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THE SOLUTIONS TO TWO PROBLEMS ON PERMUTATIONAL PRODUCTS

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Abstract

The two problems, both raised in the literature, are: (I) Is there, amongst all the permutational products (p.p.s) on the amalgam $\mathfrak{A} = (A, B; H)$ at least one which is a minimal generalized regular product? (II) If one of the p.p.s on $\mathfrak{A}$ is isomorphic to the generalized free product (g.f.p.) F on $\mathfrak{A}$ are they all? We answer both of them negatively.

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1. Preliminaries

Let F be the g.f.p. on the amalgam $\mathfrak{A} = (A, B; H)$ and let G be any group generated by $\mathfrak{A}$. Let ϕ denote the homomorphism from F onto G which extends the identity maps on A and B. Then G is called a generalized regular product (Wiegold (1961)) on $\mathfrak{A}$ if and only if $\ker \phi \subseteq [A, B]$, the cartesian subgroup of F. It was proved in Allenby (1969) that every p.p. is a generalized regular product. Further it is clear from a Zorn's Lemma argument that there exist in F normal subgroups N maximal with respect to both (i) $N \subseteq [A, B]$ and (ii) $(a \in A, b \in B$ and $ab^{-1} \in N) \to a = b \in H$. Then, for such an N, the group F/N is a minimal generalized regular product on $\mathfrak{A}$. Thus minimal generalized regular products exist on each amalgam. Gregorac (1969) asked if at least one of them was a p.p. on $\mathfrak{A}$.

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Assuming the reader to be familiar with the p.p. construction (see B. H. Neumann (1960)) problem (II) is self explanatory. It arose (Allenby and Tang (1975)) in connection with investigations into the Frattini subgroups of g.f.p.s.

2. The examples

(I) We take $A = \langle a, b; [a, b, a] = [a, b, b] = 1\rangle$, the free second nilpotent group of rank 2, $B = \langle c : \rangle$, and $H = \langle c^2\rangle = \langle b\rangle$. Theorem 2 of B. H. Neumann (1960) shows that since H is central in B, the isomorphism type of each p.p. on $\mathfrak{A} = (A, B; H)$ is independent of the choice of transversal of H in A. For each p.p. on $\mathfrak{A}$ we choose this transversal to be the set of elements $a^i[a, b]^j$. As a transversal of H in B we choose $\{c^{2m+1}, c^{2n}\}$. By a theorem of Mal'cev (1949), $\mathfrak{A}$ is embeddable in a nilpotent group of class 2. Consequently the generalized free second nilpotent product W, say, on $\mathfrak{A}$ exists (Wiegold (1959), Definition 4.1, Theorem 4.6). Further W is a generalized regular product on $\mathfrak{A}$, and is isomorphic to $F/[A, B, F]$ (Wiegold 1959, Theorem 4.6). Thus, to solve problem (I) we only need show that the kernel of the natural map ϕ_P from F onto each p.p. P on $\mathfrak{A}$ lies properly inside $[A, B, F]$.

By Allenby (1969) $\ker \phi_P \subseteq [A, B] \subseteq F$. Since F is generated by the subgroups $\langle a\rangle$ and $\langle c\rangle$ we have $[A, B] = F' = [\langle a\rangle, \langle c\rangle]$. Thus every element X of $[A, B]$ is expressible in the form

$$X = \prod_{i=1}^{t} \left[a^{A_i}, c^{2B_i}\right]\left[a^{C_i}, c^{2D_i+1}\right]\left[c^{2F_i}, a^{E_i}\right]\left[c^{2H_i+1}, a^{G_i}\right],$$

where the $A_iB_i, \ldots, H_i$ are all integers, possibly 0.

