Y). Then*

(1) Each $f \in \bar{\Omega}$ extends to $\tilde{f} \in C(X, Y^+)$.

(2) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \longrightarrow f$, then $\tilde{f}_\alpha \longrightarrow \tilde{f}$.

PROOF. For the proof of (1), let $\{f_\alpha\}$ be a net in Ω such that $f_\alpha \longrightarrow f$. By Theorem 1, $\tilde{f}_\alpha \in C(X, Y^+)$ exists for each α and there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\tilde{f}_{\alpha_\mu} \longrightarrow g \in C(X, Y^+)$. We see that $g = \tilde{f}$. As for the proof of (2), if $\{f_{\alpha_\mu}\}$ is a subnet of $\{f_\alpha\}$ such that $\tilde{f}_{\alpha_\mu} \longrightarrow g$, then $g = \tilde{f}$. Since each subnet of $\{\tilde{f}_\alpha\}$ has a convergent subnet from Theorem 1, we see that $\tilde{f}_\alpha \longrightarrow \tilde{f}$.

The next theorem provides a characterization of property κ which leads to a further extension of the theorems of Noguchi. This extension is presented in Corollary 2.

THEOREM 2. *Let X_0 be a dense subset of the space X. Then $\Omega \subset F(X_0, Y)$ satisfies property κ with respect to (X_0, X, Y) iff the following two conditions hold:*

(1) Each $f \in \Omega$ extends to $\tilde{f} \in C(X, Y^+)$, and

(2) $C[X, Y^+; \bar{\Omega}]$ is a compact subset of $C(X, Y^+)$.

PROOF. The sufficiency of the two conditions is evident from Theorem 1 since $C[X, Y^+; \Omega] \subset C[X, Y^+; \bar{\Omega}]$. To establish the necessity of the two conditions, we show that $C[X, Y^+; \bar{\Omega}] = \overline{C[X, Y^+; \Omega]}$ and employ Theorem 1 again. Let $g \in C[X, Y^+; \bar{\Omega}]$; there is a net $\{f_\alpha\}$ in Ω such that $f_\alpha \longrightarrow g$ on X_0. Since Ω satisfies property κ with respect to (X_0, X, Y) there is by Theorem 1 a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\tilde{f}_{\alpha_\mu} \longrightarrow h$; $h = g$ and $h \in \overline{C[X, Y^+; \Omega]}$. Hence one inclusion is established. Let $g \in \overline{C[X, Y^+; \Omega]}$. There is a net $\{f_\alpha\}$ in $C[X, Y^+; \Omega]$ such that $f_\alpha \longrightarrow g$. Since $f_\alpha \longrightarrow g$ on X_0, the reverse inclusion is established.

COROLLARY 2. *Let X_0 be a dense subset of the space X and let $\Omega \subset F(X_0, Y)$ satisfy property κ with respect to (X_0, X, Y). Then*

(1) Ω is relatively compact in $C(X_0, Y^+)$, and

(2) If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ and $f_\alpha \to f$, then $\tilde{f}_\alpha \to \tilde{f}$.

PROOF. Let $\{f_\alpha\}$ be a net in $\bar{\Omega}$. From Theorem 2, $\tilde{f}_\alpha \in C(X, Y^+)$ exists for each α and there is a subnet $\{\tilde{f}_{\alpha_\mu}\}$ of $\{\tilde{f}_\alpha\}$ such that $\tilde{f}_{\alpha_\mu} \to g \in C(X, Y^+)$; $f_{\alpha_\mu} \to g$ on X_0. Hence (1) holds. If $\{\tilde{f}_{\alpha_\mu}\}$ is a convergent subnet of $\{\tilde{f}_\alpha\}$ and $f_{\alpha_\mu} \to f$ then $\tilde{f}_{\alpha_\mu} \to \tilde{f}$. From Theorem 2 each subnet of $\{\tilde{f}_\alpha\}$ has a convergent subnet. Thus (2) follows.

Corollary 1 and Theorem 2 combine to produce the following corollary.

COROLLARY 3. *If X_0 is a dense subset of the space X, then $\Omega \subset F(X_0, Y)$ satisfies property κ with respect to (X_0, X, Y) iff $\bar{\Omega}$ satisfies property κ with respect to (X_0, X, Y^+).*

EXAMPLE 1. Let M be a complex manifold and let X be a relatively compact hyperbolically imbedded complex subspace of the complex space Y. Let A be a divisor on M with normal crossings. If M has dimension m, A has *normal crossings* if, locally,

$$M - A = (\mathbf{D}^*)^r \times \mathbf{D}^s \quad \text{with } r + s = m \text{ (see [L]).}$$

It is a consequence of the Big Picard theorem of Kobayashi-Kwack and Kiernan's generalization of same that $H(M - A, X)$ satisfies property κ with respect to $(M - A, M, Y)$. In fact, it is shown in [J-K] that X is hyperbolically imbedded in Y iff $H(M - A, X)$ satisfies property κ with respect to $(M-A, M, Y)$ in all such situations. Hence we have Corollary 4 (compare with Theorems 2.1, 4.1, 5.2, 5.4 and Lemma 5.1 in Chapter II of [L]).

