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Well-posedness for the 3-D generalized micropolar system in critical Fourier-Besov-Morrey spaces

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Abstract

In this paper, we focus on the Cauchy problem of the three-dimensional generalized incompressible micropolar system in critical Fourier-Besov-Morrey spaces. By using the Fourier localization argument and the Littlewood-Paley theory, we get the local well-posedness results and global well-posedness results with small initial data belonging to the critical Fourier-Besov-Morrey spaces.

Key words: generalized micropolar system, global well-posedness, Fourier-Besov-Morrey space.

MSC(2010): 35B65, 35Q35, 76D03.

1 Introduction

The 3-dimensional (3D) generalized incompressible micropolar system can be written as

$$\begin{cases} \partial_t u + u\cdot\nabla u + (\chi+\nu)(-\Delta)^\gamma u + \nabla\Pi - 2\chi\nabla\times w = 0, \\ \partial_t w + u\cdot\nabla w + \mu(-\Delta)^\gamma w + 4\chi w - \kappa\nabla\nabla\cdot w - 2\chi\nabla\times u = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x). \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^3$ with $t>0$, $u=u(x,t)$, $\Pi=\Pi(x,t)$ and $w=w(x,t)$ denote the velocity of the fluid, the scalar pressure and the micro-rotational velocity, respectively. The nonnegative parameters χ and ν are the kinematic viscosity, μ and κ are the vortex viscosity. The specific values of these parameters are not important, we assume that $\chi=\nu=\frac{1}{2}$ and $\mu=\kappa=1$ for simplicity.

When $\gamma=1$, the system (1.1) is the standard micropolar fluid system, which was first proposed by Eringen [1], it describes many physical phenomena which are difficult to be treated by the classical Navier-Stokes system for the numerous incompressible fluids. The system (1.1) was presented as an essential modification to the conventional Navier-Stokes equations for the purpose to better describe the motion of numerous real fluids (such as blood) by considering the fact that such fluids usually consist of rigid, randomly oriented (or spherical) particles suspended

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in a viscous medium, with the ignored deformation of fluid particles. Actually, there are several experiments indicating that solutions of the micropolar fluid system do better mimic behavior of many fluids than solutions of the Navier-Stokes equations, see [11, 12, 13]. For more physical background of the micropolar fluid system, refer to [14, 15].

The micropolar fluid system has recently attracted considerable attention and many interesting results have been obtained in the literature. For example, Galdi and Rionero [16] first considered some existence and uniqueness theorems of solution of the incompressible micropolar system. Baraka and Toumlilin [10] established that the global well-posedness and decay results for 3D generalized magneto-hydrodynamic equations in critical Fourier-Besov-Morrey spaces. In the critical Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3)$ for $1 \leq p < 6$, Chen and Miao [17] established global well-posedness of the micropolar fluid system (1.1) with small initial data.

Recently, Baraka and Toumlilin [18] showed that the uniform well-posedness and stability for fractional Navier-Stokes equations with coriolis force in critical Fourier-Besov-Morrey Spaces. The local well-posedness results ([18]) were established in $\mathcal{L}^4([0,T);\mathcal{F}\dot{\mathcal{N}}^{1-\frac{3}{2}\gamma+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})$ with $\max\{1,\frac{3-\lambda}{2}\} \leq p < \infty$ and $\frac{2}{3} < \gamma \leq \frac{2}{3} + \frac{1}{p'} + \frac{\lambda}{3p}$. We wonder that if we lower the requirement for $\frac{1}{2} < \gamma \leq 1$, does the local well-posedness still hold? the answer is positive. In this paper, by constructing the work space $\mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q})$ with $1 < \frac{2\gamma}{(2r)'} < \frac{5}{2} + \frac{\lambda-3}{2p}$, we can get the local well-posedness results with $\frac{1}{2} < \gamma \leq 1$. We also refer readers to see previous works [19] and [20] for Navier-Stokes equations in Fourier-Besov-Morrey spaces.

In this paper, we prove the well-posedness for the 3D micropolar fluid system in Fourier-Besov-Morrey spaces. To this end, we sketch the main difficulty and the strategy to overcome it.

Applying the Leray projection $\mathbb{P}$ to both sides of the first equation of (1.1) to eliminate the pressure Π, one has

$$\begin{cases} \partial_t u + (-\Delta)^\gamma u + \mathbb{P}(u\cdot\nabla u) - \nabla\times w = 0, \\ \partial_t w + (-\Delta)^\gamma w + u\cdot\nabla w + 2w - \nabla\nabla\cdot w - \nabla\times u = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x). \end{cases} \tag{1.2}$$

where $\mathbb{P} = (\delta_{ij} + R_iR_j)_{1\leq i,j\leq 3}$ denotes the Helmholtz projection onto the divergence-free vector fields. Inspired by [7], we will study the following linear system of (1.2):

$$\begin{cases} \partial_t u + (-\Delta)^\gamma u - \nabla\times w = 0, \\ \partial_t w + (-\Delta)^\gamma w + 2w - \nabla\nabla\cdot w - \nabla\times u = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x). \end{cases} \tag{1.3}$$

Taking the Fourier transform of $(1.3)_1$ and $(1.3)_2$, one obtains

$$\hat{u}_t + |\xi|^{2\gamma}\hat{u} + B(\xi)\hat{w} = 0, \tag{1.4}$$

and

$$\hat{w}_t + (|\xi|^{2\gamma} + 2)\hat{w} + B(\xi)\hat{u} + C(\xi)\hat{w} = 0, \tag{1.5}$$

where

$$B(\xi) = i\begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \text{ and } C(\xi) = \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix}.$$

Now, defining $G := (u,w)^T$, rewrite the system (1.4) and (1.5) as

$$\widehat{G}_t + A(\xi)\widehat{G} = 0, \tag{1.6}$$

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where

$$A(\xi) = \begin{bmatrix} |\xi|^{2\gamma} I & B(\xi) \\ B(\xi) & (|\xi|^{2\gamma}+2)I + C(\xi) \end{bmatrix}. \tag{1.7}$$

Then, we obtain $\widehat{G}(t) = e^{-A(\xi)t}\widehat{G}_0(\xi)$, where $\widehat{G}_0(\xi)$ is the Fourier transform of $G_0(x)$, $\widehat{G}_0 = \widehat{G}(\cdot,0) = (\hat{u}_0, \hat{w}_0)^T$.

