Astronomy and Astrophysics in the Gaia sky
Proceedings IAU Symposium No. 330, 2017
A. Recio-Blanco, P. de Laverny, A.G.A. Brown & T. Prusti, eds.

doi:10.1017/S1743921317005245

Light propagation in the Solar System for astrometry on sub-micro-arcsecond level

Sven Zschocke

Institute of Planetary Geodesy - Lohrmann Observatory, Dresden Technical University, Helmholtzstrasse 10, D-01069, Dresden, Germany
email: sven.zschocke@tu-dresden.de

Abstract. We report on recent advancement in the theory of light propagation in the Solar System aiming at sub-micro-arcsecond level of accuracy:

(1) A solution for the light ray in 1.5PN approximation has been obtained in the field of N arbitrarily moving bodies of arbitrary shape, inner structure, oscillations, and rotational motion.

(2) A solution for the light ray in 2PN approximation has been obtained in the field of one arbitrarily moving pointlike body.

Keywords. astrometry, relativity, gravitation

1. Introduction

In order to trace a light ray received by an observer back to the celestial light source, one has to determine the trajectory of the light ray by solving the geodesic equation, which in terms of the Barycentric Celestial Reference System $(ct, \boldsymbol{x})$ reads

$$\frac{d^2 x^i\,(t)}{c^2 dt^2} + \Gamma^i_{\mu\nu}\,\frac{dx^\mu\,(t)}{cdt}\frac{dx^\nu\,(t)}{cdt} \quad = \quad \Gamma^0_{\mu\nu}\,\frac{dx^\mu\,(t)}{cdt}\frac{dx^\nu\,(t)}{cdt}\frac{dx^i\,(t)}{cdt}\,, \tag{1.1}$$

where the Christoffel symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)/2$ are functions of the metric tensor. For a unique solution of (1.1) initial-boundary conditions must be imposed,

$$\boldsymbol{x}_0 \quad = \quad \boldsymbol{x}\,(t_0)\,, \quad \boldsymbol{\sigma} = \lim_{t\to-\infty} \frac{\dot{\boldsymbol{x}}\,(t)}{c}\,, \tag{1.2}$$

where $\boldsymbol{x}_0$ is the position of the light-source and $\boldsymbol{\sigma}$ defines the unit tangent vector of the light ray at past null-infinity. The first and second integration of geodesic equation yields the coordinate velocity and trajectory of the light signal, given by

$$\dot{\boldsymbol{x}}\,(t) \quad = \quad \int\limits_{-\infty}^{t} \ddot{\boldsymbol{x}}\,(t')\,dt' = c\,\boldsymbol{\sigma} + \Delta\dot{\boldsymbol{x}}\,(t)\,, \tag{1.3}$$

$$\boldsymbol{x}\,(t) \quad = \quad \int\limits_{t_0}^{t} \dot{\boldsymbol{x}}\,(t')\,dt' = \boldsymbol{x}_0 + c\,(t - t_0)\,\boldsymbol{\sigma} + \Delta\boldsymbol{x}\,(t)\,, \tag{1.4}$$

where $\Delta\dot{\boldsymbol{x}}$ and $\Delta\boldsymbol{x}$ are small corrections to the unperturbed (i.e. straight) light ray.

2. Light trajectory in post-Newtonian expansion

Because the gravitational fields of the Solar system are weak and velocities of the bodies are slow, $v \ll c$, one may utilize the post-Newtonian (PN) expansion of the metric,

$$g_{\alpha\beta} \quad = \quad \eta_{\alpha\beta} + h^{(2)}_{\alpha\beta} + h^{(3)}_{\alpha\beta} + h^{(4)}_{\alpha\beta} + \mathcal{O}\left(c^{-5}\right), \tag{2.1}$$

where $h_{\alpha\beta}^{(n)} = \mathcal{O}\left(c^{-n}\right)$ are metric perturbations of the flat space-time $\eta_{\alpha\beta}$. The Damour-Soffel-Xu approach as well as the Brumberg-Kopeikin formalism provide expressions for the metric perturbations in terms of intrinsic multipoles, M_L^A and S_L^A, allowing for arbitrary shape, inner structure, oscillations, and rotational motion of the Solar System bodies. Inserting (2.1) into (1.1) results in a post-Newtonian expansion of the light ray,

$$\dot{\boldsymbol{x}}(t) = c\,\boldsymbol{\sigma} + \Delta\dot{\boldsymbol{x}}_{1\mathrm{PN}}(t) + \Delta\dot{\boldsymbol{x}}_{1.5\mathrm{PN}}(t) + \Delta\dot{\boldsymbol{x}}_{2\mathrm{PN}}(t) + \mathcal{O}\left(c^{-5}\right), \quad (2.2)$$

