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QUASIHOMOGENEOUS TOEPLITZ OPERATORS WITH INTEGRABLE SYMBOLS ON THE HARMONIC BERGMAN SPACE

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Abstract

In this paper, we completely determine the commutativity of two Toeplitz operators on the harmonic Bergman space with integrable quasihomogeneous symbols, one of which is of the form $e^{ik\theta}r^m$. As an application, the problem of when their product is again a Toeplitz operator is solved. In particular, Toeplitz operators with bounded symbols on the harmonic Bergman space commute with $T_{e^{ik\theta}r^m}$ only in trivial cases, which appears quite different from results on analytic Bergman space in Čučković and Rao ['Mellin transform, monomial symbols, and commuting Toeplitz operators', *J. Funct. Anal.* **154** (1998), 195–214].

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1. Introduction

Let dA denote the Lebesgue area measure on the unit disc D, normalised so that the measure of D equals 1. $L^2(D, dA)$ is the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\, dA(z).$$

The harmonic Bergman space L^2_h is the closed subspace of $L^2(D, dA)$ consisting of all complex-valued L^2-harmonic functions on D. We will write Q for the orthogonal projection from $L^2(D, dA)$ onto L^2_h. It can be expressed as an integral operator:

$$Qf(z) = \int_D \Big(\frac{1}{(1 - z\bar{w})^2} + \frac{1}{(1 - \bar{z}w)^2} - 1\Big) f(w)\, dA(w) \quad (z \in D)$$

for $f \in L^2(D, dA)$. For $u \in L^1(D, dA)$, we define an operator T_u with symbol u on L^2_h by

$$T_u f = Q(uf) \tag{1.1}$$

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for $f \in L_h^2$. This operator is always densely defined on the polynomials and not bounded in general.

We are concerned with the problem of characterising symbols of commuting Toeplitz operators acting on L_h^2. The corresponding problem has been well studied for many years on the classical Hardy space and the analytic Bergman space; for example, see [2–4, 7, 10, 13, 14, 17, 20]. Recently, there has been an increasing interest in the present harmonic Bergman space case; see [5, 6, 8, 18, 19] and the references therein.

To state our main results we recall the following definitions, following [16].

Definition 1.1. Let $F \in L^1(D, dA)$.

(i) We say that F is a T-function if the equation (1.1), with $u = F$, defines a bounded operator on L_h^2.

(ii) If F is a T-function, we write T_F for the continuous extension of the operator defined by (1.1). We say that T_F is a Toeplitz operator if and only if T_F is defined in this way.

(iii) If there is an $r \in (0, 1)$ such that F is (essentially) bounded on the annulus $\{z : r < |z| < 1\}$, then we say that F is 'nearly bounded'.

Generally, the T-functions form a proper subset of $L^1(D, dA)$ which contains all bounded and 'nearly bounded' functions.

A function f is said to be quasihomogeneous of degree $k \in \mathbb{Z}$ if

$$f(re^{i\theta}) = e^{ik\theta}\varphi(r),$$

where φ is a radial function. In this case the associated Toeplitz operator T_f is called a quasihomogeneous Toeplitz operator of degree k. By a straightforward deduction, one can see that $e^{ik\theta}\varphi(r)$ is a T-function if and only if $\varphi(r)$ is a T-function.

In this note, we will investigate the commutativity of $T_{e^{ik_1\theta}r^m}$ and $T_{e^{ik_2\theta}\varphi(r)}$ on L_h^2, with both $e^{ik_1\theta}r^m$ and $e^{ik_2\theta}\varphi(r)$ being T-functions. Our first main result is the following theorem.