In order to derive the mild solution of the system (1.3) clearly, we denote that

$$U(x,t) = \begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix}, \tag{1.8}$$

and

$$u \otimes U = \begin{pmatrix} u \otimes u \\ u \otimes w \end{pmatrix}, \quad \mathbb{P}\nabla \cdot (u \otimes U) \triangleq \begin{pmatrix} \mathbb{P}\nabla \cdot (u \otimes u) \\ \nabla \cdot (u \otimes w) \end{pmatrix}. \tag{1.9}$$

Then by the Duhamel principle, Fourier transform and inverse Fourier transform, the solution of system (1.3) can be reduced to finding a solution U of the following integral equation:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\mathbb{P}\nabla \cdot (u \otimes U)(\tau) d\tau, \tag{1.10}$$

where $G(t)$ denotes the semigroup corresponding to the system (1.3). The semigroup $G(t)$ shall be frequently used to prove the locally well-posedness and global well-posedness with small initial data belonging to the critical Fourier-Besov-Morrey spaces. To this end we state the main results as follows:

Theorem 1.1. *Let $0 \le \lambda < 3$, $1 \le q \le \infty$, $1 \le p < \infty$ and $(u_0, w_0) \in \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$. Suppose that γ and q satisfy that $\frac{1}{2} < \gamma < \frac{5}{2} + \frac{\lambda-3}{2p}$ and $1 \le q \le \infty$. Then, there exists a constant $\varepsilon > 0$ so that for any initial data satisfying $\nabla \cdot u_0 = 0$ and*

$$\|(u_0, w_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)} < \varepsilon,$$

then the system (1.1) has a unique global mild solution such that

$$(u,w) \in C\left([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}\right) \cap \mathcal{L}^1\left([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\right) \cap \mathcal{L}^\infty\left([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}\right)$$

and

$$\|(u,w)\|_{\mathcal{L}^1([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} + \|(u,w)\|_{\mathcal{L}^\infty([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} < 2\varepsilon.$$

Remark 1.1. *Under the conditions of Theorem 1.1, if $\gamma = \frac{1}{2}$, the above global well-posedness of the system (1.1) can also be established, provided we choose $q = 1$.*

Theorem 1.2. *Let $0 \le \lambda < 3$, $1 \le q \le \infty$, $1 < p < \infty$ and $\frac{1}{2} < \gamma \le 1$, there exists a positive time $T = T(u_0, w_0)$, depending on the profile of (u_0, w_0), for any initial data satisfying $(u_0, w_0) \in \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0$, then the system (1.1) has a unique local mild solution such that*

$$(u,w) \in C\left([0,T); \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}\right) \cap \mathcal{L}^{2r}\left([0,T); \mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q}\right),$$

where $(2r)'$ is the conjugate of $2r$ satisfying $\frac{1}{2r} + \frac{1}{(2r)'} = 1$ and $1 < \frac{2\gamma}{(2r)'} < \frac{5}{2} + \frac{\lambda-3}{2p}$.

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We outline the main steps in this paper. In Section 2, we first recall the Littlewood-Paley decomposition and definitions of the Morrey spaces and homogeneous Fourier-Besov-Morrey spaces, then present some lemmas which play an important role in the proofs of the theorems. In Section 3, we finish the linear and bilinear estimates in Fourier-Besov-Morrey spaces, which will be frequently used to prove theorem. In Section 4 , we prove the Theorems 1.1 and 1.2.

2 Preliminaries

The results presented in this section are based on homogeneous Littlewood-Paley decomposition in the Fourier variables, and we introduce Fourier transform and the Littlewood-Paley decomposition, for more details refer to [2]. Given $f(x) \in \mathcal{S}(\mathbb{R}^3)$, the Schwartz class of rapidly decreasing functions, define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\mathrm{d}x,$$

and its inverse Fourier transform:

$$\check{f}(\xi) = \mathcal{F}^{-1}f(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\mathrm{d}\xi.$$

Let φ, χ be two nonnegative functions in $\mathcal{S}(\mathbb{R}^3)$ satisfying

$$\mathrm{supp}\varphi \subset \{\xi \in \mathbb{R}^3 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad for\ \xi \neq 0,$$

$$\mathrm{supp}\chi \subset \{\xi \in \mathbb{R}^3 : |\xi| \le \frac{4}{3}\}, \quad \chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad for\ \xi \in \mathbb{R}^3.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$ by

$$\dot{\Delta}_j u = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)\hat{u}] = \mathcal{F}^{-1}(\varphi_j \hat{u}),$$

$$\dot{S}_j u = \sum_{k\le j-1} \dot{\Delta}_k u.$$

From the definition, we have that

$$\dot{\Delta}_j \dot{\Delta}_k u = 0, \quad |j-k| \ge 2 \quad and \quad \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k u) = 0, \quad |j-k| \ge 5.$$

The following Bony's decomposition (see the definition in [5]) will be applied throughout the paper:

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v),$$

where

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j v = \sum_{j'=j-1}^{j+1} \dot{\Delta}_{j'} v.$$

Now, we give the definition of the Morrey space which are a complement to the L^p spaces.