COROLLARY 4. *Let M be a complex manifold and let A be a divisor on M with normal crossings. Let X be a relatively compact hyperbolically imbedded complex subspace of the complex space Y. Then*

(1) [No 2] Each $f \in \overline{H(M - A, X)}$ extends to $\tilde{f} \in H(M, Y)$, and

(2) [J-K] If $\{f_n\}$ is a sequence in $\overline{H(M - A, X)}$ and $f_n \to f$, then $\tilde{f}_n \to \tilde{f}$.

We complete this section with two theorems, a corollary, examples and remarks. In Theorem 3, three conditions found in Theorem 2 and Corollary 2 to be necessary conditions for $\Omega \subset F(X_0, Y)$ to satisfy property κ with respect to (X_0, X, Y) are shown, in combination, to be sufficient. A topological generalization of the classical theorem of Vitali in one complex variable theory [B] is provided in Theorem 4.

THEOREM 3. *Let X_0 be a dense subset of the space X. Then $\Omega \subset F(X_0, Y)$ satisfies property κ with respect to (X_0, X, Y) iff the following three conditions hold:*

(1) Ω is relatively compact in $C(X_0, Y^+)$,

(2) Each $f \in \bar{\Omega}$ extends to $\tilde{f} \in C(X, Y^+)$, and

(3) If $\{f_\alpha\}$ is a net in Ω and $f_\alpha \to f$, then $\tilde{f}_\alpha \to \tilde{f}$.

PROOF. We show that the conditions are sufficient. Let $\{f_\alpha\}$ be a net in Ω. From (1), there is a subnet $\{f_{\alpha_\mu}\}$ such that $f_{\alpha_\mu} \to f$. From (2), $\tilde{f}_{\alpha_\mu}$ and $\tilde{f}$ exist for each μ. From (3), $\tilde{f}_{\alpha_\mu} \to \tilde{f}$. The proof is completed by appeal to Theorem 1.

EXAMPLE 2. Let $X = [0,1]$, $X_0 = [0,1)$ on the real line.

(a) $C[X,X;F(X_0,X)]$ is not relatively compact in $C(X,X)$ and therefore does not satisfy property κ with respect to (X_0,X,X).

(b) If $f_n \in C(X_0,X)$ is defined by $f_n(x) = x^n$ for each positive integer n and Ω is the set of f_n, then Ω satisfies (1) and (2) of Theorem 3 but does not satisfy (3).

EXAMPLE 3. $H(\mathbf{D}^*, \mathbf{C} - \{0\})$ does not satisfy property κ with respect to $(\mathbf{D}^*, \mathbf{D}, (\mathbf{C}-\{0\})^+)$. To see this let $z_0 \in \mathbf{C}-\{0\}$ and define the sequence $\{f_k\}$ in $H(\mathbf{D}^*, \mathbf{C}-\{0\})$ by $f_k(z) = \frac{z_0}{kz}$. Then $f_k(\frac{1}{k+1}) \longrightarrow z_0$ while $f_k(\frac{1}{2(k+1)}) \longrightarrow 2z_0$.

EXAMPLE 4. Let $n \neq 1$ be a positive integer and let $M = \{(z_1,z_2,\ldots,z_n) \in \mathbf{C}^n : z_i \notin \{0,1\}\}$. Let $[w_0,w_1,\ldots,w_n]$ be homogeneous coordinates of the projective space $\mathrm{P}^n(\mathbf{C})$ and let π be the hyperplane $\{[w_0,w_1,\ldots,w_n] : w_0 = 0\}$. Define $\psi\colon M \longrightarrow \mathrm{P}^n(\mathbf{C})$ by $\psi(z_1,z_2,\ldots,z_n) = [1,z_1,\ldots,z_n]$. We show that $H\big(\mathbf{D}^*, \psi(M)\big)$ does not satisfy property κ with respect to $\big(\mathbf{D}^*, \mathbf{D}, \mathrm{P}^n(\mathbf{C})\big)$. Let $p = [0,1,w_2,\ldots,w_n]$ be a point in π with $|w_i| \geq 1$ for each i. Define a sequence $\{f_k\}$ in $H\big(\mathbf{D}^*, \psi(M)\big)$ by $f_k(z) = [1, \frac{k}{z^k}, \frac{w_2}{z^{k+1}}, \ldots, \frac{w_n}{z^{k+1}}]$. Then $f_k(z) = [\frac{z^k}{k}, 1, \frac{w_2}{kz}, \ldots, \frac{w_n}{kz}]$, $f_k(\frac{1}{k+1}) \longrightarrow p$, $f_k\big(\frac{1}{2(k+1)}\big) \longrightarrow [0,1,2w_2,\ldots,2w_n]$; for each fixed z, $f_k(z) \longrightarrow [0,1,0,\ldots,0]$, so $f_k \longrightarrow$ a constant map f on $\mathbf{D}^*$. $\tilde{f}_k, \tilde{f} \in H\big(\mathbf{D}, \mathrm{P}^n(\mathbf{C})\big)$ exist for each k but $\{\tilde{f}_k\}$ does not converge.