Theorem 1.2. *Let $k_1, k_2 \in \mathbb{Z}$ and let m be a real number greater than or equal to -1. Then for a T-function $e^{ik_2\theta}\varphi(r)$ on D, $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$ if and only if one of the following conditions holds:*

(1) *either $e^{ik_1\theta}r^m$ or $e^{ik_2\theta}\varphi$ is constant;*

(2) *both $e^{ik_1\theta}r^m$ and $e^{ik_2\theta}\varphi$ are radial;*

(3) *$e^{ik_1\theta}r^m$ and $e^{ik_2\theta}\varphi$ are linearly dependent;*

(4) *$k_1k_2 = -1$ and $\varphi = C(((m+1)/2)r^{-1} - ((m-1)/2)r)$ for some constant C.*

This has been partially proved in [11, Theorem 3.8], in the case when $|k_1| \leq |k_2|$. Also, with the additional hypothesis that the symbols are bounded, Louhichi and Zakariasy [18] proved some special cases of the above theorem, in the case when $0 < k_1 \leq k_2$. The remaining case, that is, when $|k_1| > |k_2|$, was left open.

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In this note, with extra effort, we shall prove the following statement.

With the same assumption as in Theorem 1.2, if $|k_1| > |k_2|$, *then* $T_{e^{ik_1\theta}r^m}$ *and* $T_{e^{ik_2\theta}\varphi(r)}$ *commute only when* $\varphi(r) = 0$.

This, together with [11, Theorem 3.8], completes Theorem 1.2. In [12], we proved a quite unexpected result: if the product of two quasihomogeneous Toeplitz operators on L_h^2 is equal to a Toeplitz operator, then they must be commutative. So, as an application of Theorem 1.2, we can discuss when $T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi(r)}$ is a Toeplitz operator. In fact, this problem, in some special cases, for example: $|k_1| \le |k_2|$, or $k_1k_2 > 0$ or both their symbols are bounded, has been discussed in [12]. But, now, we can solve this problem in all cases.

Theorem 1.3. *Let* $k_1, k_2 \in \mathbb{Z}$ *and let* m *be a real number greater than or equal to* -1. *Then, for a T-function* $e^{ik_2\theta}\varphi(r)$ *on* D, *there exists a T-function* ψ *such that* $T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi} = T_\psi$ *if and only if one of the following conditions holds:*

(1) *either* $e^{ik_1\theta}r^m$ *or* $e^{ik_2\theta}\varphi$ *is constant;*

(2) *both* $e^{ik_1\theta}r^m$ *and* $e^{ik_2\theta}\varphi$ *are radial. In this case,* ψ *also is a radial T-function and such that*

$$\psi(r) = \varphi(r) - mr^m \int_r^1 \frac{\varphi(t)}{t^{m+1}}\,dt;$$

(3) $k_1k_2 = -1$ *and* $\varphi = C(((m+1)/2)r^{-1} - ((m-1)/2)r)$ *for some constant* C. *In this case* $\psi = C$.

Čučković and Rao [7] characterised all Toeplitz operators on an analytic Bergman space which commute with $T_{e^{ik\theta}r^m}$ for $(k, m) \in \mathbb{N} \times \mathbb{N}$. Thinking of analytic functions being placed on the real axis, conjugate analytic functions on the imaginary axis and the radial functions on the diagonal $y = x$ in the first quadrant, then they showed for a fixed symbol $z^s\overline{z}^t$ there will be many lines parallel to the diagonal, 'holding' a symbol that gives a Toeplitz operator commuting with $T_{z^s\overline{z}^t}$. Each of these lines will hold no more than one such symbol. The following theorem will discuss the same problem on L_h^2, but for $(k, m) \in \mathbb{Z} \times \mathbb{R}^+ \cup \{0\}$. However, unlike results in [7], Toeplitz operators commute with $T_{e^{ik\theta}r^m}$ on L_h^2 only in certain trivial cases.

Theorem 1.4. *Let* $k \in \mathbb{Z}$ *and let* m *be a real number greater than or equal to* 0. *Then, for any bounded function* f *on* D,

$$T_fT_{e^{ik\theta}r^m} = T_{e^{ik\theta}r^m}T_f$$

if and only if one of the following conditions holds:

(1) *either* f *or* $e^{ik\theta}r^m$ *is constant;*

(2) *both* f *and* $e^{ik\theta}r^m$ *are radial;*

(3) f *is a linear combination of* 1 *and* $e^{ik\theta}r^m$.