Definition 2.1. *([3, 4]) For $1 \le p < \infty$, $0 \le \lambda < 3$, the Morrey spaces $M_p^\lambda = M_p^\lambda(\mathbb{R}^3)$ is defined as the set of function $f \in L^p_{loc}(\mathbb{R}^3)$ such that*

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^3} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty,$$

where $B(x_0, r)$ denotes the ball in $\mathbb{R}^3$ with center x_0 and radius r.

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Lemma 2.1. *([3, 4]) If $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1, \lambda_2, \lambda_3 < 3$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder inequality*

$$\|fg\|_{M_{p_3}^{\lambda_3}} \le \|f\|_{M_{p_1}^{\lambda_1}} \|g\|_{M_{p_2}^{\lambda_2}}. \tag{2.11}$$

Also, for $1 \le p < \infty$ and $0 \le \lambda < 3$, we have the Young inequality

$$\|\varphi * g\|_{M_p^\lambda} \le \|\varphi\|_{L^1} \|g\|_{M_p^\lambda}, \tag{2.12}$$

for all $\varphi \in L^1$ and $g \in M_p^\lambda$.

Lemma 2.2. *([6]) Let $1 \le p_2 \le p_1 < \infty$, $0 \le \lambda_1, \lambda_2 < 3$, $\frac{3-\lambda_1}{p_1} \le \frac{3-\lambda_2}{p_2}$, and let γ be a multi-index. If $\mathrm{supp}\hat{f} \subset \{|\xi| \le A2^j\}$ for $A > 0$ then there is a constant $C > 0$ independent of f and j such that*

$$\|(i\xi)^\gamma \hat{f}\|_{M_{p_2}^{\lambda_2}} \le C 2^{j|\gamma| + j(\frac{3-\lambda_2}{p_2} - \frac{3-\lambda_1}{p_1})} \|\hat{f}\|_{M_{p_1}^{\lambda_1}}. \tag{2.13}$$

Lemma 2.3. *([8, 9]) Let X be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \mapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \le \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. If there exists a constant $\varepsilon > 0$ such that $\|y\|_X < \varepsilon < \frac{1}{4\eta}$, then the equation $x = y + B(x,x)$ exists a unique solution x in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$ and $\|x'\|_X \le 2\varepsilon$, then

$$\|x - x'\|_X \le \frac{1}{1 - 4\varepsilon\eta} \|y - y'\|_X.$$

Definition 2.2. *(Homogeneous Fourier-Besov-Morrey spaces) Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q \le \infty$ and $0 \le \lambda < 3$, the spaces $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^3)$ denotes a set of all $u \in \mathcal{Z}'(\mathbb{R}^3)$ such that*

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^3)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{M_p^\lambda} \right\}^{\frac{1}{q}} < \infty \quad \text{for } q < \infty,$$

with appropriate modifications for l^q-norm when $q = \infty$. The space $\mathcal{Z}'(\mathbb{R}^3)$ denotes the topological dual of the space $\mathcal{Z}'(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); \partial^\alpha \hat{f}(0) = 0 \text{ for every multi} - \text{index } \alpha\}$, and can be identified to the quotient spaces $\mathcal{S}'(\mathbb{R}^3)/\mathcal{P}$, where $\mathcal{P}$ represents the set of all polynomials on $\mathbb{R}^3$.

Lemma 2.4. *We have the equivalence of norms that $\|\Lambda^s u\|_{\mathcal{F}\dot{\mathcal{N}}^{s'}_{p,\lambda,q}} \le C\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{s+s'}_{p,\lambda,q}}$, for $s \in \mathbb{R}$, where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $C > 0$ is a pure constant .*

Proof. Using the definition of $\mathcal{F}\dot{\mathcal{N}}^{s'}_{p,\lambda,q}(\mathbb{R}^3)$, and $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we have

$$\begin{aligned} \|\Lambda^s u\|_{\mathcal{F}\dot{\mathcal{N}}^{s'}_{p,\lambda,q}} &= \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs'} \|\widehat{\dot{\Delta}_j \Lambda^s u}\|^q_{M_p^\lambda} \right\}^{\frac{1}{q}} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs'} \||\xi|^s \widehat{\dot{\Delta}_j u}\|^q_{M_p^\lambda} \right\}^{\frac{1}{q}} \\ &\le C \left\{ \sum_{j \in \mathbb{Z}} 2^{jqs'} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{M_p^\lambda} \right\}^{\frac{1}{q}} \le C\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{s'+s}_{p,\lambda,q}}. \end{aligned}$$

□

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Definition 2.3. *Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q, \rho \le \infty$, $0 \le \lambda < 3$ and $T \in (0, \infty]$. The space-time norm is defined on $u(t,x)$ by*

$$\|u(t,x)\|_{\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \left\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{\mathcal{L}^\rho([0,T);M_p^\lambda)} \right\}^{\frac{1}{q}},$$

and denote by $\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$ the set of distributions in the space $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^3)/\mathcal{P}$ with finite $\|\cdot\|_{\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}$ norm.

3 Linear and bilinear estimates

Next, we will prove Theorems 1.1 and 1.2. As preparations, we give the following lemmas, which shall be frequently used to prove the main theorem. Using the same arguments as in [8] and observing the structure of $A(\xi)$, it can be shown the following Lemma 3.1.