REMARK 2. Corollary 2 and Theorem 3 show that Ω may be replaced by $\bar{\Omega}$ in condition (3) of Theorem 3. Note that in condition 3 we are tacitly assuming that $\tilde{f}_\alpha$ and $\tilde{f}$ exist for each α.

REMARK 3. It is easily seen from Remark 2 and Theorem 3 that if $X_0 \subset X$ is dense, then $\Omega \subset F(X_0,Y)$ satisfies property κ with respect to (X_0,X,Y) iff the following two conditions are satisfied:

(1) Ω is relatively compact in $C(X_0,Y^+)$, and

(2) The mapping $\kappa\colon \bar{\Omega} \longrightarrow C(X,Y^+)$ defined by $\kappa(f) = \tilde{f}$ is an imbedding.

THEOREM 4. *Let $\Delta \subset C(X,Y)$ be relatively compact. Let $S \subset X$ such that if $f,g \in \bar{\Delta}$ and $f = g$ on S, then $f = g$. If $\{f_\alpha\}$ is a net in $\bar{\Delta}$, $f \in \bar{\Delta}$, and $f_\alpha \longrightarrow f$ on S, then $f_\alpha \longrightarrow f$.*

PROOF. Let $\{f_{\alpha_\mu}\}$ be a subnet of $\{f_\alpha\}$ such that $f_{\alpha_\mu} \longrightarrow h$; then $f_{\alpha_\mu} \longrightarrow f$ and $f_{\alpha_\mu} \longrightarrow h$ on S, so $f_{\alpha_\mu} \longrightarrow f$. The proof is complete.

COROLLARY 5. *Let $X_0 \subset X$ be dense, let $\Omega \subset F(X_0,Y)$ satisfy property κ with respect to (X_0,X,Y). Assume there exists an $S \subset X$ such that if $f,g \in C(X,Y^+)$ and $f = g$ on S, then $f = g$. If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ and $\{\tilde{f}_\alpha\}$ converges on S, then $\{\tilde{f}_\alpha\}$ converges.*

PROOF. If $\{f_\alpha\}$ is a net in $\bar{\Omega}$, then $\tilde{f}_\alpha$ exists for each α and $\tilde{f}_{\alpha_\mu} \longrightarrow f \in C(X,Y^+)$ for some subnet $\{\tilde{f}_{\alpha_\mu}\}$ of $\{\tilde{f}_\alpha\}$. If $\{\tilde{f}_\alpha\}$ converges on S, then $\tilde{f}_\alpha \longrightarrow f$ on S since $\tilde{f}_{\alpha_\mu} \longrightarrow f$ on S; $\tilde{f}_\alpha \longrightarrow f$ from Theorem 4.

3. **More characterizations of property κ, applications of results in Section 2 and two Ascoli type theorems.** Let $\Omega \subset F(X_0, Y)$ where X_0 is a dense subset of X. The major aim of this section is to produce a number of necessary and sufficient conditions for Ω to satisfy property κ with respect to (X_0, X, Y) and to apply these in the complex space context. Most of these conditions have as bases criteria for extendability of a function from a dense subset of a space to a continuous function on the space. The condition for extendability in the following proposition is suggestive that such criteria might be formulated. In all results below, closures of inverse images are computed in the topology of the parent space rather than in that of the dense subset.

PROPOSITION 3 [B-D]. *Let X_0 be a dense subset of the space X, let Y be a regular space and let $f: X_0 \longrightarrow Y$ be a function. Then f has a continuous extension $\tilde{f}: X \longrightarrow Y$ iff for each $x \in X$, there is a $y \in Y$ such that if $\{x_\alpha\}$ is a net in X_0 and $x_\alpha \longrightarrow x$, then $f(x_\alpha) \longrightarrow y$.*

We see readily that if f satisfies the hypothesis of Proposition 3, then f has a continuous extension $\tilde{f}: X \longrightarrow Y$ iff

(a) $\{\bigcap_{\Sigma(y)} \overline{f^{-1}(W)} : y \in Y\}$ covers X, and

(b) If $x \in \bigcap_{\Sigma(y)} \overline{f^{-1}(W)}$ and $\{x_\alpha\}$ is a net in X_0 such that $x_\alpha \longrightarrow x$, then $f(x_\alpha) \longrightarrow y$.

In Theorem 5, we produce other necessary and sufficient conditions for the existence of continuous extensions from dense subsets as motivations for our characterizations of property κ. In this theorem, neither X nor Y is assumed to be a k-space.

THEOREM 5. *Let Y be regular, let $X_0 \subset X$ be dense and let $f: X_0 \longrightarrow Y$ be a function. The following statements are equivalent:*

(1) The function f extends to a continuous $\tilde{f}: X \longrightarrow Y$.