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2. Some preliminary results

We start this section with the concept of the Mellin transform. For a function $f \in L^1([0,1], rdr)$, the Mellin transform of f is the function $\widehat{f}$ defined by

$$\widehat{f}(z) = \int_0^1 f(s) s^{z-1}\, ds.$$

It is known that $\widehat{f}$ is well defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$.

When considering the product of two Toeplitz operators, we need a known fact about the Mellin convolution of their symbols. If f and g are defined on $[0, 1)$, then their Mellin convolution is defined by

$$(f *_M g)(r) = \int_r^1 f\Big(\frac{r}{t}\Big) g(t) \frac{dt}{t}, \quad 0 \leq r < 1.$$

It is known that if f and g are in $L^1([0,1], r\,dr)$, then so is $f *_M g$.

In [9], we proved the following results, which we shall use frequently in this paper.

Lemma 2.1. *Let $k \in \mathbb{Z}$ and let φ be a radial T-function. Then, for each $n \in \mathbb{N}$,*

$$T_{e^{ik\theta}\varphi}(z^n) = \begin{cases} 2(n+k+1)\widehat{\varphi}(2n+k+2) z^{n+k} & \text{if } n \geq -k, \\ 2(-n-k+1)\widehat{\varphi}(-k+2)\bar{z}^{-n-k} & \text{if } n < -k; \end{cases}$$

$$T_{e^{ik\theta}\varphi}(\bar{z}^n) = \begin{cases} 2(n-k+1)\widehat{\varphi}(2n-k+2)\bar{z}^{n-k} & \text{if } n \geq k, \\ 2(k-n+1)\widehat{\varphi}(k+2) z^{k-n} & \text{if } n < k. \end{cases}$$

The next lemma will much simplify our arguments in the proof of Theorem 1.2.

Lemma 2.2. *Let $k_1,\ k_2 \in \mathbb{Z}$ be such that $k_1 > |k_2|$ and let $m \in \mathbb{R}$, $m \geq -1$. Then, for a radial function $\varphi \in L^1(D, dA)$,*

$$\widehat{\varphi}(2n+2k_1+k_2+2) = \widehat{\varphi}(2n+k_2+2)\frac{(2n+2k_2+2)(2n+k_1+m+2)}{(2n+2k_1+2)(2n+k_1+2k_2+m+2)} \tag{2.1}$$

holds for any $n \in \mathbb{N}$ such that $n \geq -k_2$ if and only if

$$\widehat{\varphi}(z) = C\frac{\Gamma(\frac{z+k_2}{2k_1})\Gamma(\frac{z+m+k_1-k_2}{2k_1})}{\Gamma(\frac{z+2k_1-k_2}{2k_1})\Gamma(\frac{z+m+k_1+k_2}{2k_1})}, \qquad \operatorname{Re} z > 2$$

for some constant C.

Proof. It is well known that a bounded analytic function is uniquely determined by its value on an arithmetic sequence of integers, so (2.1) implies that

$$\widehat{\varphi}(z+2k_1) = \widehat{\varphi}(z)\frac{(z+k_2)(z+m+k_1-k_2)}{(z+2k_1-k_2)(z+m+k_1+k_2)} \tag{2.2}$$

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for $\operatorname{Re} z > 2$. Denote

$$F(z) = \frac{\Gamma(\frac{z+k_2}{2k_1})\Gamma(\frac{z+m+k_1-k_2}{2k_1})}{\Gamma(\frac{z+2k_1-k_2}{2k_1})\Gamma(\frac{z+m+k_1+k_2}{2k_1})}.$$

Using the well-known identity $\Gamma(z+1) = z\Gamma(z)$, we can easily see that

$$F(z+2k_1) = F(z)\frac{(\frac{z+k_2}{2k_1})(\frac{z+m+k_1-k_2}{2k_1})}{(\frac{z+2k_1-k_2}{2k_1})(\frac{z+m+k_1+k_2}{2k_1})}.$$

Then it follows from (2.2) that

$$\widehat{\varphi}(z+2k_1)F(z) = \widehat{\varphi}(z)F(z+2k_1)$$

and by [15, Lemma 6], we get $\widehat{\varphi}(z) = CF(z)$ for some constant C. This completes the proof. □

In fact, (2.1) is the same as (2.4) of [7], the only difference is the range of m, and here we only simplify the proof of [7].