Lemma 3.1. *([8]) Let $A(\xi)$ be defined in (1.7), then we have*

$$\|e^{-A(\xi)t}\| \le e^{-C(\gamma)|\xi|^{2\gamma}t}, \ \forall t > 0, \ \xi \in \mathbb{R}^3. \tag{3.14}$$

For some constant $C(\gamma) = 1 - \frac{1}{2\gamma} > 0$ independent of t and ξ, where $\|\cdot\|$ denotes the maximum value norm.

Lemma 3.2. *Let $0 < T \le \infty$, $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \lambda < 3$ and $U_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$. Then there exists a constant $C > 0$ such that*

$$\|G(t)U_0\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}, \tag{3.15}$$

$$\|G(t)U_0\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}, \tag{3.16}$$

$$\|G(t)U_0\|_{\mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q})} \le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}. \tag{3.17}$$

Proof. To prove the first inequality (3.15), from Definition 2.3, it is easy to see that

$$\begin{aligned}
\|G(t)U_0\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} &\le C\left(\sum_{j\in\mathbb{Z}} 2^{jq(4+\frac{\lambda-3}{p})} \|e^{-t2^{2\gamma j}}\widehat{\dot{\Delta}_j U_0}\|^q_{L^1([0,T);M_p^\lambda)}\right)^{\frac{1}{q}} \\
&\le C\left\{\sum_{j\in\mathbb{Z}} 2^{jq(4+\frac{\lambda-3}{p})} \left(\int_0^T e^{-t2^{2\gamma j}} \|\widehat{\dot{\Delta}_j U_0}\|_{M_p^\lambda} dt\right)^q\right\}^{\frac{1}{q}} \\
&\le C\left(\sum_{j\in\mathbb{Z}} 2^{jq(4+\frac{\lambda-3}{p})} 2^{-2\gamma jq} \|\widehat{\dot{\Delta}_j U_0}\|^q_{M_p^\lambda}\right)^{\frac{1}{q}} \\
&\le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}.
\end{aligned}$$

Similarly, for (3.16), using $e^{-t2^{2\gamma j}} \le 1$, we have

$$\begin{aligned}
\|G(t)U_0\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} &\le \left(\sum_{j\in\mathbb{Z}} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \|\widehat{\dot{\Delta}_j U_0}\|^q_{M_p^\lambda}\right)^{\frac{1}{q}} \\
&\le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}.
\end{aligned}$$

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To estimate the inequality (3.17), it suffices to write that

$$\begin{aligned}\|G(t)U_0\|_{\mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q})} &\le C\left(\sum_{j\in\mathbb{Z}}2^{jq(4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'})}\|e^{-t2^{2\gamma j}}\widehat{\dot{\Delta}_jU_0}\|^q_{L^{2r}([0,T);M_p^\lambda)}\right)^{\frac{1}{q}}\\ &\le C\left\{\sum_{j\in\mathbb{Z}}2^{jq(4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'})}\left(\int_0^T e^{-2rt2^{2\gamma j}}\|\widehat{\dot{\Delta}_jU_0}\|^{2r}_{M_p^\lambda}dt\right)^{\frac{q}{2r}}\right\}^{\frac{1}{q}}\\ &\le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}},\end{aligned}$$

which finish the proof. □

Lemma 3.3. *Let $0<T\le\infty$, $s\in\mathbb{R}$, $1\le p<\infty$, $1\le q,\rho\le\infty$, $1\le r\le\rho$, $0\le\lambda<3$ and $f\in\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$. Then there exists a constant $C>0$ such that*

$$\|\int_0^t G(t-\tau)f(\tau)d\tau\|_{\mathcal{L}^\rho\left([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}\right)}\le C\|f\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}^{s-2\gamma(1+\frac{1}{\rho}-\frac{1}{r})}_{p,\lambda,q})}.$$

Proof. Set $1+\frac{1}{\rho}=\frac{1}{\tilde{\rho}}+\frac{1}{r}$. By Definition 2.3 and the Young's inequality, we obtain that

$$\begin{aligned}&\|\int_0^t G(t-\tau)f(\tau)d\tau\|_{\mathcal{L}^\rho\left([0,T);\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}\right)}\\ &\le\left\{\sum_{j\in\mathbb{Z}}2^{jqs}\left(\int_0^T\|\int_0^t e^{-(t-\tau)2^{2\gamma j}}\widehat{\dot{\Delta}_jf}(\tau)d\tau\|^\rho_{M_p^\lambda}dt\right)^{\frac{q}{\rho}}\right\}^{\frac{1}{q}}\\ &\le\left\{\sum_{j\in\mathbb{Z}}2^{jqs}\left(\int_0^T e^{-\tilde{\rho}t2^{2\gamma j}}dt\right)^{\frac{q}{\tilde{\rho}}}\|\widehat{\dot{\Delta}_jf}(\tau)d\tau\|^q_{L^r([0,T);M_p^\lambda)}dt\right\}^{\frac{1}{q}}\\ &\le C\left\{\sum_{j\in\mathbb{Z}}2^{jq(s-2\gamma(1+\frac{1}{\rho}-\frac{1}{r}))}\|\widehat{\dot{\Delta}_jf}(\tau)d\tau\|^q_{L^r([0,T);M_p^\lambda)}dt\right\}^{\frac{1}{q}}\\ &\le C\|f\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}^{s-2\gamma(1+\frac{1}{\rho}-\frac{1}{r})}_{p,\lambda,q})}.\end{aligned}$$