(2) For each $x \in X$, $\bigcap_{\Sigma(x)} \overline{f(V \cap X_0)} \cap \bar{K}$ is a singleton whenever $K \subset Y$ and $f(V \cap X_0) \cap K \neq \emptyset$ for each $V \in \Sigma(x)$.

(3) The function f satisfies the following two conditions:

(a) $\overline{f^{-1}(K)} \cap \overline{f^{-1}(M)} = \emptyset$ if K, M are disjoint closed subsets of Y, and

(b) $\bigcap_{\Sigma(x)} \overline{f(V \cap X_0)} \neq \emptyset$ is satisfied for each $x \in X$.

PROOF. (1) $\Rightarrow$ (2). For $x \in X$,

$$\{\tilde{f}(x)\} \subset \bigcap_{\Sigma(x)} \tilde{f}(V) \subset \bigcap_{\Sigma(x)} \overline{f(V \cap X_0)} \subset \bigcap_{\Sigma(x)} \overline{\tilde{f}(V)} = \{\tilde{f}(x)\}.$$

Furthermore, if $K \subset Y$ and $f(V \cap X_0) \cap K \neq \emptyset$ is satisfied for each $V \in \Sigma(x)$ we have $V \cap \tilde{f}^{-1}(K) \neq \emptyset$ satisfied for each $V \in \Sigma(x)$, so $x \in \overline{\tilde{f}^{-1}(K)}$ and $\tilde{f}(x) \in \bar{K}$. Hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). If $K \subset Y$ is closed and $x \in \overline{\tilde{f}^{-1}(K)}$, then $f(V \cap X_0) \cap K \neq \emptyset$ is satisfied for all $V \in \Sigma(x)$. Thus $\bigcap_{\Sigma(x)} \overline{f(V \cap X_0)} \cap K$ is a singleton. Hence $x \notin \overline{f^{-1}(M)}$ if M is closed and $M \cap K = \emptyset$ since $f(V \cap X_0) \cap Y \neq \emptyset$ satisfied for each $V \in \Sigma(x)$ implies $\bigcap_{\Sigma(x)} \overline{f(V \cap X_0)}$ is a singleton. Thus (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). We show first that $\bigcap_{\Sigma(x)} \overline{f(V \cap X_0)}$ is a singleton for each $x \in X$. Suppose $y, z \in \bigcap_{\Sigma(x)} \overline{f(V \cap X_0)}$ and $y \neq z$. Let W, H be closed disjoint neighborhoods of y, z, respectively. Then $x \in \overline{f^{-1}(W)} \cap \overline{f^{-1}(H)}$, contradicting condition (3a). Now, if $\{y\} =$

https://doi.org/10.4153/CJM-1995-063-6 Published online by Cambridge University Press

$\bigcap_{\Sigma(x)}\overline{f(V\cap X_0)}$, define $\tilde{f}(x) = y$; if $x \in X_0, f(x) \in \bigcap_{\Sigma(x)}\overline{f(V\cap X_0)}$, so $\tilde{f}(x) = f(x)$. Let $x \in X$, let $W \in \Sigma(\tilde{f}(x))$ and let $W_1, W_2 \in \Sigma(\tilde{f}(x))$ such that $\bar{W}_1 \subset W_2 \subset \bar{W}_2 \subset$ W. Then $x \in \overline{f^{-1}(\overline{W}_1)}$ and, from condition (3a), $x \notin \overline{f^{-1}(Y - W_2)}$. Hence some $V \in \Sigma(x)$ satisfies $V \cap X_0 \cap f^{-1}(Y - W_2) = \emptyset$ and, equivalently, $f(V \cap X_0) \subset W_2$. For such a V, $\tilde{f}(V) = \bigcup_{z\in V}\bigcap_{Q\in\Sigma(z)}\overline{f(Q\cap X_0)} \subset \overline{f(V\cap X_0)} \subset \bar{W}_2 \subset W$. The proof is complete.

A proof of Theorem 5 has been provided since the equivalences therein, although possibly part of the folklore of topology, do not seem to be prevalent in the literature. The following two propositions are presented without proof; again neither X nor Y is assumed to be a k-space unless otherwise stated.

PROPOSITION 4 [D]. *Let X_0 be a dense subspace of the space X, let Y be a regular space and let $f \in C(X_0, Y)$. Then f has a continuous extension $\tilde{f} \in C(X, Y)$ iff the filterbase $\{f(V\cap X_0) : V \in \Sigma(x)\}$ converges for each $x \in X$.*

PROPOSITION 5. *Let Y be a compact space, let $X_0 \subset X$ be dense and let $f\colon X_0 \longrightarrow Y$ be a function. The following statements are equivalent:*

(1) The function f extends to $\tilde{f} \in C(X, Y)$.

(2) $\bigcap_{\Sigma(x)}\overline{f(V\cap X_0)}$ is a singleton for each $x \in X$.