The next lemma plays the key role in proving Theorem 1.2.

Lemma 2.3. *For each $a \in (0, 1)$, the function*

$$x \longmapsto \frac{\Gamma(x+1-a)\Gamma(x+a)}{\Gamma(x+1)\Gamma(x)}$$

is strictly monotone increasing on $(0, +\infty)$.

Proof. Define

$$g(x, y) := \psi(x+1-y) + \psi(x+y) - \psi(x+1) - \psi(x)$$

for $x \in (0, +\infty)$ and $y \in [0, 1]$, where ψ is defined by

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

known in the literature as the psi or digamma function. The derivatives $\psi', \psi'', \psi''', \ldots$ are known as the tri-, tetra- or pentagamma functions or, generally, the polygamma functions. We refer the reader to [1, page 260] for the properties of these functions.

Now, we fix x and denote $h(y) = g(x, y)$. Note that

$$h''(y) = \psi''(x+1-y) + \psi''(x+y) < 0$$

for all $y \in [0, 1]$. Here we used the formula [1, page 260, 6.4.6]

$$\psi''(s) = -\int_0^{+\infty} \frac{t^2 e^{-st}}{1-e^{-t}}\,dt, \qquad s \in (0, \infty).$$

Note also that

$$h(0) = h(1) = 0.$$

Hence,

$$h(a) > 0 \quad \text{for all } a \in (0, 1).$$

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It follows that $g(x, a) > 0$ for all $x \in (0, +\infty)$ and all $a \in (0, 1)$. But note that

$$g(x, a) = (\log G(x))' = \frac{G'(x)}{G(x)},$$

where

$$G(x) := \frac{\Gamma(x + 1 - a)\Gamma(x + a)}{\Gamma(x + 1)\Gamma(x)}.$$

This implies that $G(x)$ is strictly monotone increasing on $(0, +\infty)$, as desired. □

3. Proofs of the theorems

In this section we will prove our main theorems.

Proof of Theorem 1.2. Assume $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$. If $|k_1| \le |k_2|$, then it follows from [11, Theorem 3.8] that one of conditions (1)–(4) holds. If $|k_1| > |k_2| = 0$, then it follows from [11, Lemma 3.5] that φ is constant and hence condition (1) holds.

Now we assume $|k_1| > |k_2| > 0$. Without loss of generality, we can also assume $k_1 > 0$, for otherwise we could take the adjoints. Then for each $n \in \mathbb{N}$ such that $n \ge -k_2$, the equality

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}(z^n) = T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}(z^n)$$

together with Lemma 2.1 gives

$$\widehat{\varphi}(2n + 2k_1 + k_2 + 2) = \widehat{\varphi}(2n + k_2 + 2)\frac{(2n + 2k_2 + 2)(2n + k_1 + m + 2)}{(2n + 2k_1 + 2)(2n + k_1 + 2k_2 + m + 2)}.$$

Thus, Lemma 2.2 implies

$$\widehat{\varphi}(z) = C\frac{\Gamma(\frac{z+k_2}{2k_1})\Gamma(\frac{z+m+k_1-k_2}{2k_1})}{\Gamma(\frac{z+2k_1-k_2}{2k_1})\Gamma(\frac{z+m+k_1+k_2}{2k_1})}$$

for some constant C. In what follows, we will show $C = 0$ and hence condition (1) holds.

So, assume $C \ne 0$. We split the proof into two cases.

