ON OPERATORS AND DISTRIBUTIONS

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Mikusinski [1] has extended the operational calculus by methods which are essentially algebraic. He considers the family C of continuous complex valued functions on the half-line $[0,\infty)$. Under addition and convolution C becomes a commutative ring. Titchmarsh's theorem [2] shows that the ring has no divisors of zero and, hence, that it may be imbedded in its quotient field Q whose elements are then called operators. Included in the field are the integral, differential and translational operators of analysis as well as certain generalized functions, such as the Dirac delta function. An alternate approach [3] yields a rather interesting result which we shall now describe briefly.

Consider the family R of infinitely differentiable functions whose supports are bounded on the left. Under addition and convolution this family also becomes a commutative ring without zero divisors. It may be imbedded in the family E(R) of all maximal homomorphic mappings of nonzero ideals of R into R (the extended centralizer of R over the regular R-module R [4]). Under addition and composition the family E(R) becomes a field which is isomorphic to the quotient field of R. It is easy to see that E(R) is also isomorphic to the Mikusinski operator field. Indeed, if R_o denotes the ring of infinitely differentiable functions, whose supports are in the half line $[0, \infty)$, under addition and convolution, then for each $\phi \varepsilon R$ there exists a $\psi \varepsilon R_o$ such that the convolution $\phi * \psi \varepsilon R_o$. It follows from this that $E(R_o)$ and E(R) are isomorphic [5; Proposition 5.8]. Moreover, R_o is an ideal in C and, thus, E(R) and Q are isomorphic.

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Now consider the family D' of all Schwartz distributions whose supports are bounded on the left [6]. Under addition and convolution this family becomes a commutative ring without zero divisors. Furthermore, the ring R may be considered an ideal in D' and, therefore, D' itself is isomorphically contained in the field E(R). (Of course, the quotient field of D' is isomorphic to E(R).) In particular, $D' \subset \mathrm{Hom}_R(R, R)$, which is the subalgebra of homomorphic mappings defined on all of R (the centralizer of R over the regular R-module R). The interesting result alluded to is the fact that $\mathrm{Hom}_R(R, R)$ is precisely the family of distributions D'. To establish this result we consider the topological structure of R.

A sequence $\{\phi_n\}$ of elements of the ring R is said to converge in R if the supports of the functions are uniformly bounded on the left and if $\{\phi_n\}$, as well as all sequences of derivatives $\{\phi_n^{(k)}\}$, converges uniformly on compact sets. The ordinary limit function ϕ is an element of R and we shall use the ordinary notation, $\phi_n \to \phi$, to indicate that the sequence $\{\phi_n\}$ converges to ϕ in R. Using this convergence concept we may consider R to be the space of test functions for the distributions in D'. Thus if $\phi \varepsilon R$ and $f \varepsilon D'$, we let $<f(t), \phi(-t)>$ denote the value of the functional f at ϕ. The convolution of f and ϕ is the function ψ whose value at τ is $\psi(\tau) = <f(t), \phi(\tau - t)>$. The function ψ is an element of the ring R and the mapping which sends ϕ to ψ characterizes f as an element of $\mathrm{Hom}_R(R, R)$.

THEOREM. The family D' of Schwartz distributions whose supports are bounded on the left may be identified both algebraically and topologically with the subalgebra $\mathrm{Hom}_R(R, R)$.

Proof. As indicated above, there is a natural imbedding of D' in $\mathrm{Hom}_R(R, R)$. We shall establish the inverse imbedding. Let $f \varepsilon \mathrm{Hom}_R(R, R)$ and let $f(\phi)$ denote the image of ϕ for some $\phi \varepsilon R$. Then $f(\phi)$ is an element of R whose value at τ we shall denote by $f(\phi)(\tau)$. As a homomorphism, f is a weak limit in R, that is, there exists a sequence $\{\phi_n\}$ in R such that

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$\phi_n(\phi) = \phi_n * \phi \rightarrow f(\phi)$ for all $\phi \varepsilon R$. In fact, if $\{\delta_n\}$ is any "delta function" sequence in R, then f is the weak limit of the sequence $\{f(\delta_n)\}$ since $\delta_n * f(\phi) = f(\delta_n) * \phi$ for all $\phi \varepsilon R$. In particular, this implies that f is translation invariant in as much as convolution by a fixed element of R is translation invariant. Thus f is uniquely defined, as a linear functional, through the convolution equation $<f(t), \phi(\tau - t)> = f(\phi)(\tau)$. Moreover, it follows that the functional f is a weak limit in R in the distributional sense, that is, there exists a sequence $\{\phi_n\}$ in R (for example, the above sequence $\{f(\delta_n)\}$) such that $<\phi_n(t), \phi(-t)> \rightarrow <f(t), \phi(-t)>$, as a number sequence, for all $\phi \varepsilon R$. It is a well-known result of Schwartz [7] that such weak limits are indeed distributions and, furthermore, that every distribution is such a weak limit. This not only establishes the desired imbedding of $\mathrm{Hom}_R(R, R)$ in D' but also justifies their identification topologically.

Recently Norris [8] recognized the overriding importance of the subalgebra $\mathrm{Hom}_R(R, R)$ in the operational calculus and introduced the weak topology for it, but did not attempt to identify it further. Since this is precisely the family of (right-sided) distributions it is not surprising to find that it contains essentially all of the operators which are actually used in the operational calculus. It is interesting to note that although the elements of the subalgebra $\mathrm{Hom}_R(R, R)$ are continuous in an appropriate weak sense and that there is a suitable topology for them, they may be obtained without topological considerations. It follows from the above theorem that the (right-sided) distributions also may be obtained without topological considerations.

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