(3) [T] If K, M are disjoint closed subsets of Y, $\overline{f^{-1}(K)} \cap \overline{f^{-1}(M)} = \emptyset$.

Preliminary to our characterizations of property κ, we give some additional definitions and notation. Let $\{A_\alpha\}_\Lambda$ be a net of subsets of a topological space X. We recall that the *limit superior* of $\{A_\alpha\}_\Lambda$ is $\bigcap_{\mu\in\Lambda}\overline{\bigcup_{\alpha\geq\mu}A_\alpha}$ [K-T]. This set will be denoted by $\overline{\lim}_\Lambda A_\alpha$ or simply by $\overline{\lim}A_\alpha$ if no confusion is possible. We say that $x \in X$ is a *strong limit point* for $\{A_\alpha\}_\Lambda$ if for each $V \in \Sigma(x)$, $A_\alpha \subset V$ holds eventually. If $\{f_\alpha\}_\Lambda$ is a net of functions from a space X to a space Y, $X_0 \subset X$ and $x \in X$, when we speak of the net $\{f_\alpha(V\cap X_0)\}_x$ we will be using as directed set $\Lambda \times \Sigma(x)$, ordered by $(\alpha, V) \leq (\mu, W)$ iff $\alpha \leq \mu$ and $W \subset V$. In what follows, each limit superior is computed in the topology of the parent space.

THEOREM 6. *Let $X_0 \subset X$ be dense in X and let $\Omega \subset F(X_0, Y)$. The following statements are equivalent:*

(1) Ω satisfies property κ with respect to (X_0, X, Y).

(2) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\overline{\lim}f_{\alpha_\mu}^{-1}(K) \cap \overline{\lim}f_{\alpha_\mu}^{-1}(M) = \emptyset$ for each pair of disjoint subsets K, M of Y with K compact and M closed.

(3) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each pair of disjoint subsets K, M of Y with K compact and M closed, some neighborhood W of K satisfies $\overline{\lim}f_{\alpha_\mu}^{-1}(W) \cap \overline{\lim}f_{\alpha_\mu}^{-1}(M) = \emptyset$.

(4) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each closed subset M of Y and $y \in Y - M$, some $W \in \Sigma(y)$ satisfies $\overline{\lim}f_{\alpha_\mu}^{-1}(W) \cap \overline{\lim}f_{\alpha_\mu}^{-1}(M) = \emptyset$.

(5) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each $x \in X$ and compact $K \subset Y$, either $\{f_{\alpha_\mu}(V\cap X_0)\}_x$ has a strong limit point in Y or $f_{\alpha_\mu}(V\cap X_0) \subset Y - K$ eventually.

(6) For each $x \in X$ and net $\{f_\alpha\}$ in $\bar{\Omega}$, either some subnet of $\{f_\alpha(V\cap X_0)\}_x$ has a strong limit point in Y or for each compact $K \subset Y$, $f_\alpha(V\cap X_0) \subset Y - K$ eventually.

(7) If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each $x \in X$ and compact $K \subset Y$, each $y \in K \cap \overline{\lim} f_{\alpha_\mu}(V \cap X_0)$ is a strong limit point for $\{f_{\alpha_\mu}(V \cap X_0)\}_x$.

(8) If $\{f_\alpha\}$ is a net in Ω there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each $x \in X$ and compact $K \subset Y$, each $y \in K \cap \overline{\lim} f_{\alpha_\mu}(V \cap X_0)$ is a strong limit point for $\{f_{\alpha_\mu}(V \cap X_0)\}_x$.

PROOF. It is obvious that (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6), and (7) $\Rightarrow$ (8); and (2) $\Rightarrow$ (3) is fairly immediate since K must have a compact neighborhood W such that $W \cap M = \emptyset$.

(1) $\Rightarrow$ (2). Suppose (1) holds and let K, M be disjoint, respectively, compact and closed subsets of Y. Let $\{f_\alpha\}$ be a net in $\bar{\Omega}$. From Corollary 1 and Theorem 2, $\tilde{f}_\alpha \in C(X, Y^+)$ exists for each α and there is a subnet of $\{f_\alpha\}$, called again $\{f_\alpha\}$, such that $\tilde{f}_\alpha \longrightarrow g \in C(X, Y^+)$. Suppose $x \in \overline{\lim} f_\alpha^{-1}(K) \cap \overline{\lim} f_\alpha^{-1}(M)$. It follows that $\tilde{f}_\alpha(x) \longrightarrow g(x)$ and, by the even continuity of $C[X, Y; \bar{\Omega}]$ and compactness of K in Y, that $g(x) \in K \cap \bar{M}$ in Y^+. Hence $g(x) \in K \cap M$, a contradiction.