Case 1. Suppose $k_2 < 0$. Noting that $k_1 > 0$, $k_2 < 0$ and $|k_1| > |k_2|$, by Lemma 2.1,

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}(z^0) = T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}(z^0)$$

gives

$$(-k_2 + 1)\widehat{\varphi}(-k_2 + 2) = (k_1 + 1)\widehat{\varphi}(2k_1 + k_2 + 2).$$

Then it follows that

$$(-k_2 + 1)\frac{\Gamma(\frac{2}{2k_1})\Gamma(\frac{m+k_1-2k_2+2}{2k_1})}{\Gamma(\frac{2k_1-2k_2+2}{2k_1})\Gamma(\frac{m+k_1+2}{2k_1})} = (k_1 + 1)\frac{\Gamma(\frac{2k_1+2k_2+2}{2k_1})\Gamma(\frac{m+3k_1+2}{2k_1})}{\Gamma(\frac{4k_1+2}{2k_1})\Gamma(\frac{m+3k_1+2k_2+2}{2k_1})}.$$

Denote

$$x = \frac{m + k_1 + 2}{2k_1} \quad \text{and} \quad a = \frac{-k_2}{k_1};$$

then, from the above equation,

$$\Big(\frac{1}{k_1}+a\Big)\frac{\Gamma(\frac{1}{k_1})\Gamma(x+a)}{\Gamma(\frac{1}{k_1}+a+1)\Gamma(x)}=\Big(\frac{1}{k_1}+1\Big)\frac{\Gamma(\frac{1}{k_1}+1-a)\Gamma(x+1)}{\Gamma(\frac{1}{k_1}+2)\Gamma(x+1-a)}$$

and using the identity $\Gamma(z+1)=z\Gamma(z)$, we get

$$\frac{\Gamma(x+1-a)\Gamma(x+a)}{\Gamma(x+1)\Gamma(x)}=\frac{\Gamma(\frac{1}{k_1}+1-a)\Gamma(\frac{1}{k_1}+a)}{\Gamma(\frac{1}{k_1}+1)\Gamma(\frac{1}{k_1})}. \tag{3.1}$$

Since $k_1>-k_2>0$, $k_1\in\mathbb{Z}$ and $m\geq -1$,

$$a\in(0,1)\quad\text{and}\quad x>\frac{1}{k_1}>0.$$

Therefore, (3.1) contradicts Lemma 2.3 and hence $C=0$.
Case 2. Suppose $k_2>0$. So, $k_1>k_2>0$. Similarly, by Lemma 2.1,

$$T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}(\overline{z}^{k_2})=T_{e^{ik_2\theta}\varphi}T_{e^{ik_1\theta}r^m}(\overline{z}^{k_2})$$

gives

$$\widehat{\varphi}(k_2+2)=(k_1-k_2+1)\widehat{\varphi}(2k_1-k_2+2).$$

Then it follows that

$$\frac{\Gamma(\frac{2k_2+2}{2k_1})\Gamma(\frac{m+k_1+2}{2k_1})}{\Gamma(\frac{2k_1+2}{2k_1})\Gamma(\frac{m+k_1+2k_2+2}{2k_1})}=(k_1-k_2+1)\frac{\Gamma(\frac{2k_1+2}{2k_1})\Gamma(\frac{m+3k_1-2k_2+2}{2k_1})}{\Gamma(\frac{4k_1-2k_2+2}{2k_1})\Gamma(\frac{m+3k_1+2}{2k_1})}.$$

Denote

$$x=\frac{m+k_1+2}{2k_1}\quad\text{and}\quad a=\frac{k_2}{k_1}\in(0,1);$$

then the above equation implies that

$$\frac{\Gamma(x+1-a)\Gamma(x+a)}{\Gamma(x+1)\Gamma(x)}=\frac{\Gamma(\frac{1}{k_1}+1-a)\Gamma(\frac{1}{k_1}+a)}{\Gamma(\frac{1}{k_1}+1)\Gamma(\frac{1}{k_1})},$$

which contradicts Lemma 2.3 since $x>1/k_1>0$ and hence $C=0$.