Now each p.p. P on $\mathfrak{A}$ is a permutation group on a set of triples of the form $(a^i[a, b]^j, c^{2m+1}, b^k)$ or $(a^i[a, b]^j, c^{2n}, b^k)$. The effect on these triples of the permutation

$$\rho(X) = \prod_{i=1}^{t} \left[\rho(a^{A_i}), \rho(c^{2B_i})\right]\left[\rho(a^{C_i}), \rho(c^{2D_i+1})\right] \cdot \left[\rho(c^{2F_i}), \rho(a^{E_i})\right]\left[\rho(c^{2H_i+1}), \rho(a^{G_i})\right]$$

is, on the one hand, the identity permutation (since $X \in \ker \phi_P$). On the other, one readily checks that

$$\left(a^i[a, b]^j, c^{2m+1}, b^k\right)^{\rho(X)} = \left(a^i[a, b]^{j+U}, c^{2m+1}, b^k\right)$$

whilst

$$\left(a^i[a, b]^j, c^{2n}, b^k\right)^{\rho(X)} = \left(a^i[a, b]^{j+V}, c^{2n}, b^k\right),$$

where

$$U = \sum_{i=1}^{t} A_iB_i + C_iD_i - E_iF_i - G_iH_i + (m-n)(G_i - C_i)$$

and

$$V = \sum_{i=1}^{t} A_iB_i + C_iD_i - E_iF_i - G_iH_i + (n-m-1)(G_i - C_i).$$

Since $[a, b]$ has infinite order we have $U = V = 0$. Hence $\Sigma_{i=1}^{t}(G_i - C_i) = 0$ and $Z = \Sigma_{i=1}^{t}(A_iB_i + C_iD_i - E_iF_i - G_iH_i) = 0$. Let $\bar{X}$ be the image of X in $\bar{F} = F/[A, B, F]$. Since $[\bar{A}, \bar{B}]$ is central in $\bar{F}$ we have

$$\bar{X} = \sum_{i=1}^{t} [\bar{a}, \bar{c}]^{2A_iB_i}[\bar{a}, \bar{c}]^{C_i(2D_i+1)}[\bar{c}, \bar{a}]^{2F_iE_i}[\bar{c}, \bar{a}]^{(2H_i+1)G_i}.$$

But then $\bar{X} = \bar{1}$ in $\bar{F}$–since $Z = \Sigma_{i=1}^{t}(G_i - C_i) = 0$. Thus $\ker \phi_P \subseteq [A, B, F]$.

Finally, this inequality is strict. For the permutation $[\rho(a), \rho(c), \ldots, \rho(c)]$, with $\alpha + 2$ copies of $\rho(c)$, maps

$$\left(a^i[a, b]^j, c^{2m+1}, b^k\right) \quad \text{to} \quad \left(a^i[a, b]^{j+Y}, c^{2m+1}, b^k\right),$$

where $Y = (-1)^\alpha 2^\alpha(2m - 2n + 1)$. This suffices.

(II) Here we take $A = \langle a : \rangle \times H$, $B = H \times \langle b : \rangle$ where H is free of countably infinite rank on the free generators

$$\{h_1, h_2, \ldots, k_1, k_2, \ldots, l_1, l_2, \ldots, m_1, m_2, \ldots\}.$$

For the first p.p., P_1, on $\mathfrak{A} = (A, B; H)$ we choose $\{1, a, a^{-1}, a^2, a^{-2}, \ldots\}$, $\{1, b, b^{-1}, b^2, b^{-2}, \ldots\}$ as transversals for A and B respectively modulo H. It is easy to check that P_1 is isomorphic to $\langle a \rangle \times H \times \langle b \rangle$.