(4) $\Rightarrow$ (1). If $x \in X, y \in Y$ and $\{(f_\alpha, x_\alpha, v_\alpha)\}$ is a net in $\Omega \times X_0 \times X_0$ such that $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x, f_\alpha(x_\alpha) \longrightarrow y$ and $f_\alpha(v_\alpha) \not\to y$, to reach a contradiction we choose $H \in \Sigma(y)$ such that some subnet of $\{(f_\alpha, x_\alpha, v_\alpha)\}$, called again $\{(f_\alpha, x_\alpha, v_\alpha)\}$, satisfies $f_\alpha(x_\alpha) \longrightarrow y$ and $f_\alpha(v_\alpha) \in Y - H$ for each α. Then if $W \in \Sigma(y)$ and $\{f_{\alpha_\mu}\}$ is any subnet of $\{f_\alpha\}$, we have

$$x \in \overline{\lim} f_{\alpha_\mu}^{-1}(W \cap H) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(Y - H) \subset \overline{\lim} f_{\alpha_\mu}^{-1}(W) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(Y - H).$$

(1) $\Rightarrow$ (5). Suppose $K \subset Y$ is compact and $\{f_\alpha\}$ is a net in $\bar{\Omega}$ such that $f_\alpha(V \cap X_0) \cap K \neq \emptyset$ is satisfied for each pair (α, V). By Corollary 1 and Theorem 2, there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\tilde{f}_{\alpha_\mu} \longrightarrow g \in C(X, Y^+)$. It follows that $g(x) \in K$. Let $W \in \Sigma\big(g(x)\big)$ in Y; eventually $\tilde{f}_{\alpha_\mu}(V) \subset W$ and $f_{\alpha_\mu}(V \cap X_0) \subset W$ by the even continuity of $C[X, Y; \bar{\Omega}]$.

(1) $\Rightarrow$ (7). If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\tilde{f}_\alpha \in C(X, Y^+)$ exists for each μ and $\tilde{f}_{\alpha_\mu} \longrightarrow g \in C(X, Y^+)$. If $x \in X$, $K \subset Y$ is compact and $y \in K \cap \overline{\lim} f_{\alpha_\mu}(V \cap X_0)$, it follows that $y = g(x) \in Y$ is a strong limit point for $\{f_{\alpha_\mu}(V \cap X_0)\}_x$.

(6) or (8) $\Rightarrow$ (1). Let $x \in X, y \in Y$, W be a compact neighborhood of y, and let $\{(f_\alpha, x_\alpha, v_\alpha)\}$ be a net in $\Omega \times X_0 \times X_0$ such that $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x, f_\alpha(v_\alpha) \in Y - W$ for each α, and $f_\alpha(x_\alpha) \longrightarrow y$; $y \in W \cap \overline{\lim} f_\alpha(V \cap X_0)$ and $f_\alpha(V \cap X_0) \cap W \neq \emptyset$ is satisfied frequently; by (6) or (8), there is a subnet $\{f_{\alpha_\mu}(V \cap X_0)\}$ of $\{f_\alpha(V \cap X_0)\}_x$ such that $f_{\alpha_\mu}(V \cap X_0) \subset W$ eventually. This is a contradiction since $v_\alpha \longrightarrow x$ and $f_\alpha(v_\alpha) \in Y - W$ for each α.

The proof of Theorem 6 is complete.

COROLLARY 6. *Let Y be compact, let $X_0 \subset X$ be dense and let $\Omega \subset F(X_0, Y)$. The following statements are equivalent:*

(1) Ω satisfies property κ with respect to (X_0, X, Y).

(2) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, for each $x \in X$, $Y \cap \overline{\lim} f_{\alpha_\mu}(V \cap X_0)$ is a singleton.

(3) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\{f_{\alpha_\mu}(V \cap X_0)\}_x$ has a strong limit point in Y for each $x \in X$.

(4) For each net $\{f_\alpha\}$ in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\overline{\lim} f_{\alpha_\mu}^{-1}(K) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(M) = \emptyset$ for each pair of disjoint closed subsets K and M of Y.

(5) If $\{f_\alpha\}$ is a net in $\bar{\Omega}$ there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, if $y,z \in Y$ and $y \neq z$, some open sets $W \in \Sigma(y)$, $H \in \Sigma(z)$ satisfy $\overline{\lim} f_{\alpha_\mu}^{-1}(W) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(H) = \emptyset$.

(6) If $x \in X$, $y,z \in Y$ and $\{(f_\alpha, x_\alpha, v_\alpha)\}$ is a net in $\Omega \times X_0 \times X_0$ such that $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x$, $f_\alpha(x_\alpha) \longrightarrow y$ and $f_\alpha(v_\alpha) \longrightarrow z$, then $y = z$.

(7) Ω satisfies

(a) Each $f \in \Omega$ extends to $\tilde{f} \in C(X,Y)$, and

(b) $\{\tilde{f} : f \in \Omega\}$ is relatively compact in $C(X,Y)$.

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Follows from the equivalence of (1), (5), and (6) of Theorem 6 and the compactness of Y.

(1) $\Leftrightarrow$ (4). Follows from equivalence of (1) and (4) of Theorem 6.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (7) $\Rightarrow$ (1). Clear from Theorem 1.