Thus the proof is completed. □

Lemma 3.4. *Let $0<T\le\infty$, $1\le p<\infty$, $\frac{1}{2}<\gamma<\frac{5+\frac{\lambda-3}{p}}{4-\frac{2}{\sigma}}$, $1\le q\le\infty$, $1\le\sigma\le\infty$ and $0\le\lambda<3$, then there exists a constant $C>0$ such that*

$$\begin{aligned}\|uv\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})}&\le C\|u\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|v\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\ &\quad+C\|v\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|u\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.\end{aligned}$$

Remark 3.1. *The above Lemma 3.4 also hold true for the case $\gamma=\frac{1}{2}$, provided we choose $q=1$.*

Proof. Applying Bony's decomposition, we have

$$\begin{aligned}\dot{\Delta}_j(uv)&=\sum_{|k-j|\le4}\dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_kv)+\sum_{|k-j|\le4}\dot{\Delta}_j(\dot{S}_{k-1}v\dot{\Delta}_ku)+\sum_{k\ge j-3}\dot{\Delta}_j(\dot{\Delta}_ku\tilde{\dot{\Delta}}_kv)\\ &:=I_1+I_2+I_3.\end{aligned}$$

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By the triangular inequality, one finds that

$$\begin{aligned}\|uv\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})} &\le C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}\|\hat{I}_1\|^q_{L^\sigma([0,T);M^\lambda_p)}\right)^{\frac{1}{q}}\\ &+C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}\|\hat{I}_2\|^q_{L^\sigma([0,T);M^\lambda_p)}\right)^{\frac{1}{q}}\\ &+C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}\|\hat{I}_3\|^q_{L^\sigma([0,T);M^\lambda_p)}\right)^{\frac{1}{q}}\\ &:= J_1+J_2+J_3.\end{aligned} \tag{3.18}$$

We firstly evaluate J_1. By using Young's inequality (2.12) and Hölder inequality, one finds

$$\begin{aligned}&\|\hat{I}_1\|_{L^\sigma([0,T);M^\lambda_p)}\\ &\le \sum_{|k-j|\le 4}\|\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\|_{L^\sigma([0,T);M^\lambda_p)}\\ &\le \sum_{|k-j|\le 4}\|\hat{v}_k\|_{L^\sigma([0,T);M^\lambda_p)}\sum_{l\le k-2}2^{l(3+\frac{\lambda-3}{p})}\|\hat{u}_l\|_{L^\infty([0,T);M^\lambda_p)}\\ &\le \sum_{|k-j|\le 4}\|\hat{v}_k\|_{L^\sigma([0,T);M^\lambda_p)}\left(\sum_{l\le k-2}2^{lq(4-2\gamma+\frac{\lambda-3}{p})}\|\hat{u}_l\|^q_{L^\infty([0,T);M^\lambda_p)}\right)^{\frac{1}{q}}\left(\sum_{l\le k-2}2^{(2\gamma-1)lq'}\right)^{\frac{1}{q'}}\\ &\le C\sum_{|k-j|\le 4}2^{(2\gamma-1)k}\|\hat{v}_k\|_{L^\sigma([0,T);M^\lambda_p)}\|u\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.\end{aligned}$$

Taking l^q-norm, we obtain

$$\begin{aligned}J_1&\le C\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{|k-j|\le 4}2^{(j-k)(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}2^{(4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})k}\|\hat{v}_k\|_{L^\sigma([0,T);M^\lambda_p)}\right)^q\right\}^{\frac{1}{q}}\|u\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\ &\le C\|v\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|u\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.\end{aligned}$$

Similarly, we have

$$J_2\le C\|u\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|v\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.$$

To estimate I_3, we use a different approach. First, we use the Young's inequality (2.12) to get

$$\begin{aligned}&\|\hat{I}_3\|_{L^\sigma([0,T);M^\lambda_p)}\\ &\le \sum_{k\ge j-3}\sum_{|k-i|\le 1}\|\hat{u}_k\|_{L^\sigma([0,T);M^\lambda_p)}2^{i(3+\frac{\lambda-3}{p})}\|\hat{v}_i\|_{L^\infty([0,T);M^\lambda_p)}\\ &\le C\sum_{k\ge j-3}\left(\sum_{|k-i|\le 1}2^{(i-k)(2\gamma-1)}2^{i(4-2\gamma+\frac{\lambda-3}{p})}\|\hat{v}_i\|_{L^\infty([0,T);M^\lambda_p)}\right)2^{(2\gamma-1)k}\|\hat{u}_k\|_{L^\sigma([0,T);M^\lambda_p)}\\ &\le C\sum_{k\ge j-3}2^{(2\gamma-1)k}\|\hat{u}_k\|_{L^\sigma([0,T);M^\lambda_p)}\|v\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.\end{aligned}$$

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Then we have

$$\begin{aligned}
&2^{j(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}\|\hat{I}_3\|_{L^\sigma([0,T);M_p^\lambda)} \\
&\leq C\|v\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}})}\sum_{k\geq j-3}2^{(j-k)(5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})}2^{(4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p})k}\|\hat{u}_k\|_{L^\sigma([0,T);M_p^\lambda)}.
\end{aligned}$$

Since $\gamma < \frac{5+\frac{\lambda-3}{p}}{4-\frac{2}{\sigma}}$, we have $5-4\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p} > 0$. By taking l^q-norm on both sides in the above estimate, and then we apply Young's inequality for series to get

$$J_3 \leq C\|u\|_{\mathcal{L}^\sigma([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{2\gamma}{\sigma}+\frac{\lambda-3}{p}})}\|v\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}})}.$$

Thus the proof is complete. □

Lemma 3.5. *Let $0 < T \leq \infty$, $1 < p < \infty$, $\gamma > \frac{1}{2}$, $1 \leq q \leq \infty$ and $0 \leq \lambda < 3$, and set*

$$Y = \mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}),$$

there exists a constant $C = C(p,q) > 0$ depending on p, q such that

$$\|uv\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}})} \leq C\|u\|_Y\|v\|_Y.$$

where $(2r)'$ is the conjugate of $2r$ satisfying $\frac{1}{2r}+\frac{1}{(2r)'} = 1$ for $1 < r < \infty$. We choose r satisfying $1-\frac{1}{2\gamma}(\frac{5}{2}+\frac{\lambda-3}{2p}) < \frac{1}{2r} < 1-\frac{1}{2\gamma}$ such that $1 < \frac{2\gamma}{(2r)'} < \frac{5}{2}+\frac{\lambda-3}{2p}$.