For the second p.p., P_2, we choose, as transversals for A and B respectively modulo H, the sets

$$\left\{1, ah_1, a^{-1}k_1, a^2h_2, a^{-2}k_2, \ldots\right\} \quad \text{and} \quad \left\{1, bl_1, b^{-1}m_1, b^2l_2, b^{-2}m_2, \ldots\right\}.$$

Recall that $\ker \phi_{P_2}: F \to P_2$ is contained in $[A, B] \subseteq F$. We shall show that $\ker \phi_{P_2}$ is trivial. Thus let

$$X = \prod_{i=1}^{t} \left[a^{\alpha_i}H(i), b^{\beta_i}K(i)\right]\left[b^{\delta_i}M(i), a^{\gamma_i}L(i)\right]$$

be a typical element of $[A, B]$. Here the $H(i)$, $K(i)$, $L(i)$, $M(i)$ are from H and hence products of *hs*, *ks*, *ls*, *ms* and their inverses. Since a, b centralize H, X can be re-written

$$\begin{aligned} X &= \prod_{i=1}^{t} \left[a^{\alpha_i}, b^{\beta_i}\right]\left[a^{\gamma_i}, b^{\delta_i}\right]^{-1} \cdot \prod_{i=1}^{t} \left[H(i), K(i)\right]\left[L(i), M(i)\right]^{-1} \\ &= Y \cdot Z \quad \text{say.} \end{aligned}$$

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Suppose that the maximum suffix attached to any h, k, l or m occurring in the $H(i)$, $K(i)$, $L(i)$, $M(i)$ $(1 \leqslant i \leqslant t)$ is v. Set

$$u = \max\left(\sum_{i=1}^{t} |\alpha_i| + |\gamma_i| \quad \text{and} \quad \sum_{i=1}^{t} |\beta_i| + |\delta_i| \right)$$

and set $\lambda > 2\max\{u, v\}$.

Consider the effect of $\rho(X) \in P_2$ on $(a^\lambda h_\lambda, b^\lambda l_\lambda, 1)$. Now

$$\begin{aligned} \rho(X) &= \prod_{i=1}^{t} [\rho(a^{\alpha_i}), \rho(b^{\beta_i})][\rho(a^{\gamma_i}), \rho(b^{\delta_i})]^{-1} \\ &\quad \cdot \prod_{i=1}^{t} [\rho(H(i)), \rho(K(i))][\rho(L(i)), \rho(M(i))]^{-1} \\ &= \rho(Y) \cdot \rho(Z). \end{aligned}$$

One can check, recalling that $\lambda > 2\max\{u, v\}$ and that a and b centralize H, that $(a^\lambda h_\lambda, b^\lambda l_\lambda, 1)^{\rho(Y)} = (a^\lambda h_\lambda, b^\lambda l_\lambda, U)$ where

$$(*) \qquad U^{-1} = \prod_{i=1}^{t} [h_{\lambda-\alpha_i}^{-1} h_\lambda, l_{\lambda-\beta_i}^{-1} l_\lambda][h_{\lambda-\gamma_i}^{-1} h_\lambda, l_{\lambda-\delta_i}^{-1} l_\lambda]^{-1}.$$

(Notice that no k_i nor m_i appears in $(*)$ precisely because λ was chosen sufficiently large.) Now $(*)$ is a product of commutators (and their inverses) each belonging to the natural free generating set for the derived group of the free subgroup of H generated by

$$\{h_\lambda, k_\lambda\} \cup \{h_{\lambda-\alpha_i}^{-1} h_\lambda, h_{\lambda-\gamma_i}^{-1} h_\lambda, l_{\lambda-\beta_i}^{-1} l_\lambda, l_{\lambda-\delta_i}^{-1} l_\lambda \quad (1 \leqslant i \leqslant t)\}.$$

It follows that $U = 1$ in H if and only if $Y = 1$ in F.

Finally, if $Y = 1$, then $\rho(X) = \rho(Z)$ acts trivially on $(a^\lambda h_\lambda, b^\lambda l_\lambda, 1)$ if and only if $Z = 1$. If $Y \neq 1$ then $(a^\lambda h_\lambda, b^\lambda l_\lambda, 1)^{\rho(X)} = (a^\lambda h_\lambda, b^\lambda l_\lambda, V)$ where $V = U^{-1}Z$. But $V \neq 1_H$ since, by choice of λ, the suffices occurring on the hs and ls in U are disjoint from those occurring on the hs, ks, ls and ms in Z.

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