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ON STRONGLY EXPOSING FUNCTIONALS

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Abstract

Let X be a real Banach space and let K be a bounded closed convex subset of X. We prove that the set of strongly exposing functions $K^\wedge$ of K is a (norm) dense G_δ in X^* if and only if for any bounded closed convex subset C such that $K \not\subset C$, there exists a point x in K which is a strongly exposed point of $\overline{\text{conv}}\,(C \cup K)$. As an application, we show that if X^* is weakly compact generated, then for any weakly compact subset K in X, the set $K^\wedge$ is a dense G_δ in X^*.

1. Introduction

Throughout we assume that X is a real Banach space. Suppose K is a bounded closed convex subset in X, a point $x \in K$ is called a *support point* of K if there exists a non-zero functional f in X^* such that $f(x) = \sup_{y \in K} f(y)$; a point x in K is called a *strongly exposed point* of K if $f(x) > f(y)$ for all $y \in K$ with $y \neq x$ and if $\{x_n\}$ is a sequence in K such that $\{f(x_n)\}$ converges to $f(x)$, then $\{x_n\}$ converges to x (in norm). The corresponding functionals $f \in X^*$ in the above definitions will be called *support functional* and *strongly exposing functional* of K at x respectively. A well known result on the support functionals is (Bishop and Phelps (1962)): *if K is a bounded closed convex subset in a Banach space X, then the set of support functionals is dense in X^*.* Recently, Huff and Morris observed if X has the Radon-Nikodym property, then the set of strongly exposing functionals of a bounded closed convex subset is a dense G_δ in X^*. In this note, we give a necessary and sufficient condition on bounded closed convex sets so that the above assertion holds (Theorem 2.4). By using the concept of farthest points, we show further that if X is a Banach space with X^* weakly compact generated, then any weakly compact convex subset K of X will satisfy the condition and hence the set of strongly exposing functionals is a dense G_δ in X^* (Theorem 3.5).

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2. The main theorem

Let X be a Banach space, we use B to denote the closed unit ball of X. For a closed convex subset K in X, we let $K^\wedge$ denote the set of strongly exposing functionals of K in X^*.

PROPOSITION 2.1. *Let X be a Banach space and let K be a bounded closed convex subset of X, then K is a G_δ subset in X^* (may be empty).*

PROOF. We need only observe that the set

$$K_n = \{f \in X^*: \operatorname{diam}\{x \in K: f(x) > \sup_{y \in K} f(y) - a\} < \frac{1}{n} \text{ for some } a > 0\}$$

is open in X^* and $K = \bigcap_{n=1}^{\infty} K_n$. C. f. also Anantharaman (to appear), Huff and Morris (to appear).

LEMMA 2.2. *Let X be a Banach space and let K be a closed convex subset of X. For any $x \in X$ and $\alpha > 0$, we have $(x + K)^\wedge = K^\wedge$ and $(\alpha K)^\wedge = K^\wedge$.*

LEMMA 2.3. Phelps (1960). *Let X be a normed linear space. For any $\varepsilon > 0$, suppose there exists f, $g \in X^*$ with $\|f\| = \|g\| = 1$ and suppose $f(x) \leqq \epsilon/2$ whenever $g(x) = 0$ and $\|x\| \leqq 1$. Then either $\|f - g\| \leqq \varepsilon$ or $\|f + g\| \leqq \epsilon$.*

THEOREM 2.4. *Let K be a bounded closed convex subset in a Banach space X. Then $K^\wedge$ is dense in X^* if and only if for any bounded closed convex subset C such that $K \not\subset C$ there exists a point x in K which is a strongly exposed point of $\overline{\text{conv}}\,(C \cup K)$.*

PROOF. We first prove the sufficiency. Suppose we are given $1 > \varepsilon > 0$ and $g \in X^*$. Without loss of generality, assume that $\|g\| = 1$, that K is a subset of the closed unit ball B and $g(x) > \varepsilon$ for some $x \in K$ (for otherwise, we can consider a homothetic image of the set K as in Lemma 2.2). Let

$$C = g^{-1}(0) \cap 2\varepsilon^{-1}B, \; D = \overline{\text{conv}}\,(C \cup K).$$