Proof. Using the same methods as in (3.18), we have

$$\begin{aligned}
\|uv\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}})} &\leq C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_1\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}} \\
&+ C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_2\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}} \\
&+ C\left(\sum_{j\in\mathbb{Z}}2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_3\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}}.
\end{aligned}$$

We evaluate the above three terms separately, first, using Young's inequality (2.12) and Hölder inequality, one finds

$$\begin{aligned}
&\|\hat{I}_1\|_{L^r([0,T);M_p^\lambda)} \\
&\leq \sum_{|k-j|\leq 4}\|\hat{v}_k\|_{L^{2r}([0,T);M_p^\lambda)}\sum_{l\leq k-2}\|\hat{u}_l\|_{L^{2r}([0,T);L^1)} \\
&\leq \sum_{|k-j|\leq 4}\|\hat{v}_k\|_{L^{2r}([0,T);M_p^\lambda)}\sum_{l\leq k-2}2^{l(3+\frac{\lambda-3}{p})}\|\hat{u}_l\|_{L^{2r}([0,T);M_p^\lambda)} \\
&\leq \sum_{|k-j|\leq 4}\|\hat{v}_k\|_{L^{2r}([0,T);M_p^\lambda)}\left(\sum_{l\leq k-2}2^{lq(4-\frac{2\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{u}_l\|^q_{L^{2r}([0,T);M_p^\lambda)}\right)^{\frac{1}{q}}\left(\sum_{l\leq k-2}2^{(\frac{2\gamma}{(2r)'}-1)lq'}\right)^{\frac{1}{q'}} \\
&\leq C\sum_{|k-j|\leq 4}2^{(\frac{2\gamma}{(2r)'}-1)k}\|\hat{v}_k\|_{L^{2r}([0,T);M_p^\lambda)}\|u\|_Y.
\end{aligned}$$

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Taking l^q-norm, we obtain

$$\left(\sum_{j\in\mathbb{Z}} 2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_1\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}}$$
$$\leq C\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{|k-j|\leq 4} 2^{(j-k)(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}2^{(4-\frac{2\gamma}{(2r)'}+\frac{\lambda-3}{p})k}\|\hat{v}_k\|_{L^{2r}([0,T);M_p^\lambda)}\right)^q\right\}^{\frac{1}{q}}\|u\|_Y$$
$$\leq C\|v\|_Y\|u\|_Y.$$

Similarly, we have

$$\left(\sum_{j\in\mathbb{Z}} 2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_2\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}} \leq C\|u\|_Y\|v\|_Y.$$

To estimate I_3, we use a different approach. First, we use the Young's inequality (2.12) to get

$$\|\hat{I}_3\|_{L^r([0,T);M_p^\lambda)}$$
$$\leq \sum_{k\geq j-3}\sum_{|k-i|\leq 1}\|\hat{u}_k\|_{L^{2r}([0,T);M_p^\lambda)}2^{i(3+\frac{\lambda-3}{p})}\|\hat{v}_i\|_{L^{2r}([0,T);M_p^\lambda)}$$
$$\leq C\sum_{k\geq j-3}\left(\sum_{|k-i|\leq 1}2^{(1-\frac{2\gamma}{(2r)'})(k-i)}2^{i(4-\frac{2\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{v}_i\|_{L^{2r}([0,T);M_p^\lambda)}\right)2^{(\frac{2\gamma}{(2r)'}-1)k}\|\hat{u}_k\|_{L^{2r}([0,T);M_p^\lambda)}$$
$$\leq C\sum_{k\geq j-3}2^{(\frac{2\gamma}{(2r)'}-1)k}\|\hat{u}_k\|_{L^{2r}([0,T);M_p^\lambda)}\|v\|_Y.$$

Then we have

$$2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_3\|_{L^r([0,T);M_p^\lambda)}$$
$$\leq C\|v\|_Y\sum_{k\geq j-3}2^{(j-k)(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}2^{(4-\frac{2\gamma}{(2r)'}+\frac{\lambda-3}{p})k}\|\hat{u}_k\|_{L^{2r}([0,T);M_p^\lambda)}.$$

Since $1<p<\infty$ and $1<\frac{2\gamma}{(2r)'}<\frac{5}{2}+\frac{\lambda-3}{2p}$, we have $5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}>0$. By taking l^q-norm on both sides in the above estimate, and then we apply Young's inequality for series to get

$$\left(\sum_{j\in\mathbb{Z}} 2^{jq(5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p})}\|\hat{I}_3\|^q_{L^r([0,T);M_p^\lambda)}\right)^{\frac{1}{q}} \lesssim \|u\|_Y\|v\|_Y.$$

Thus the proof is complete. □

4 Proof of Theorems 1.1 and 1.2

To ensure the existence of a global mild solution with small initial data of the system (1.1), we will use the linear and bilinear estimates that we have established in Sections 3.