The converse implication is clear. This completes the proof. □

Proof of Theorem 1.3. First we suppose $T_{e^{ik_1\theta}r^m}T_{e^{ik_2\theta}\varphi}$ is equal to a Toeplitz operator; then [12, Theorem 1.2] implies that $T_{e^{ik_1\theta}r^m}$ commutes with $T_{e^{ik_2\theta}\varphi}$. In view of Theorem 1.2, we only need to discuss the case where $e^{ik_1\theta}r^m$ and $e^{ik_2\theta}\varphi$ are linearly dependent. However, in this case [12, Corollary 3.3] implies either $k_1=k_2=0$ or $\varphi=0$ and hence one of conditions (1) or (2) holds.

Conversely, if condition (1) holds, then the desired result is obvious.

Now assume (2) holds. Then, by [12, Corollary 3.1], we need to show that ψ is a solution of the equation

$$\mathbb{I}*_M\psi=r^m*_M\varphi,$$

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which is equivalent to

$$\int_r^1 \frac{\psi(t)}{t} dt = r^m \int_r^1 \frac{\varphi(t)}{t^{m+1}} dt.$$

By differentiating both sides,

$$\psi(r) = \varphi(r) - mr^m \int_r^1 \frac{\varphi(t)}{t^{m+1}} dt.$$

Since φ is a radial T-function, it follows that

$$\|\varphi\|_{L^1} = \int_0^1 |\varphi(t)| t\, dt < \infty.$$

Thus,

$$\begin{aligned}\int_0^1 \left| r^m \int_r^1 \frac{\varphi(t)}{t^{m+1}} dt \right| r\, dr &\le \int_0^1 r^{m+1}\, dr \int_r^1 \frac{|\varphi(t)|}{t^{m+1}} dt \\ &= \int_0^1 \frac{|\varphi(t)|}{t^{m+1}} dt \int_0^t r^{m+1}\, dr \\ &= \frac{1}{m+2} \|\varphi\|_{L^1} < \infty.\end{aligned}$$

Moreover,

$$\left| r^m \int_r^1 \frac{\varphi(t)}{t^{m+1}} dt \right| \le \int_r^1 |\varphi(t)| \frac{dt}{t} \le \frac{1}{r^2} \|\varphi\|_{L^1}.$$

Therefore, the radial function

$$r^m \int_r^1 \frac{\varphi(t)}{t^{m+1}} dt$$

is 'nearly bounded' on D and hence ψ is a T-function.

If condition (3) holds, a direct calculation shows that

$$(r^m) *_M \left(\frac{m+1}{2} r^{-1} - \frac{m-1}{2} r \right) = \frac{1}{2} \left(\frac{1}{r} - r \right) = r *_M r^{-1}.$$

Hence, by [12, Corollary 3.1], the desired result is obvious. □

Proof of Theorem 1.4. Assume T_f and $T_{e^{ik\theta} r^m}$ commute. If $k = 0$, then [11, Theorem 4.3] shows that either r^m is constant or f is radial and hence one of conditions (1) or (2) holds. Now we suppose $k \neq 0$. Let

$$f(re^{i\theta}) = \sum_{l \in \mathbb{Z}} e^{il\theta} f_l(r);$$

then [11, Lemma 4.1] implies $T_{e^{il\theta} f_l}$ and $T_{e^{ik\theta} r^m}$ commute for any $l \in \mathbb{Z}$. Then, by Theorem 1.2, one can easily get that:

(a) if $l = 0$, then $f_l = C_1$ for some constant C_1;
(b) if $l = k$, then $f_l = C_2 r^m$ for some constant C_2;
(c) if $l \neq 0$ and $l \neq k$, then $f_l = 0$.

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In summary,

$$f(re^{i\theta}) = C_1 + C_2 e^{ik\theta} r^m$$

and hence condition (3) holds.

The converse implication is clear. This completes the proof. □

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