By assumption, there exists a strongly exposed point x_0 of D and $x_0 \in K$. Let $f \in X^*$, $\|f\| = 1$, strongly exposed D at x_0, then f also strongly exposes K at x_0, hence $f \in K^\wedge$. To show that $\|f - g\| \leqq \varepsilon$, we observe that

$$f(x) \leqq f(x_0) \leqq \|f\| \cdot \|x_0\| \leqq 1 \text{ for all } x \in C.$$

Thus for $x \in g^{-1}(0)$ and $\|x\| \leqq 1$, $2\varepsilon^{-1}x$ is in C and $f(2\varepsilon^{-1}x) \leqq 1$, i.e. $f(x) \leqq \varepsilon/2$. By Lemma 2.3, either $\|f - g\| \leqq \varepsilon$ or $\|f + g\| \leqq \varepsilon$. Since $f(x_0) > f(0) = 0$, we have

$$\| f + g \| \geqq f(x_0) + g(x_0) > g(x_0) > \varepsilon.$$

Thus we have $\| f - g \| \leqq \varepsilon$ for some $f \in K$. This shows that $K^\wedge$ is dense in X^*.

The prove the necessity, let C, K be bounded closed convex subset such that $K \not\subset C$. By the separation theorem and the density of $K^\wedge$, we may assume that there exists $f \in K$, $x_0 \in K$ such that f exposes K at x_0 and

$$\sup_{y \in C} f(y) = \alpha < f(x_0).$$

We claim that x_0 is a strongly exposed point of $\overline{\text{conv}}(C \cup K)$. Indeed, let $\{x_n\}$ be a sequence in $\overline{\text{conv}}(C \cup K)$ and $f(x_n) \to f(x_0)$. Without loss of generality, we assume that $\{x_n\} \subset \text{conv}(C \cup K)$, then

$$x_n = \lambda_n y_n + (1 - \lambda_n) z_n, \; y_n \in C, z_n \in K \text{ and } 0 \leqq \lambda_n \leqq 1.$$

Since $f(y_n) \leqq \alpha < f(x_0)$ and $f(x_n) \to f(x_0)$, we have $\lambda_n \to 0$ and $f(z_n) \to f(x_0)$. Thus by definition of x_0, $\{z_n\}$ converges to x_0 (in norm). It follows that $\{x_n\}$ converges to x_0.

It is shown in (Davis and Phelps (1974), Huff (1974), Phelps (1974)) that a Banach space has the Radon-Nikodym property if and only if every bounded closed convex subset is the closed convex hull of its strongly exposed points. This class contains the reflexive spaces, dual Banach spaces which are weakly compact generated.

COROLLARY 2.5 Huff and Morris (to appear). *Let X be a Banach space which has the Radon Nikodym property, then for any bounded closed convex subset K of X, $K^\wedge$ is a dense G_δ.*

PROOF. The second equivalent condition of the above theorem is satisfied.

3. Some applications

Let X be a Banach space, we say that the norm on X is *Fréchet differentiable* if

$$\lim_{y \to 0} \frac{\| x + y \| - \| x \|}{\| y \|}$$

exists for each nonzero x in X. The norm is called *locally uniformly convex* if for each $x \in X$ with $\| x \| = 1$ and for $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $y \in X$ with $\| y \| \leqq 1$ and $\| x - y \| \geqq \varepsilon$, then $\| x + y \| \leqq 2(1 - \delta)$.

PROPOSITION 3.1 Asplund (1967). *Let X be a Banach space. Suppose there exists two equivalent norms on X under which X and X^* are locally uniformly convex respectively. Then X has an equivalent norm which is both locally uniformly convex and Fréchet differentiable.*

A Banach space X is called *weakly compact generated* if there exists a weakly compact subset K in X such that the closed linear subspace generated by K is X.

PROPOSITION 3.2 Trojanski (1971). *Suppose X is a Banach space which is weakly compact generated, then X has an equivalent locally uniformly convex norm.*

Let K be a bounded closed subset of X. A point $x \in K$ is called a *farthest point* of K if there exists a point $y \in X$ such that

$$\| y - x \| = \sup\{\| y - z \|: z \in K\}.$$

It is easy to see that if X has a locally uniformly convex norm, then each farthest point of K is also a strongly exposed point of K, the converse is not true in general (Edelstein and Lewis (1971)).