In the situation where $X = \mathbf{D}$ and $X_0 = \mathbf{D}^*$, or where M is a complex manifold, $X = M$ and $X_0 = M - A$ for some divisor A on M with normal crossings, at each $x \in X$ we have a base of open sets $\theta(x)$ such that $V \cap X_0$ is connected for each $V \in \theta(x)$. Our next theorem offers characterizations of property κ in this framework.

THEOREM 7. *Let $X_0 \subset X$ be dense and suppose at each $x \in X$ there is a base $\theta(x)$ of open sets such that $V \cap X_0$ is connected for each $V \in \theta(x)$. The following statements are equivalent for $\Omega \subset F(X_0, Y)$:*

(1) Ω satisfies property κ with respect to (X_0, X, Y).

(2) For each $x \in X$, $y,z \in Y$ and net $\{(f_\alpha, x_\alpha, v_\alpha)\}$ in $\Omega \times X_0 \times X_0$ satisfying $x_\alpha \longrightarrow x$, $v_\alpha \longrightarrow x$, $f_\alpha(x_\alpha) \longrightarrow y$, $f_\alpha(v_\alpha) \longrightarrow z$, we have $y = z$.

(3) For each net $\{f_\alpha\}$ in $\bar{\Omega}$, there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\{f_{\alpha_\mu}(V \cap X_0)\}_x$ has a strong limit point in Y for each $x \in X$.

(4) For each net $\{f_\alpha\}$ in $\bar{\Omega}$, there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\overline{\lim} f_{\alpha_\mu}^{-1}(K) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(M) = \emptyset$ for each pair of disjoint compact $K, M \subset Y$.

(5) For each net $\{f_\alpha\}$ in $\bar{\Omega}$, there is a subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that, if $y,z \in Y$, $y \neq z$, some $W \in \Sigma(y)$, $H \in \Sigma(z)$ satisfy $\overline{\lim} f_{\alpha_\mu}^{-1}(K) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(H) = \emptyset$.

PROOF. We show only that (4) $\Rightarrow$ (1) and that (2) $\Rightarrow$ (1). Suppose (1) does not hold. Then we may assume a net $\{(f_\alpha, x_\alpha, v_\alpha)\}$ in $\Omega \times X_0 \times X_0$, $x \in X$, $y \in Y$, $f_\alpha(x_\alpha) \longrightarrow y$, $x_\alpha \longrightarrow x$, $W, H \in \Sigma(y)$ such that $\bar{W} \subset H$, $\bar{H}$ compact, $f_\alpha(x_\alpha) \in W$ and $f_\alpha(v_\alpha) \in Y - \bar{H}$ for each α. For any subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ we have $x \in \overline{\lim} f_{\alpha_\mu}^{-1}(\partial W) \cap f_{\alpha_\mu}^{-1}(\partial H)$, where ∂W and ∂H represent the boundary of W and H, respectively. Hence there is a net $\{(f_\alpha, x_\alpha, q_\alpha)\}$ in $\Omega \times X_0 \times X_0$ and $q \in \partial W$ such that $x_\alpha \longrightarrow x$, $q_\alpha \longrightarrow x$, $f_\alpha(x_\alpha) \longrightarrow y$, $f_\alpha(q_\alpha) \longrightarrow q$, $q \neq y$, so neither (4) nor (2) holds.

In Corollaries 7, 8 and Remark 4, we apply our results to give some characterizations of hyperbolic imbeddedness of complex subspaces.

COROLLARY 7. *The following statements are equivalent for a complex subspace X of a complex space Y:*

(1) X is hyperbolically imbedded in Y.

(2) [J-K] For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}^{-1}(K_1)\cap\overline{\lim}f_{n_k}^{-1}(K_2)=\emptyset$ for each pair of disjoint subsets K_1, K_2 of Y with K_1 compact and K_2 closed.

(3) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$ there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that, for each pair of disjoint subsets K_1, K_2 of Y with K_1 compact and K_2 closed, some neighborhood W of K_1 satisfies $\overline{\lim}f_{n_k}^{-1}(W)\cap\overline{\lim}f_{n_k}^{-1}(K_2)=\emptyset$.

(4) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$ there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that, for each $x\in\mathbf{D}$ and compact $K\subset Y$, either $\{f_{n_k}(V\cap X_0)\}_x$ has a strong limit point in Y or $f_{n_k}(V\cap X_0)\subset Y-K$ eventually.

(5) For each $x\in\mathbf{D}$ and sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$, either some subnet of $\{f_n(V\cap\mathbf{D}^*)\}_x$ has a strong limit point in Y or for each compact subset K of Y, $f_n(V\cap\mathbf{D}^*)\subset Y-K$ eventually.

(6) If $\{f_n\}$ is a sequence in $\overline{H(\mathbf{D}^*,X)}$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that, for each $x\in\mathbf{D}$ and compact $K\subset Y$, each $y\in K\cap\overline{\lim}f_{n_k}(V\cap\mathbf{D}^*)$ is a strong limit point for $\{f_{n_k}(V\cap\mathbf{D}^*)\}_x$.