Proof of Theorem 1.1. Let $0<T\leq\infty$, $1\leq p<\infty$, $1\leq q\leq\infty$, $0\leq\lambda<3$, and set

$$X:=\mathcal{L}^1\left([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\right)\cap\mathcal{L}^\infty\left([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}\right),$$

with the norm

$$\|u\|_X = \|u\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} + \|u\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.$$

Then, we begin with the integral equation

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u\otimes U)(\tau)d\tau, \tag{4.19}$$

and we consider the bilinear operator B given by

$$B(U_1,U_2) = \int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u_1\otimes U_2)(\tau)d\tau, \tag{4.20}$$

where

$$U_1(x,t) = \begin{pmatrix} u_1(x,t) \\ w_1(x,t)\end{pmatrix}, \quad U_2(x,t) = \begin{pmatrix} u_2(x,t) \\ w_2(x,t)\end{pmatrix}.$$

By Lemma 3.3, we have

$$\|\int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u_1\otimes U_2)d\tau\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|\mathbb{P}\nabla\cdot(u_1\otimes U_2)\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})},$$

and

$$\|\int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u_1\otimes U_2)d\tau\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|\mathbb{P}\nabla\cdot(u_1\otimes U_2)\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.$$

Then, according to Lemmas 2.4, 3.3 and 3.4, we get

$$\begin{aligned}\|B(U_1,U_2)\|_X &\le C\|\mathbb{P}\nabla\cdot(u_1\otimes U_2)\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\le C\|u_1\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|U_2\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\quad + C\|U_2\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|u_1\|_{\mathcal{L}^1([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\le C_2\|U_1\|_X\|U_2\|_X.\end{aligned} \tag{4.21}$$

Lemma 3.2 yields

$$\|G(t)U_0\|_X \le C_1\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}. \tag{4.22}$$

If $\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}} < \frac{1}{4C_1C_2}$, then by Lemma 2.3, there has a unique global solution $U\in X$ satisfying

$$\|U\|_X \le 2C_1\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}.$$

Thus the proof of the global existence is complete.

As for continuity, by using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned}&\|u(t_1)-u(t_2)\|^q_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}} \\ &\le \sum_{j\le N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})}\|\hat{u}_j(t_1)-\hat{u}_j(t_2)\|^q_{M^\lambda} + 2\sum_{j>N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})}\|\hat{u}_j(t)\|^q_{L^\infty([0,T);M_p^\lambda)},\end{aligned}$$

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where we denote $\hat{u}_j = \varphi_j \hat{u}$. For any small constant $\varepsilon > 0$, let N be large enough so that

$$\sum_{j>N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \|\hat{u}_j(t)\|^q_{L^\infty([0,T);M_p^\lambda)} \le \frac{\varepsilon}{4}.$$

Using the same arguments as in [21], we get

$$\begin{aligned}
&\sum_{j\le N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \|\hat{u}_j(t_1) - \hat{u}_j(t_2)\|^q_{M_p^\lambda} \\
&\le C|t_1 - t_2|^q \sum_{j\le N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \sup_{t_1\le t\le t_2} \|\widehat{(\partial_t u)}_j\|^q_{M_p^\lambda} \\
&\le C|t_1 - t_2|^q \sum_{j\le N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \\
&\quad \times \sup_{t_1\le t\le t_2} \left(\|\widehat{(-\Delta)^\gamma} u_j\|^q_{M_p^\lambda} + \|\mathbb{P}\nabla\cdot \widehat{(u_j\otimes u_j)}\|^q_{M_p^\lambda} + \|\widehat{\nabla\times w_j}\|^q_{M_p^\lambda} \right) \\
&\le C|t_1 - t_2|^q \left(2^{2\gamma q N} \|u\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} + 2^{2\gamma q N} \|\nabla\cdot(u\otimes u)\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-4\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \right) \\
&\quad + C|t_1 - t_2|^q 2^{Nq} \|w\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.
\end{aligned} \tag{4.23}$$

Applying Lemma 2.4 and taking $\sigma = \infty$ in Lemma 3.4, we have

$$\begin{aligned}
&\|\nabla\cdot(u\otimes u)\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-4\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\le C\|u\otimes u\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-4\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\le C\|u\|^{2q}_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})}.
\end{aligned} \tag{4.24}$$

Substituting (4.24) into (4.23) to obtain that

$$\begin{aligned}
&\sum_{j\le N} 2^{jq(4-2\gamma+\frac{\lambda-3}{p})} \|\hat{u}_j(t_1) - \hat{u}_j(t_2)\|^q_{M_p^\lambda} \\
&\le C|t_1 - t_2|^q \left(2^{2\gamma q N} \|u\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} + 2^{2\gamma q N} \|u\|^{2q}_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \right) \\
&\quad + C|t_1 - t_2|^q 2^{Nq} \|w\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\le C|t_1 - t_2|^q \left(\|u_0\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} + \|u_0\|^{2q}_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \|w_0\|^q_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q})} \right) \\
&\le \left(\frac{\varepsilon}{2}\right)^q
\end{aligned}$$

provided $|t_1 - t_2| < \delta$ for some a small enough δ. Thus, $\|u(t_1) - u(t_2)\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}} \le \varepsilon$, we obtain the continuity of u in time t. Similarly, we use the same discussion to get the continuity of w in time t. Hence $(u, w) \in \mathcal{C}\left([0,T); \mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}\right)$. Thus the proof of the Theorem 1.1 is complete. □

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Proof of Theorem 1.2. Now, we will use Lemma 3.5 to prove the local existence. We set

$$Y = \mathcal{L}^{2r}([0,T); \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}),$$