PROPOSITION 3.3 Lau (to appear). *Let X be a Banach space and let K be a weakly compact subset of X. Then the set*

$$\{y \in X: \| y - x \| = \sup\{\| y - z \|: z \in K\} \text{ for some } x \in K\}$$

is a dense G_δ in X.

Suppose Y is a closed subspace of a Banach space X, let i: $Y \to X$ be the identity map and let i^* be the adjoint of i. It may be shown that if X has an equivalent norm such that X^* is locally uniformly convex, so does the subspace Y. Let K be a bounded closed convex subset in Y. By the Hahn-Banach Theorem, the map i^* is onto. Let $f \in Y^*$ be a strongly exposing functional of K, then for any $g \in i^{*-1}(f)$, g is also a strongly exposing functional of K in X^*. Moreover, if the set of strongly exposing functionals of K in Y^* is dense in Y^*, by the open mapping theorem, the set of strongly exposing functionals of K in X^* is dense in X^*.

THEOREM 3.4. *Let X be a Banach space such that X^* is locally uniformly convex (with the dual norm) and let K be a weakly compact convex subset in X. Then the set of strongly exposing functionals $K^\wedge$ of K is a dense G_δ in X.*

PROOF. In view of Proposition 2.1 and Theorem 2.4, it suffices to show that for any bounded closed convex subset C of X such that $K \not\subset C$, there exists an $x_1 \in K$ which is a strongly exposed point of conv $(C \cup K)$ (since K is weakly compact, this set is closed). By the above remark and Proposition 3.1, 3.2, we may assume that K generates X and the norm on X is both locally uniformly convex and Fréchet differentiable. The Fréchet differentiability implies that every bounded closed convex set can be represented as the intersection of a family of closed balls (Theorem 2.2 in (Phelps (1960)). Choose a point

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$x_0 \in K \backslash C$, there exists a closed ball $B(y_0, r_0) = \{y : \|y - y_0\| \leqq r_0\}$ such that $B(y_0, r_0) \supseteq C$ but $\|x_0 - y_0\| > r_0 + \alpha$ for some $\alpha > 0$. Since the set

$$G = \{y \in X : \|y - x\| = \sup\{\|y - z\| : z \in K\} \text{ for some } x \in K\}$$

is a dense G_δ in X (Proposition 3.3). We may choose $y_1 \in G$ such that $\|y_1 - y_0\| < \alpha/2$ and $x_1 \in K$ with

$$\|y_1 - x_1\| = \sup\{\|y_1 - z\| : z \in K\} = r_1,$$

i.e., x_1 is a farther point in K. Note that $B(y_1, r_1)$ contains C and K, and x_1 is on the boundary of $B(y_1, r_1)$. By the locally uniformly convexity of the norm, $x_1 \in K$ is a strongly exposed point of $B(y_1, r_1)$, hence a strongly exposed point of conv $(C \cup K)$. This completes the proof.

THEOREM 3.5. *Let X be a Banach space such that X^* is weakly compact generated, then for any weakly compact subset K in X, the set of strongly exposing functionals $K^\wedge$ of K is a dense G_δ in X^*.*

PROOF. Again, without loss of generality, we may assume that X is weakly compact generated. By a theorem of John and Zizler (1972), X has an equivalent locally uniformly convex norm such that the dual norm is also a locally uniformly convex. Hence the result follows from the above theorem.

COROLLARY 3.6. *Let X be a Banach space whose dual is separable. Let K be a weakly compact convex subset in K, then $K^\wedge$ is a dense G_δ in X^*.*

COROLLARY 3.7. *Let X be a Banach space such that X^* has the Radon-Nikodym property. Then for each weakly compact, separable convex subset K of X, the set $K^\wedge$ is a dense G_δ in X^*.*

PROOF. Note that X^* has the Radon-Nikodym property if and only if for any separable subspace Y of X, Y^* is separable (Stegall (to appear), Uhr (1972)). Let Y be a closed subspace generated by K, then Y^* is separable. Hence by Corollary 3.6 and the previous remark, the set $K^\wedge$ is G_δ in X^*.

We do not know whether Theorem 3.4 will hold without the restriction on the dual of the Banach space X. Another particular case (Anantharaman (to appear)) is that if K is the closed convex hull of the range of a vector valued measure from a σ-algebra into X (hence K is weakly compact), then $K^\wedge$ is a dense G_δ in X^*.

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