(7) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\overline{\lim}f_{n_k}^{-1}(K_1)\cap\overline{\lim}f_{n_k}^{-1}(K_2)=\emptyset$ for each pair of disjoint compact $K_1,K_2\subset Y$.

(8) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$ there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that if $y,z\in Y$, $y\neq z$, some $W\in\Sigma(y)$, $H\in\Sigma(z)$ satisfy $\overline{\lim}f_{n_k}^{-1}(W)\cap\overline{\lim}f_{n_k}^{-1}(H)=\emptyset$.

COROLLARY 8. *The following statements are equivalent for a relatively compact complex subspace X of a complex space Y:*

(1) X is hyperbolically imbedded in Y.

(2) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$, there is a subnet $\{f_m\}$ of $\{f_n\}$ such that $\overline{\lim}f_m(V\cap\mathbf{D}^*)$ is a singleton for each $x\in\mathbf{D}$.

(3) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$ there is a subnet $\{f_m\}$ of $\{f_n\}$ such that $\{f_m(V\cap\mathbf{D}^*)\}_x$ has a strong limit point in Y for each $x\in\mathbf{D}$.

(4) For each sequence $\{f_n\}$ in $\overline{H(\mathbf{D}^*,X)}$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\overline{\lim}f_{n_k}^{-1}(K_1)\cap\overline{\lim}f_{n_k}^{-1}(K_2)=\emptyset$ for each pair of disjoint closed $K_1,K_2\subset Y$.

(5) If $\{f_n\}$ is a sequence in $\overline{H(\mathbf{D}^*,X)}$ there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that, if $y,z\in Y$, $y\neq z$, some $W\in\Sigma(y)$, $H\in\Sigma(z)$ satisfy $\overline{\lim}f_{n_k}^{-1}(W)\cap\overline{\lim}f_{n_k}^{-1}(H)=\emptyset$.

REMARK 4. In Corollaries 7 and 8, the requirement that $\mathbf{D}^*$ satisfy the given condition may be replaced by the requirement that $M-A$ satisfy the condition for every complex manifold M and divisor A on M with normal crossings. The subnets in equivalences (4) and (5) of Corollary 7 and in equivalences (2) and (3) of Corollary 8 may be taken to be sequences.

We close with "Ascoli type" theorems modeled after some of the equivalences to property κ given in previous results in the section.

THEOREM 8. *Let Y be regular (not necessarily a k-space) and let* $\Omega \subset C(X, Y)$ *satisfy* $\Omega(x)$ *is relatively compact in Y for each* $x \in X$. *The following statements are equivalent:*

(1) Ω *is evenly continuous.*

(2) For each net $\{f_\alpha\}$ *in* Ω, *there is a subnet* $\{f_{\alpha_\mu}\}$ *of* $\{f_\alpha\}$ *such that* $\overline{\lim} f_{\alpha_\mu}^{-1}(K) \cap \overline{\lim} f_{\alpha_\mu}^{-1}(M) = \emptyset$ *for each pair of closed disjoint subsets* K, M *of* Y.

(3) For each net $\{f_\alpha\}$ *in* Ω, *there is a subnet* $\{f_{\alpha_\mu}\}$ *of* $\{f_\alpha\}$ *such that for each* $x \in X$, *the net* $\{f_{\alpha_\mu}(V)\}_x$ *has a strong limit point in* Y.

PROOF. $(1) \Rightarrow (2)$. We may adapt the proof of $(1) \Rightarrow (2)$ of Theorem 6.

$(2) \Rightarrow (3)$. Assume $f_\alpha(x) \longrightarrow y$, let $W \in \Sigma(y)$ and let $H \in \Sigma(y)$ such that $\bar{H} \subset W; x \in \overline{\lim} f_{\alpha_\mu}^{-1}(\bar{H})$ for each subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$. For the subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ insured by (2), we must have $x \notin \overline{\lim} f_{\alpha_\mu}^{-1}(Y - W)$. Hence $f_{\alpha_\mu}(V) \subset W$ eventually.

$(3) \Rightarrow (1)$. Assume $x \in X, y \in Y$, a net $\{(f_\alpha, x_\alpha)\}$ in $\Omega \times X$ and $W \in \Sigma(y)$ such that $x_\alpha \longrightarrow x, f_\alpha(x_\alpha) \in Y - W$ for all α and $f_\alpha(x) \longrightarrow y$. There is no subnet $\{f_{\alpha_\mu}\}$ of $\{f_\alpha\}$ such that $\{f_{\alpha_\mu}(V)\}$ has a strong limit point.

COROLLARY 9. *Let Y be regular (not necessarily a k-space). Then* $\Omega \subset C(X, Y)$ *is relatively compact iff* $\Omega(x)$ *is relatively compact in Y for each* $x \in X$ *and* Ω *satisfies either condition (2) or condition (3) of Theorem 8.*

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