Similarly, according to Lemmas 2.4, 3.3 and 3.5, we get

$$\begin{aligned}\|B(U_1,U_2)\|_Y &\le C\|\mathbb{P}\nabla\cdot(u_1\otimes U_2)\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}})}\\ &\le C\|u_1U_2\|_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}})}\\ &\le C\|U_1\|_Y\|U_2\|_Y.\end{aligned}\tag{4.25}$$

Lemma 3.2 yields

$$\|G(t)U_0\|_Y \le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}}.\tag{4.26}$$

We divide the initial data U_0 into two terms:

$$U_0 = \mathcal{F}^{-1}(\chi_{B(0,\delta)}\widehat{U}_0) + \mathcal{F}^{-1}(\chi_{B^C(0,\delta)}\widehat{U}_0) := U_{0,1} + U_{0,2},$$

where $\delta = \delta(U_0) > 0$ is a real number to be determined later. Since $U_0 \in \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}$, by (4.26) we choose δ large enough satisfying

$$\|G(t)U_{0,2}\|_Y \le \frac{\varepsilon}{2},$$

thus, we get

$$\|G(t)U_0\|_Y \le \|G(t)U_{0,1}\|_Y + \frac{\varepsilon}{2}.$$

For the first term $U_{0,1}$, applying l^q-norm to it and using the fact that $|\xi| \approx 2^j \le \delta$ for every $j \in \mathbb{Z}$, we have

$$\begin{aligned}\|G(t)U_{0,1}\|_Y &\le \left\{\sum_{j\in\mathbb{Z}} 2^{jq(4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'})}\|e^{-t|\xi|^{2\gamma}}\widehat{\dot{\Delta}_jU_{0,1}}\|^q_{L^{2r}([0,T);M_p^\lambda)}\right\}^{\frac{1}{q}}\\ &\le \left\{\sum_{j\in\mathbb{Z}} 2^{jq(4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'})}\|\widehat{\dot{\Delta}_jU_0}\|^q_{L^{2r}([0,T);M_p^\lambda)}\right\}^{\frac{1}{q}}\\ &\le C_3\delta^{\frac{2\gamma}{2r}}T^{\frac{1}{2r}}\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}}.\end{aligned}$$

Therefore, if we choose T such that

$$C_3\delta^{\frac{2\gamma}{2r}}T^{\frac{1}{2r}}\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}} \le \frac{\varepsilon}{2},\tag{4.27}$$

then we have

$$\|G(t)U_0\|_Y \le \varepsilon.$$

Applying Lemma 2.3, we get a fixed point in the closed ball $\overline{B}(0,2\varepsilon) = \{x \in Y : \|x\|_Y \le 2\varepsilon\}$. Thus for arbitrary U_0 in $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}$, (4.19) has a unique local solution in Y on $[0,T)$, where

$$T \le \left(\frac{\varepsilon}{2C_3\delta^{\frac{2\gamma}{2r}}\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{4-2\gamma+\frac{\lambda-3}{p}}}}\right)^{2r}.$$

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We can use the same discussion to get the continuity of (u,w) in time t. For the integral equation

$$U(t)=G(t)U_0-\int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u\otimes U)(\tau)d\tau,$$

by Lemma 3.2, we have that

$$\|G(t)U_0\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-2\gamma}_{p,\lambda,q})}\le C\|U_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\gamma+\frac{\lambda-3}{p}}_{p,\lambda,q}}.$$

Then, we estimate the bilinear part. By Definition 2.3, Lemma 3.5 and the Young's inequality, set $1+\frac{1}{\infty}=\frac{1}{r'}+\frac{1}{r}$, we obtain that

$$\begin{aligned}
&\|\int_0^t G(t-\tau)\mathbb{P}\nabla\cdot(u\otimes U)(\tau)d\tau\|_{\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-2\gamma}_{p,\lambda,q})}\\
&\le\left\{\sum_{j\in\mathbb{Z}}2^{jq(4+\frac{\lambda-3}{p}-2\gamma)}\left(\int_0^T e^{-r't2^{2\gamma j}}dt\right)^{\frac{q}{r'}}\|\dot{\Delta}_j\mathbb{P}\widehat{\nabla\cdot(u}\otimes U)(\tau)d\tau\|^q_{L^r([0,T);M_p^\lambda)}dt\right\}^{\frac{1}{q}}\\
&\le C\left\{\sum_{j\in\mathbb{Z}}2^{jq(4+\frac{\lambda-3}{p}-2\gamma-2\gamma(1-\frac{1}{r}))}\|\dot{\Delta}_j\mathbb{P}\widehat{\nabla\cdot(u}\otimes U)(\tau)d\tau\|^q_{L^r([0,T);M_p^\lambda)}dt\right\}^{\frac{1}{q}}\\
&\le C||u\otimes U||_{\mathcal{L}^r([0,T);\mathcal{F}\dot{\mathcal{N}}^{5-\frac{4\gamma}{(2r)'}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C||U||_{\mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q})}||U||_{\mathcal{L}^{2r}([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-\frac{2\gamma}{(2r)'}}_{p,\lambda,q})}.
\end{aligned}$$

Therefore, $(u,w)\in\mathcal{L}^\infty([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-2\gamma}_{p,\lambda,q})$. Same as the proof of the Theorem 1.1, we have the continuity of (u,w) in time t. Hence $(u,w)\in\mathcal{C}\left([0,T);\mathcal{F}\dot{\mathcal{N}}^{4+\frac{\lambda-3}{p}-2\gamma}_{p,\lambda,q}\right)$. Thus the proof of the Theorem 1.2 is complete. □

Declarations

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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