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + c\,(t - t_0)\,\boldsymbol{\sigma} + \Delta\boldsymbol{x}_{1\mathrm{PN}}(t) + \Delta\boldsymbol{x}_{1.5\mathrm{PN}}(t) + \Delta\boldsymbol{x}_{2\mathrm{PN}}(t) + \mathcal{O}\left(c^{-5}\right), \quad (2.3)$$

where $\Delta\boldsymbol{x}_{1\mathrm{PN}} = \mathcal{O}\left(c^{-2}\right)$, $\Delta\boldsymbol{x}_{1.5\mathrm{PN}} = \mathcal{O}\left(c^{-3}\right)$, and $\Delta\boldsymbol{x}_{2\mathrm{PN}} = \mathcal{O}\left(c^{-4}\right)$.

3. Light trajectory in 1PN and 1.5PN approximation

In the investigations of Zschocke (2015) and Zschocke (2016 a) a solution for $\Delta\dot{\boldsymbol{x}}_{1\mathrm{PN}}$, $\Delta\boldsymbol{x}_{1\mathrm{PN}}$ and $\Delta\dot{\boldsymbol{x}}_{1.5\mathrm{PN}}$, $\Delta\boldsymbol{x}_{1.5\mathrm{PN}}$, respectively, has been determined for the case of N arbitrarily moving bodies with full intrinsic multipole structure.

4. Light trajectory in 2PN approximation

A solution for $\Delta\dot{\boldsymbol{x}}_{2\mathrm{PN}}$, $\Delta\boldsymbol{x}_{2\mathrm{PN}}$ for one arbitrarily moving pointlike body has been given by Zschocke (2016 b) in terms of vectorial functions $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3$ and $\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3$. The vectorial functions $\boldsymbol{A}_2$ and $\boldsymbol{B}_2$ contain tiny vectorial parameters, $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, whose expressions were not presented by Zschocke (2016 b) and will, therefore, be given here:

$$\begin{aligned}\boldsymbol{\epsilon}_1 = & -\frac{v^2}{c^2}\,\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}\,\frac{1}{x} + 2\left(\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c\,x}\right)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}\,\frac{1}{x} - 2\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}}{c}\right)^2\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}\,\frac{1}{x}\\ & +4\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}}{c}\right)\left(\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c\,x}\right)\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}}\,\frac{1}{x} + 4\,\frac{\boldsymbol{v}}{c}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c\,x}\right)\frac{1}{x} - 4\,\frac{\boldsymbol{v}}{c}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}}{c}\right)\frac{1}{x}\\ & -\frac{v^2}{c^2}\,\frac{\boldsymbol{\sigma}}{x} + 2\left(\frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c\,x}\right)^2\frac{\boldsymbol{\sigma}}{x} + 2\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}}{c}\right)^2\frac{\boldsymbol{\sigma}}{x}\,,\end{aligned} \quad (4.1)$$

$$\boldsymbol{\epsilon}_2 = -\frac{v^2}{c^2}\,\frac{\boldsymbol{\sigma}\times(\boldsymbol{x}\times\boldsymbol{\sigma})}{x-\boldsymbol{\sigma}\cdot\boldsymbol{x}} + \frac{v^2}{c^2}\,\boldsymbol{\sigma}\,\ln\left(x-\boldsymbol{\sigma}\cdot\boldsymbol{x}\right). \quad (4.2)$$

The upper limit for their absolute values can be estimated as follows:

$$|\boldsymbol{\epsilon}_1| \leqslant \frac{10}{|\boldsymbol{\sigma}\times\boldsymbol{x}|}\,\frac{v^2}{c^2} \quad \text{and} \quad |\boldsymbol{\epsilon}_2| \leqslant \frac{v^2}{c^2}\sqrt{\frac{4\,x^2}{|\boldsymbol{\sigma}\times\boldsymbol{x}|^2} + \ln^2\left(x-\boldsymbol{\sigma}\cdot\boldsymbol{x}\right)}\,. \quad (4.3)$$

As noticed by Zschocke (2016 b), $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are negligible for light deflection measurements on nano-arcsecond level and time delay measurements on pico-second level, respectively.

5. Outlook

For sub-micro-arcsecond astrometry the approach needs to be further developed. Especially, the following issues have to be treated: (i) retardation effects, (ii) light ray in the observers reference system, (iii) 2PN light ray in the field of arbitrarily shaped bodies.

References

Zschocke, S. 2015, *Phys. Rev. D*, 92, 063015
Zschocke, S. 2016 a, *Phys. Rev. D*, 93, 103010
Zschocke, S. 2016 b, *Phys. Rev. D*, 94, 124007